Proseminar Functionalanalysis
Problems 5 18.4.2005

30. Let $X = C([0,1])$, $W = \{f \in X \colon f' \in X, f'' \in X, f(0) = f(1) = 0\}$, endow X with the sup-norm and define $||f|| = \max\{||f||_\infty, ||f'||_\infty, ||f''||_\infty\}$ for $f \in W$. Consider the operator $T \colon W \to X$
$$(Tf)(t) = a_0(t)f''(t) + a_1(t)f'(t) + a_2(t)f(t), \qquad t \in [0,1]$$
with $a_i \in X$, $i = 0, 1, 2$.
Prove: T^{-1} exists and is continuous if the boundary value problem $Tf = g$ has a unique solution $f \in W$ for every choice of g.

31. Let X, Y be Banach spaces and $T \colon X \to Y$ (not a priori linear!). If $y^* \circ T \in X^*$ for all $y^* \in Y^*$, then T is linear and continuous.
Hint: for continuity you can use the closed graph theorem

32. Let X, Y be normed spaces and choose $\tilde{x} \in X$, $\tilde{x} \neq 0$, $\tilde{y} \in Y$. Show, that there is an operator $T \in \mathcal{L}(X, Y)$ with $T\tilde{x} = \tilde{y}$.
Hint: Look for an operator of the form $Tx = x^*(x)\tilde{y}$.

33. For each $x \in L^2([0,1])$ let $y = Tx$ be the solution of $y' + ay = x$ that satisfies $y(0) = 0$, where a is a real constant. Determine the adjoint T^*.

34. For each $x \in L^2([0,1])$ let $y = Tx$ be the solution of $y'' + ay' + by = x$ that satisfies $y(0) = y(1) = 0$, where a and b are constants. Determine the adjoint T^*. Is it ever true that $T = T^*$.

35. Let $I = [0, T]$ and $k \colon I \times I \to \mathbb{C}$ satisfy $\int_I \int_I |k(s,t)|^2 \, ds \, dt < \infty$. Define $K \colon L^2(I) \to L^2(I)$ by
$$(Kx)(t) = \int_0^t k(t,s)x(s) \, ds.$$
(The operator K is an integraloperator of Volterra type). Determine the adjoint K^*.

36. Let $T \in \mathcal{L}(X)$ be an isomorphism, i.e. T^{-1} exists and is continuous. Show that $(T^{-1})^* = (T^*)^{-1}$.