Proseminar Functionalanalysis
Hints to Problems 8 18.5.2005

51. Let M be a subset of a Hilbert space H, and let v, $w \in H$. Suppose that $(v, x) = (w, x)$ for all $x \in M$ implies $v = w$. If this holds for all v, $w \in H$, show that M is total (complete) in H, i.e $\overline{\text{span}} M = H$.

   Equivalent formulation: If $y \in M^\perp$ then $y = 0$. Show this implies $\overline{\text{span}} M = H$

52. Let $x_n(t) = t^n$, $n \in \mathbb{N}_0$. Show that $\{x_n : n \in \mathbb{N}_0\}$ is complete in $L^2(-1, 1)$.
   Hint: Assume $f \in L^2(-1, 1)$ satisfies $f \perp x_n$, $n \in \mathbb{N}_0$. Show $F \perp x_n$, $n \in \mathbb{N}_0$, where $F(t) = \int_{-1}^{t} f(s)\, ds$. Approximate F by a polynomial Q using the theorem of Weierstraß and use

   $$\int_{-1}^{1} F(t)^2\, dt = \int_{-1}^{1} F(t)(F(t) - Q(t))\, dt$$

   (justify) to argue $F = 0$, which implies $f = 0$.

   Step1: $f \perp x_0 \Leftrightarrow \int_{-1}^{1} f(t)\, dt = 0$
   Step2: $f \perp x_n$, $n \geq 1$ implies by integrating by parts

   $$0 = \int_{-1}^{1} f(t) x_n(t)\, dt = \cdots = -n \int_{-1}^{1} F(t) x_{n-1}\, dt.$$

   Hence $F \perp x_n$, $n \in \mathbb{N}$. Note $F(0) = F(1) = 0$
   Step3: Choose $\varepsilon > 0$ and a polynomial Q (by the Weierstraß Theorem) s.th. $||F - Q|| < \varepsilon/\sqrt{2}$. By the hint already given $||F||_{L^2} < \varepsilon$ follows. Since ε is arbitrary $F = 0$ follows. Differentiate formally to obtain $f = 0$.

53. On $\mathbb{C}^2$ let the operator $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $Tx = (\xi_1 + i\xi_2, \xi_1 - i\xi_2)$, where $x = (\xi_1, \xi_2)$. Find T'. Show that we have $T'T = TT' = 2\text{id}$.

54. Let $\{y_k : 1 \leq k \leq n\}$ be linearly independent elements of a Hilbert space H and define $\mathcal{K} = \{x \in H : (x, y_k) = c_k, 1 \leq k \leq n\}$, where $c_k \in \mathbb{R}$ are fixed constants. Show that there exists a unique element $x_0 \in \mathcal{K}$ with minimum norm. Furthermore show that x_0 is given by

   $$x_0 = \sum_{i=1}^{n} \beta_i y_i$$

   where the coefficients β_i are given by the solution of the system

   $$\begin{aligned} (y_1, y_1)\beta_1 + (y_2, y_1)\beta_2 + \cdots + (y_n, y_1)\beta_n &= c_1 \\ (y_1, y_2)\beta_1 + (y_2, y_2)\beta_2 + \cdots + (y_n, y_2)\beta_n &= c_2 \\ &\vdots \\ (y_1, y_n)\beta_1 + (y_2, y_n)\beta_2 + \cdots + (y_n, y_n)\beta_n &= c_n. \end{aligned}$$

Observe that $\mathcal{K}$ is the intersection of closed hyperplanes. We want to use the theorem on the projection onto a convex set. Show that $\mathcal{K}$ is not empty: we show more: define $\phi : H \to \mathbb{R}^n$, $\phi(x) = ((x, y_1), \ldots, (x, y_n))$. Claim: ϕ is surjective. Assume ϕ is not surjective, then there exists $\alpha \notin \phi(X)$. By the strict separation theorem there exists $\beta \in \mathbb{R}^n$, $\beta \neq 0$ with

$$\beta \cdot \phi(X) < \beta \cdot \alpha,$$

Deduce from this $\sum \beta_i y_i = 0$ which by linear independence of y_i implies $\beta = 0$.
Use the projection theorem to argue that the element x_0 of minimal norm is characterized by

$$(x_0, z) \geq 0, \qquad \text{for all } z \in \mathcal{K} - x_0.$$

Show $\mathcal{K} - x_0 = M^\perp$, with $M = span\{y_1, \ldots, y_n\}$ and deduce from this that x_0 is characterized by

$$(x_0, z) = 0 \qquad \text{for all } z \in M^\perp,$$

hence $x_0 \in M^{\perp\perp} = M$. This implies the desired representation.

55. The shaft angular velocity of an d–c motor driven from a variable current source u is governed by the following first order differential equation

$$\dot{\omega}(t) + \omega(t) = u(t).$$

The angular position θ of the motor shaft is the time integral of ω. Assume that the motor is initially at rest $\theta(0) = \omega(0) = 0$. Determine the field current u of minimum energy which rotates the shaft within one second to the new rest position $\theta = 1$, $\omega = 0$. The energy is measured by $\int_0^1 u^2(t)\, dt$.
Hint: Use problem 54
Show $\theta(t) = \int_0^t (1 - e^{-(t-s)}) u(s)\, ds$. the constraints $\theta(1) = 1$, $\omega(1) = 0$ lead to

$$\int_0^1 e^s u(s)\, ds = 0$$
$$\int_0^1 u(s)\, ds = 0$$

Use problem 54 with $y_1(t) = 1$, $y_2(t) = e^t$.