Proseminar Functionalanalysis
Problems 2 8.3.2005

7. Let X be a vector space, $f \in L(X,\mathbb{R})$ and $x_0 \notin \ker f$. Show

$$X = \ker f \bigoplus \operatorname{span}\{x_0\},$$

i.e. any $x \in X$ has a unique representation $x = \alpha x_0 + y$, $\alpha \in \mathbb{K}$, $y \in \ker f$.

8. Show that two functionals f_1 and f_2 which are defined on the same vector space and have the same kernel are linearly dependent.

9. Consider the vector space $C^{0,\beta}([0,1])$ of Hölder continuous functions $x\colon [0,1] \to \mathbb{R}$ with exponent $\beta \in (0,1]$ and define

$$||x|| = |x(0)| + \sup_{s \neq t} \frac{|x(s) - x(t)|}{|s-t|^\beta}$$

Show that $||\cdot||$ defines a norm on $C^{0,\beta}([0,1])$ and verify $||x||_\infty \leq ||x||$. Furthermore, show that $(C^{0,\beta}([0,1]), ||\cdot||)$ is complete. What can you say about the spaces $C^{0,\beta}([0,1])$ with $\beta > 1$.
Recall: a function $x\colon [0,1] \to \mathbb{K}$ is called Hölder continuous with exponent $\beta \in (0,1]$ if there is a constant $l > 0$ such that

$$|x(s) - x(t)| \leq l|s-t|^\beta$$

holds for all s, $t \in [0,1]$. If $\beta = 1$ the function is called Lipschitz continuous.

10. Let $v_n(x) = \frac{1}{n}\sin nx$. Show that $v_n \to 0$ in $C^{0,\beta}([0,1])$ for any $\beta \in (0,1)$, but $v_n \nrightarrow 0$ in $C^{0,1}([0,1])$.

11. Consider $v(x) = x^\alpha$ for some $\alpha \in (0,1)$. For which $\beta \in (0,1]$ is it true that $v \in C^{0,\beta}([0,1])$.

12. Show: $T \in \mathcal{L}(X)$ if and only if T maps bounded sets into bounded sets.

13. Let $T \in L(X,Y)$ and $\dim X < \infty$. Show that T is continuous.

14. A matrix $A \in \mathbb{K}^{m\times n}$ defines a linear map $\mathbb{K}^n \to \mathbb{K}^m$. Calculate the norm of this map, $||A||$, when in both spaces $\mathbb{K}^n$ and $\mathbb{K}^m$ either $||\cdot||_1$ or $||\cdot||_\infty$ is used.

15. Let $X = C([0,1])$ be endowed with the natural norm and define $T \in L(X)$ by

$$(Tx)(t) = \frac{t}{1+t^2}x(t).$$

Show $T \in \mathcal{L}(X)$ and compute $||T||$.

16. Consider the functional $s\colon \ell^1 \to \mathbb{C}$ defined by $s(x) = \sum_{i=1}^\infty x_i$. Is s continuous?

17. In a normed space $(X, ||\cdot||)$ a series $\sum_{i=1}^\infty x_i$ converges absolutely if $\sum_{i=1}^\infty ||x_i||$ converges. Show that absolute convergence implies convergence if and only if $(X, ||\cdot||)$ is complete. Give an example of a sequence which converges absolutely but does not converge.
Hint: In order to show completeness given a Cauchy sequence try to construct a subsequence which satisfies $||x_{n_k} - x_{n_{k-1}}|| < \frac{1}{2^{k-1}}$ and use a telescoping series.