Proseminar Functionalanalysis
Problems 6 26.4.2005

37. Let X be a finite dimensional normed space. Show $\dim X = \dim X^*$.

38. Let X be a finite dimensional normed space. Show that a sequence converges strongly if and only if it converges weakly.

39. (a) Let X be a separable metric space and $A \subset X$. Then A is separable.
    (b) In a normed space X the following statements are equivalent.
        i. X is separable.
        ii. $B(0,1)$ is separable.
        iii. $S(0,1) = \{x \in X : ||x|| = 1\}$ is separable.

40. Let (x_n) be a sequence in a normed space X. Show
    (a) If (x_n) converges strongly to $x \in X$, then $x_n \rightharpoonup x$.
    (b) If (x_n) converges weakly to $x \in X$, then (x_n) is bounded and
    $$||x|| \leq \liminf_{n\to\infty} ||x_n||.$$
    (c) If (x_n) converges weakly to $x \in X$ and (z_n^*) converges strongly to $z^* \in X^*$, then
    $$\lim_{n\to\infty} z_n^*(x_n) = z^*(x).$$

41. A sequence (x_n) in a normed space X is called a weak Cauchy sequence, if for all $x^* \in X^*$ the sequence $(x^*(x_n))$ of scalars is a Cauchy sequence.
    (a) Construct in c_0, respectively $C([0,1]$ an example of a weak Cauchy sequence, which does not converge weakly.
    (b) A weak Cauchy sequence is bounded.
    (c) In a reflexive space every weak Cauchy sequence converge weakly.

42. Endow $X = C([0,1])$ with a complete norm such that (f_n) converges to $f \in X$ pointwise whenever $||f_n - f|| \to 0$. Show that such a norm is equivalent to the usual supremum norm.