Proseminar Functionalanalysis
Problems 4 8.4.2005

24. Let X be a complex normed space.

    (a) Let $l\colon X \to \mathbb{R}$ be $\mathbb{R}$-linear and define

    $$\tilde{l}(x) = l(x) - il(ix), \quad x \in X.$$

    Show $\tilde{l}$ is $\mathbb{C}$-linear and $l = \operatorname{Re}\tilde{l}$.

    (b) Let $h\colon X \to \mathbb{C}$ be $\mathbb{C}$-linear, $l = \operatorname{Re} h$ and $\tilde{l}$ as above, then l is $\mathbb{R}$-linear and $\tilde{l} = h$.

    (c) Let $l\colon X \to \mathbb{C}$ be $\mathbb{C}$-linear. Show that l is continuous if and only if $\operatorname{Re} l$ is continuous.

    (d) Let $x^* \in X^*$. Show $||x^*|| = ||\operatorname{Re} x^*||$.

25. Let $(X, ||\cdot||$ be a normed space over $\mathbb{C}$ and x', $y' \in L(X, \mathbb{C})$. Show

    (a) $\operatorname{Re} x'(x) \leq \operatorname{Re} y'(x)$ for all $x \in X$ implies $x' = y'$.

    (b) $|\operatorname{Re} x'(x)| \leq |\operatorname{Re} y'(x)|$ for all $x \in X$ implies $x' = \alpha y'$ for some $\alpha \in \mathbb{R}$.

26. Show that a closed set F is nowhere dense if and only if it contains no open set.

27. Prove that a set M is nowhere dense if and only if for any nonempty open set O there is a ball contained in $O \setminus M$.

28. Let $\mathcal{P}$ be the space of all polynomials on $\mathbb{R}$ and $||\cdot||$ any norm on $\mathcal{P}$. Use Baire's category theorem to show that $\mathcal{P}$ is not complete.

29. Let $f\colon [0, \infty) \to \mathbb{R}$ be continuous and satisfy $\lim_{n\to\infty} f(nt) = 0$ for all $t \geq 0$. Show $\lim_{t\to\infty} f(t) = 0$.