Proseminar Functionalanalysis
Problems 7 4.5.2005

43. Show that in a Hilbert space the orthogonal projection onto a nonempty, closed and convex subset is a contraction.

44. Let (x_n) be a sequence in a Hilbert space H which converges weakly to $x \in H$. If $\lim_{n\to\infty} ||x_n|| = ||x||$ then (x_n) converges strongly to x.

45. Let H_i, $i = 1, 2$, be Hilbert spaces and $H = H_1 \times H_2$. Endow H with the norm $||(x, y)|| = \max(||x||_1, ||x||_2)$, $x \in H_1$, $y \in H_2$ where $||\cdot||_i$ denote the norms induced by the inner products on H_i,, $i = 1, 2$. Is H a Hilbert space?

46. Let X be a real inner product space. Verify the polarization identity

$$4(x, y) = ||x + y||^2 - ||x - y||^2, \qquad x, y \in X.$$

47. Let $H = L^2(-a, a)$, $0 < a \le \infty$ and define $Px(t) = \frac{1}{2}(x(t) + x(-t))$. Find the domain, range and kernel of P. Show that P is an orthogonal projection.

48. The norm in a Banach space X is induced by an inner product if and only if it satisfies the parallelogram law

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2, \qquad x, y \in X.$$

49. Let M, N be closed subspaces of a Hilbert space H.

   (a) If $M \perp N$ then $M + N$ is a closed subspace of H.

   (b) If only $M \cap N = \emptyset$ holds the sum $M+N$ need not be closed. Hint: $M = \overline{\text{span}}\{e_1, e_3, e_5 \ldots\}$, $N = \overline{\text{span}}\{z_1, z_2, z_3, \ldots\}$, $z_n = \alpha_n e_{2n-1} + \frac{1}{n} e_{2n}$ with $\alpha_n = \sqrt{1 - \frac{1}{n^2}}$.

      i. Show $(z_n, z_m) = \delta_{nm}$, $M \cap N = \{0\}$.

      ii. Show $y = \sum_{n=1}^{\infty} \frac{1}{n} e_{2n}$ does not belong to $M + N$, however is the limit of elements in $M + N$.