Proseminar Functionalanalysis
Problems 3 29.3.2005

18. * Consider the weakly singular operator with domain $C([0,1])$ defined by $(Tx)(t) = \int_0^1 k(t,s)x(s)\,ds$, $t \in [a,b]$. The kernel k is given by

$$k(t,s) = \begin{cases} |t-s|^{-\alpha} & s \neq t, \\ 0 & \text{else} \end{cases}$$

Show that $T \in \mathcal{L}(C([0,1]))$ for $\alpha \in (0,1)$.

19. Let $T\colon \mathbb{R} \to \mathbb{R}$ be a continuous and additive mapping, i.e. $T(x+y) = T(x) + T(y)$ holds for all x, $y \in \mathbb{R}$. Show that T is linear.

20. Let $X = (\mathbb{R}^3, ||\cdot||_1$ and $W = \{(x,y,z)\colon x + 2y = z = 0\}$ and let $x^* \in W^*$ be the functional $x^*((x,y,z)) = x$. Construct an extension of x^* from W to X which preserves the norm.

21. Let $(X, ||\cdot||$ be a normed space and $x^* \in X^*$. Show

$$||x^*|| = \sup_{x \in M} \frac{|x^*(x)|}{||x||},$$

where M is an arbitrary hyperplane $\{x \in X\colon x^*(x) = \alpha\}$, $\alpha \neq 0$.

22. Show that $(l^1)^*$ is isometrically isomorphic to l^∞.

23. Let $c = \{x = (a_i)_{i=1}^\infty\colon \lim_{i\to\infty} a_i \text{ exists}\}$, $||x|| = \sup_i |a_i|$. Find a characterization of c^*.
Hint: Set $e_0 = (1,1,1,\dots)$ and $e_k = (0,\dots,0,1,0,\dots)$, and $a_0 = \lim_{k\to\infty} a_k$ then

$$x = a_0 e_0 + \sum_{k=1}^{\infty} (a_k - a_0) e_k.$$