Proseminar Functionalanalysis
Problems 10 14.6.2005

63. Let $X = C([0,1])$ and $K \in \mathcal{L}(X)$ be defined by $(Kf)(x) = \int_0^x f(t)\,dt$. Compute the spectral radius of K using $r(K) = \lim_{n\to\infty} ||K^n||^{1/n}$.

64. Let $X = L^2(0,1)$ and define $Tf(t) = tf(t)$. Investigate the spectrum of T.

65. Let H be a complex Hilbert space and $T \in \mathcal{L}(H)$ be selfadjoint. Then $\lambda \in \rho(T)$ if and only if there exists $m > 0$ such that

$$||(\lambda - T)x|| \geq m||x||$$

holds for all $x \in H$.

66. Let $X = C([0,1])$ and $K \in \mathcal{L}(X)$ be defined by $(Kf)(x) = \int_0^{1-x} f(t)\,dt$. Determine the spectrum of T.

67. Let (α_n) be a sequence of complex numbers and $p \in [1,\infty)$. Define an operator T on l^p by

$$(Tu)_n = \alpha_n u_n, \qquad n \in \mathbb{N},$$

where $u = (u_1, u_2, \dots)$.

   (a) Show that T is continuous if and only if the sequence (α_n) is bounded.
   (b) When T is continuous, compute its eigenvalues and spectrum.

68. Suppose $p \in [1,\infty]$. Consider the left shift T on l^p defined by

$$(Tu)_n = u_{n+1}, \qquad n \in \mathbb{N},$$

where $u = (u_1, u_2, \dots)$, $u_i \in \mathbb{K}$.

   (a) If $p < \infty$ show that $\sigma_p(T) = \{\lambda \in \mathbb{K} \colon |\lambda| < 1\}$. If $p = \infty$ show that $\sigma_p(T) = \{\lambda \in \mathbb{K} \colon |\lambda| \leq 1\}$.
   (b) Deduce that $\sigma(T) = \{\lambda \in \mathbb{K} \colon |\lambda| \leq 1\}$ in both cases

69. Let $X = l^p$, $p \in [1,\infty]$ and consider the right shift T defined by

$$Tu = (0, u_1, u_2, \dots), \quad u \in X.$$

Show that $\sigma(T) = \{\lambda \in \mathbb{K} \colon |\lambda| \leq 1\}$ and that T has no eigenvalues.