Proseminar
Functionalanalysis
Problems 1 2.3.2005

1. The distance $\operatorname{dist}(A, B)$ between two nonempty subsets A and B of a metric space (X, d) is defined as
$$\operatorname{dist}(A, B) = \inf\{d(a, b) \colon a \in A, b \in B\}.$$
Show that dist does not define a metric on the powerset of X.

2. Show that the metric space $(C([a, b]), d_\infty)$ is complete.

3. Endow $\ell^\infty = \{x = (x_n)_{n\in\mathbb{N}} \colon x_n \in \mathbb{C}, (x_n) \text{ is bounded}\}$ with the metric $d(x, y) = \sup\{|x_n - y_n| \colon n \in \mathbb{N}\}$. Show that the metric space (ℓ^∞, d) is complete.

4. A subspace M of a complete metric space (X, d) is complete if and only if M is closed in X.

5. Let $c_0 = \{(x_n) \in \ell^\infty \colon \lim_{n\to\infty} x_n = 0\}$. Show that (c_0, d) is complete.

6. Show that $d \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ given by
$$d(x, y) = |\arctan(x) - \arctan(y)|$$
defines a metric on $\mathbb{R}$. Is $(\mathbb{R}, d)$ complete?