THE MONOID OF REGULAR ELEMENTS IN COMMUTATIVE RINGS WITH ZERO DIVISORS

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ABSTRACT. Let R be a commutative ring with identity, $R^\bullet$ be the multiplicative monoid of regular elements in R, t be the so-called t-operation on R or $R^\bullet$. A Marot ring is a ring whose regular ideals are generated by their regular elements. Marot rings were introduced by J. Marot in 1969 and have been playing a key role in the study of rings with zero divisors. The notion of Marot rings can be extended to t-Marot rings such that Marot rings are t-Marot rings. In this paper, we study some ideal-theoretic relationships between a t-Marot ring R and the monoid $R^\bullet$. We first construct an example of a t-Marot ring that is not Marot. This also serves as an example of a rank-one DVR of reg-dimension ≥ 2. Let R be a t-Marot ring, t-spec(R) (resp., t-spec$(R^\bullet)$) be the set of regular prime t-ideals of R (resp., the set of non-empty prime t-ideals of $R^\bullet$), and $Cl(A)$ be the class group of A for $A = R$ or $R^\bullet$. Then, among other things, we prove that the map $\varphi : t\text{-spec}(R) \to t\text{-spec}(R^\bullet)$ given by $\varphi(P) = P^\bullet$ is bijective; $Cl(R) \cong Cl(R^\bullet)$; and R is a factorial ring if and only if $R^\bullet$ is a factorial monoid.

INTRODUCTION

All rings considered in this paper are commutative rings with identity. Throughout, we denote by R a ring, by $\mathsf{T}(R)$ the total quotient ring of R, and by $\mathsf{Z}(R)$ the set of zero divisors in R. An element which is not a zero divisor is said to be regular. For a subset $X \subseteq \mathsf{T}(R)$, we let $X^\bullet = X \setminus \mathsf{Z}\big(\mathsf{T}(R)\big)$ be the set of all regular elements in X, and we say that X is regular if $X^\bullet \neq \emptyset$. In particular, an ideal is called a regular ideal if it contains a regular element. Clearly, $R^\bullet$ is a monoid under the multiplication of R. We say that $R^\bullet$ is the monoid of regular elements of R, and we let $\mathsf{q}(R^\bullet)$ denote the quotient group of $R^\bullet$; so $\mathsf{q}(R^\bullet) = \mathsf{T}(R)^\bullet$. Other definitions and notations will be reviewed in Section 1.

We say that R is Marot if each regular ideal of R is generated by its regular elements. The notion of Marot rings was introduced by Marot [21]. The Marot property is very useful when we study the ideal-theoretic properties of rings with zero divisors, and many ring-theoretic properties of integral domains can be generalized to Marot rings. Furthermore, many important classes of rings with zero divisors (e.g., Noetherian rings, polynomial rings, overrings of a Marot ring) have the Marot property [16]. It is well known that an integral domain R is a Krull domain if and only if $R^\bullet$ is a Krull monoid [20, Proposition]. Halter-Koch formulated these equivalent conditions on Marot rings (i.e., he proved that if R is a Marot ring, then R is a Krull ring if and only if $R^\bullet$ is a Krull monoid [14, Theorem]). Then, in [12, Theorem 3.5], the authors introduced the notion of v-Marot rings and showed that a v-Marot ring R is a Krull ring if and only if $R^\bullet$ is a Krull monoid.

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It is well known that $I_t = I_v$ for every regular fractional ideal I of a Krull ring, and the t-operation is a very useful tool for the study of ideal-theoretic characterizations of integral domains. For example, by making use of the t-operation, we can generalize Dedekind domains, principal ideal domains (PIDs), and Prüfer domains to Krull domains, UFDs, and PvMDs, respectively, as follows: (i) D is a Krull domain if and only if every nonzero ideal of D is t-invertible, (ii) D is a UFD if and only if every t-ideal of D is principal, and (iii) D is a PvMD if and only if every nonzero finitely generated ideal of D is t-invertible. Thus, it is natural to consider the t-analog of the Marot property when we study commutative rings with zero divisors. Recently, in [8], Elliott introduced the notion of t-Marot rings. In this paper, we study the ideal-theoretic relationships between a t-Marot ring R and the monoid $R^\bullet$.

This paper consists of five sections including the introduction. Let t-spec(R) (resp., t-spec$(R^\bullet)$) be the set of regular prime t-ideals of R (resp., non-empty prime t-ideals of $R^\bullet$), and R be a t-Marot ring. In Section 1, we first review some definitions and preliminary results for better understanding of the paper. In Section 2, we study some basic properties of t-Marot rings. Among them, we first construct a t-Marot ring that is not a Marot ring. This example also serves to show that rank-one DVRs need not be of reg-dimension one. We show that if $P \in t$-spec(R), then $P^\bullet \in t$-spec$(R^\bullet)$, and conversely, if $I \in t$-spec$(R^\bullet)$, then $(IR)_t \in t$-spec(R), and hence the map $\varphi : t\text{-spec}(R) \to t\text{-spec}(R^\bullet)$ given by $\varphi(P) = P^\bullet$ is an order-preserving bijection. In Section 3, we show that if I is an ideal of $R^\bullet$, then I is t-invertible if and only if IR is t-invertible. Hence, R is a PvMR if and only if $R^\bullet$ is a PvMM. We also show that $Cl(R) \cong Cl(R^\bullet)$. Finally, in Section 4, we show that R is a weakly Krull ring (resp., Krull ring, weakly factorial ring) if and only if $R^\bullet$ is a weakly Krull monoid (resp., Krull monoid, weakly factorial monoid).

1. Definitions and preliminary results

Let R be a commutative ring with identity and $\mathsf{T}(R)$ be the total quotient ring of R. An overring of R is a subring of $\mathsf{T}(R)$ containing R. A *fractional ideal* I of R is an R-submodule of $\mathsf{T}(R)$ such that $dI \subseteq R$ for some $d \in R^\bullet$, and an *(integral) ideal* I of R is a fractional ideal of R with $I \subseteq R$. Throughout this paper, by a monoid, we always means a commutative cancellative monoid, so we can consider the quotient group of a monoid. Let H be a monoid. Then a subset A of H is an (semigroup) *ideal* if $AH = \{ah \mid a \in A, h \in H\} = A$. An ideal A is *finitely generated* if $A = EH$ for some finite subset E of A.

1.1. **General definitions of rings.** Let P be a regular prime ideal of R. The regular-height of P is defined by reg-ht$P = \sup\{n \mid P_1 \subsetneq \cdots \subsetneq P_n = P$ and each P_i is a regular prime ideal of $R\}$. Then the regular-dimension of R is defined by

$$\text{reg-dim}(R) = \sup\{\text{reg-ht}P \mid P \text{ is a regular prime ideal of } R\}.$$

Thus, reg-ht$P \leq$ ht$P$, reg-dim$(R) \leq \dim(R)$, and equalities hold if R is an integral domain. Let $X_r^1(R)$ be the set of regular height-one prime ideals of R.

Let S be a multiplicative set of R. Then there are two types of localizations of R with respect to S;

(1) $R_{(S)} = \{\frac{a}{b} \,|\, a \in R \text{ and } b \in S^\bullet\}$.

(2) $R_{[S]} = \{z \in \mathsf{T}(R) \mid zs \in R \text{ for some } s \in S\}$.

Clearly, $R_{(S)}$ and $R_{[S]}$ are overrings of R, $R_{(S)} \subseteq R_{[S]}$, and if $S \subseteq R^\bullet$, then $R_{(S)} = R_S$. If P is a prime ideal of R, then we set $R_{(P)} = R_{(R \setminus P)}$ and $R_{[P]} = R_{[R \setminus P]}$. It is well known that if R is a Marot ring, then $R_{[S]} = R_{(S)}$ [16, Theorem 7.6]. Moreover, if I is an ideal of R, then $[I]R_{[P]} = \{x \in \mathsf{T}(R) \mid xa \in I \text{ for some } a \in R \setminus P\}$ is an ideal of $R_{[P]}$ such that $IR_{[P]} \subseteq [I]R_{[P]}$.

1.2. **The t-operation.** Let $F(R)$ be the set of R-submodules of $\mathsf{T}(R)$. For $I \in F(R)$, let $I^{-1} = \{x \in \mathsf{T}(R) \mid xI \subseteq R\}$; then $I^{-1} \in F(R)$. Hence, $I_v := (I^{-1})^{-1}$ and $I_t := \bigcup\{J_v \mid J \text{ is a finitely generated fractional subideal of } I\}$ are well-defined. Let $* = v$ or t. Then, for any $a \in \mathsf{T}(R)$ and $I, J \in F(R)$;

(1) $aI_* \subseteq (aI)_*$, and equality holds if a is regular.
(2) $I \subseteq I_*$; $I \subseteq J$ implies $I_* \subseteq J_*$.
(3) $(I_*)_* = I_*$.
(4) $(IJ)_* = (IJ_*)_*$.

A fractional ideal I of R is said to be *regular* if $I \cap \mathsf{T}(R)^\bullet \neq \emptyset$, so I is regular if and only if dI is a regular ideal of R for some $d \in R^\bullet$. Let $F_r(R)$ be the set of regular fractional ideals of R. We say that $I \in F_r(R)$ is a *regular fractional $*$-ideal* if $I_* = I$. Moreover, an ideal I of R is called an *(integral) $*$-ideal* if $I_* = I$. A $*$-ideal I of R is *of finite type* if $I = J_*$ for some finitely generated ideal J of R. A *maximal t-ideal* of R is a t-ideal that is maximal among proper integral t-ideals of R. It is easy to see that each maximal t-ideal is a prime ideal, each regular integral t-ideal is contained in a maximal t-ideal, a prime ideal minimal over an integral t-ideal is a t-ideal, each regular principal fractional ideal is a v-ideal, each v-ideal is a t-ideal, $I \subseteq I_t \subseteq I_v$ for all $I \in F_r(R)$, and $I_t = I_v$ if I is finitely generated. We say that R is a *Mori ring* (or *v-Noetherian ring*) if R satisfies the ascending chain condition on regular integral v-ideals of R, and in this case, $I_t = I_v$ for all $I \in F_r(R)$.

Lemma 1.1. *Let A be a regular fractional ideal of a ring R and I be a fractional ideal of $R^\bullet$.*

1. $A_t = \bigcup\{J_v \mid J \subseteq A$ *is a finitely generated regular fractional ideal of* $R\}$.
2. $(IR)_t = \bigcup\{(JR)_v \mid J \subseteq I$ *is a finitely generated fractional ideal of* $R^\bullet\}$.

Proof. 1. Let B be a finitely generated fractional subideal of A and $a \in A^\bullet$. Then $J := B + aR$ is a finitely generated regular fractional ideal of R such that $B_v \subseteq J_v \subseteq A_t$. Thus,

$$\begin{aligned} A_t &= \bigcup \{B_v \mid B \subseteq A \text{ is a finitely generated fractional ideal of } R\} \\ &= \bigcup \{J_v \mid J \subseteq A \text{ is a finitely generated regular fractional ideal of } R\}. \end{aligned}$$

2. Let $x_1, \ldots, x_n \in IR$. Then there are some $a_i \in I$ and $r_{ij} \in R$ such that $x_j = \sum_i a_i r_{ij}$. Hence, if J is the fractional ideal of $R^\bullet$ generated by $\{a_i\}$, then J is finitely generated and $(x_1, \ldots, x_n)R \subseteq JR$; so $((x_1, \ldots, x_n)R)_v \subseteq (JR)_v$. Thus, the result follows. □

Let H be a monoid and $\mathsf{q}(H)$ be the quotient group of H. The v- and t-operations on H can be defined as in commutative rings with identity. The reader can refer to [11, 15] for more on the v- and t-operation on H.

1.3. **The class groups of R and $R^\bullet$.** An $I \in F_r(R)$ is said to be *t-invertible* if $(II^{-1})_t = R$. Let $Tinv(R)$ be the set of t-invertible regular fractional t-ideals of R. Then $Tinv(R)$ is an abelian group under $I \cdot_t J = (IJ)_t$. Let $Prin(R)$ be its subgroup of regular principal fractional ideals, and

$$Cl(R) = Tinv(R)/Prin(R).$$

We say that $Cl(R)$ is the *t-class group* or the *class group* of R (see, for example, [8, Definition 2.5.21]). Hence if R is a Krull ring, then $Cl(R)$ is the divisor class group of R. For $I \in \mathit{Tinv}(R)$, let $[I]$ be the class in $\mathit{Cl}(R)$ containing I. Hence, if $I, J \in \mathit{Tinv}(R)$, then $[I] = [J]$ if and only if $I = qJ$ for some $q \in \mathsf{q}(R^\bullet)$. In a similar way, we define $\mathit{Tinv}(R^\bullet)$, $\mathit{Prin}(R^\bullet)$, and the class group $\mathit{Cl}(R^\bullet)$ for the monoid $R^\bullet$ of regular elements of R (see, [15, Chapter 12]).

1.4. **Rank-one DVRs and DVMs.** Let $\mathbb{Z}$ be the additive group of integers. Extend $\mathbb{Z}$ by the symbol ∞ by defining $n < \infty$, $n + \infty = \infty + \infty = \infty$ for all $n \in \mathbb{Z}$, and $\infty - \infty$ undefined. Let T be a commutative ring with identity. A *rank-one discrete valuation* on T is a mapping v from T onto $\mathbb{Z} \cup \{\infty\}$ with the following properties for all $x, y \in T$;

(1) $v(xy) = v(x) + v(y)$.
(2) $v(x + y) \geq \min\{v(x), v(y)\}$.
(3) $v(1) = 0$ and $v(0) = \infty$.

If there is a rank-one discrete valuation v on $\mathsf{T}(R)$ such that

$$R = \{x \in \mathsf{T}(R) \mid v(x) \geq 0\} \quad \text{and} \quad P = \{x \in \mathsf{T}(R) \mid v(x) > 0\},$$

then (R, P) is called a rank-one discrete valuation pair of $\mathsf{T}(R)$, and R is called a *rank-one discrete valuation ring* (rank-one DVR). Clearly, if P is regular, then reg-ht$P = 1$. Moreover, if R is a Marot ring such that P is regular, then P is principal and a unique regular maximal ideal of R, and thus reg-dim$(R) = 1$. However, this is not true in general (see, for example, [3, Example 5.4] and Example 2.2).

Let H be a monoid and $H^\times$ be the group of units of H. Then $H_{\text{red}} = H/H^\times$ is a monoid. Let $\mathbb{N}$ be the additive monoid of nonnegative integers. We say that H is a *rank-one discrete valuation monoid* (rank-one DVM) if $H_{\text{red}} \cong \mathbb{N}$ as monoids.

1.5. **Krull rings and monoids.** We say that R is a *Krull ring* if there exists a family $\{(V_\alpha, P_\alpha) \mid \alpha \in \Lambda\}$ of rank-one discrete valuation pairs of $\mathsf{T}(R)$ with associated valuations $\{v_\alpha \mid \alpha \in \Lambda\}$ such that

(i) $R = \bigcap\{V_\alpha \mid \alpha \in \Lambda\}$,
(ii) for each $a \in \mathsf{T}(R)^\bullet$, $v_\alpha(a) = 0$ for almost all $\alpha \in \Lambda$ and P_α is a regular ideal for all $\alpha \in \Lambda$.

It is known that the integral closure of a ring whose regular ideals are finitely generated is a Krull ring [6, Theorem 13], and the polynomial ring $R[X]$ is a Krull ring if and only if R is a finite direct sum of Krull domains [16, Theorem 8.16]. It is also known that R is a Krull ring if and only if R is a completely integrally closed Mori ring ([19, Proposition 2.2] and [22, Theorem 5]), if and only if every regular ideal of R is t-invertible [18, Theorem 13].

Let H be a monoid with quotient group $\mathsf{q}(H)$. We say that H is a *Krull monoid* if there exists a family $\{V_\alpha \mid \alpha \in I\}$ of rank-one DVMs such that

(i) $H = \bigcap\{V_\alpha \mid \alpha \in I\}$.
(ii) for each $z \in \mathsf{q}(H)$, the set $\{V_\alpha \mid \alpha \in I,\, z \notin V_\alpha^\times\}$ is finite.

Then H is a Krull monoid if and only if H is a completely integrally closed Mori monoid, if and only if each non-empty ideal of H is t-invertible [15, Theorem 22.8].

It is known that if R is a Krull ring, then $R^\bullet$ is a Krull monoid ([14, Proof of the Theorem (Part I)] or [3, Theorem 5.1(1)]). However, $R^\bullet$ being a Krull monoid does not imply that R is a Krull ring (see, for example, [3, Example 5.2]).

1.6. **Idealization.** Let M be a unitary R-module, and consider

$$R(+)M = \{(r, m) \mid r \in R \text{ and } m \in M\}.$$

For all elements (r, a) and (s, b) of $R(+)M$, if we define

- $(r, a) = (s, b)$ if and only if $r = s$ and $a = b$,

- $(r, a) + (s, b) = (r + s, a + b)$, and
- $(r, a)(s, b) = (rs, rb + sa)$,

then $R(+)M$, called the *idealization* of M in R, becomes a commutative ring with identity. There exists a canonical map from R into $R(+)M$ given by $r \mapsto (r, 0)$, and hence R can be embedded into $R(+)M$. The set $(0)(+)M$ is an ideal of $R(+)M$, giving rise to the name idealization. For more on basic properties of idealizations, see [16, Section 25] and [4].

The reader can refer to [16] for commutative rings with zero divisors and [11, 15] for monoids.

2. t-Marot rings

Let R be a commutative ring with identity and $T = \mathsf{T}(R)$ be the total quotient ring of R. A *Marot ring* is a ring in which every regular ideal is generated by a set of regular elements. Hence, R is Marot if and only if $I = I^\bullet R$ for all regular ideals I of R. As the v-operation analog, in [12], the authors called R a *v-Marot ring* if $I = (I^\bullet R)_v$ for all regular v-ideals I of R. They also showed that a Marot ring is v-Marot (in fact, this is clear by definition), and R is a v-Marot ring if and only if $I_v = \bigcap_{\substack{z \in T^\bullet \\ zR \supseteq I}} zR$ for every regular ideal I of R [12, Lemma 3.1 and Proposition 3.3]. Anderson and Markanda first noted that there is a ring R with a regular ideal I such that $I_v \subsetneq \bigcap_{\substack{z \in T^\bullet \\ zR \supseteq I}} zR$ [2, Example].

Definition 2.1. We will say that R is a *t-Marot ring* if $I = (I^\bullet R)_t$ for all regular t-ideals I of R.

The notion of t-Marot rings was introduced by Elliott [8, Definition 2.7.21] in a more general setting of semistar operations. Clearly, Marot rings are t-Marot rings, and since every v-ideal is a t-ideal, t-Marot rings are v-Marot rings, i.e.,

$$\text{Marot} \quad \Rightarrow \quad t\text{-Marot} \quad \Rightarrow \quad v\text{-Marot}.$$

We next give an example of a t-Marot ring that is not Marot. This example also shows that rank-one DVRs need not be of reg-dimension one. However, we don't know an example of a v-Marot ring that is not t-Marot (cf. [8, Open Problem 2.7.24]).

Example 2.2. Let K be a field, X, Y be indeterminates over K, and $D = K[X, Y]$ be the polynomial ring over K. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be the set of maximal ideals of D not containing Y, $M = \sum_{\lambda \in \Lambda} D/M_\lambda$ be the direct sum of D-modules $\{D/M_\lambda\}_{\lambda \in \Lambda}$, and $R = D(+)M$ be the idealization of M in D. Then

1. R is a t-Marot ring that is not Marot.
2. R has a regular ideal A such that $(A^\bullet R)_t \subsetneq A \subsetneq A_t$.
3. reg-dim$(R) = \dim(R) = 2$.
4. R is a rank-one DVR.

Proof. 1. (i) Let $\mathsf{Z}(M) = \{x \in D \mid xm = 0 \text{ for some } 0 \neq m \in M\}$. Then $\mathsf{Z}(M) = \bigcup_{\lambda \in \Lambda} M_\lambda$. Moreover, if p is a prime element of D with $pD \neq YD$, then $p \in \bigcup_{\lambda \in \Lambda} M_\lambda$.

(ii) $R^\bullet = \{(\alpha Y^n, m) \mid 0 \neq \alpha \in K, n \geq 0 \text{ is an integer, and } m \in M\}$, and hence $\{(Y^n, 0)R \mid n \geq 0\}$ is the set of ideals of R generated by a set of regular elements.

(iii) Note that $(X, Y)(+)M$ is a regular ideal of R but it is not generated by a set of regular elements. Thus, R is not a Marot ring. (For a complete proof of (i)-(iii), see [13]).

(iv) Let $S = D \setminus \mathsf{Z}(M)$. Then $\mathsf{T}(R) = D_S(+)M_S$ [16, Corollary 25.5]. Let $f \in S$, and assume that p is a prime divisor of f in D. If $pD \neq YD$, then $p \in \bigcup_{\lambda\in\Lambda} M_\lambda$ by (i), and hence $f \in \mathsf{Z}(M)$, a contradiction. Note that D is a UFD and Y is a prime element of D; hence, $f = \alpha Y^n$ for some $0 \neq \alpha \in K$ and an integer $n \geq 0$. Note also that $\alpha Y^n M = M$. Thus, $M_S = M$ and $\mathsf{T}(R) = D_S(+)M$.

(v) Let $A = (\{(a_1, m_1), \ldots, (a_k, m_k)\})$ be a finitely generated regular ideal of R with $(a_1, m_1) \in R^\bullet$, and let I be the ideal of D generated by $\{a_1, \ldots, a_k\}$. Then $A = I(+)M$ [16, Theorem 25.1(1)] because $a_1 M = M$. Hence, $A^{-1} = I^{-1}(+)M$ [16, Theorem 25.10], and thus $A_v = I_v(+)M$. Note that D is a UFD; so $I_v = gD$ for some $g \in D$, whence by (ii), $A_v = (Y^n)(+)M = (Y^n, 0)R$ for some integer $n \geq 0$. Therefore, if B is a regular ideal of R, then by Lemma 1.1,

$$\begin{aligned} B_t &= \bigcup \{A_v \mid A \subseteq B \text{ is a finitely generated regular ideal of } R\} \\ &= (Y^n, 0)R \end{aligned}$$

for some integer $n \geq 0$ by the previous paragraph. Thus, R is a t-Marot ring.

2. Let $A = (X, Y)(+)M$. Then $A^\bullet R = (Y)(+)M$ and $A_t = R$. Thus, $(A^\bullet R)_t = (Y)(+)M \subsetneq A \subsetneq A_t$.

3. Note that $(Y)(+)M \subsetneq (X, Y)(+)M$ is a chain of regular prime ideals of R by 1.(ii) and [16, Theorem 25.1(3)]. Hence, $2 \leq \text{reg-dim}(R) \leq \dim(R) = \dim(D) = 2$ [16, Theorem 25.1(3)]. Thus, $\text{reg-dim}(R) = \dim(R) = 2$.

4. By 1.(ii), $R^\bullet$ is a rank-one DVM. Thus, R is a rank-one DVR by 1.(v) and Proposition 2.11. □

Example 2.3. Let R be a v-Marot ring such that $R^\bullet$ is a Mori monoid. Then R is a Mori ring by [12, Theorem 3.5]. Hence, if A is a regular t-ideal of R, then $A = A_v$, and since R is v-Marot, $A = (A^\bullet R)_v = (A^\bullet R)_t$. Thus, R is a t-Marot ring.

Given a t-Marot ring R, we can construct two types of t-Marot overrings of R for which we first need a lemma.

Lemma 2.4. (cf. [17, Lemma 3.4] for integral domains) *Let R be a ring, $S \subseteq R^\bullet$ be a multiplicative set, and A be an ideal of R.*

1. *If A is finitely generated, then $(AR_S)^{-1} = A^{-1}R_S$.*
2. *$(AR_S)_t = (A_t R_S)_t$.*

Proof. 1. Clearly, $A^{-1}R_S \subseteq (AR_S)^{-1}$. For the reverse containment, let $x \in (AR_S)^{-1}$. Then $xA \subseteq xAR_S \subseteq R_S$, and since A is finitely generated, there exists $s \in S$ such that $xsA \subseteq R$. Hence, $xs \in A^{-1}$, and thus $x \in A^{-1}R_S$.

2. Let $x \in A_t$. Then $x \in I_v$ for some finitely generated subideal I of A. Hence, $xI^{-1} \subseteq R$, and since I is finitely generated, $x(IR_S)^{-1} = xI^{-1}R_S \subseteq R_S$ by 1. Hence $x \in (IR_S)_v \subseteq (AR_S)_t$, and thus $A_t \subseteq (AR_S)_t$. Therefore, $(A_t R_S)_t = (AR_S)_t$. □

Proposition 2.5. *Let R be a t-Marot ring and D be an overring of R. Then D is a t-Marot ring if D is one of the following rings:*

1. *$D = R_S$ for some multiplicative set S of R with $S \subseteq R^\bullet$.*
2. *D is a regular fractional v-ideal of R.*

Proof. 1. Let A be a regular t-ideal of R_S and $I = A \cap R$. Then $A = IR_S$, and hence $A = A_t = (IR_S)_t = (I_t R_S)_t$ by Lemma 2.4(2). Thus $I = I_t$, and since R is t-Marot, $I = (I^\bullet R)_t$, whence

$$A = (I_t R_S)_t = \big((I^\bullet R)_t R_S\big)_t = \big((I^\bullet R)R_S\big)_t = (I^\bullet R_S)_t = (A^\bullet R_S)_t\,.$$

Therefore, R_S is a t-Marot ring.

2. Let t_R, v_R, t_D, and v_D be the t- and v-operations on R and D, respectively. Let A be a regular fractional t-ideal of D and $J \subseteq A$ be a finitely generated regular

fractional ideal of D. Since D is a regular fractional v-ideal of R, [12, Lemma 2.1(6)] implies that $(D : J_{v_R}) = (D : J)$, whence $J_{v_R} \subseteq (J_{v_R})_{v_D} = J_{v_D}$. Thus,

$$\begin{aligned} A_{t_R} &= \bigcup \{J_{v_R} \mid J \subseteq A \text{ is a finitely generated regular fractional ideal of } R\} \\ &\subseteq \bigcup \{J_{v_D} \mid J \subseteq A \text{ is a finitely generated regular fractional ideal of } D\} \\ &= A_{t_D} = A. \end{aligned}$$

Note that D is a fractional ideal of R; so A is a regular fractional t-ideal of R. Hence, $(A^\bullet D)_{t_D} \subseteq A = (A^\bullet R)_{t_R} \subseteq (A^\bullet D)_{t_R} \subseteq (A^\bullet D)_{t_D}$, and thus $A = (A^\bullet D)_{t_D}$. Thus, D is a t-Marot ring. □

We next study the relationship between the regular fractional t-ideals of R and the fractional t-ideals of $R^\bullet$ when R is a t-Marot ring.

Lemma 2.6. *Let R be a v-Marot ring, and I be a fractional ideal of $R^\bullet$.*

1. $(IR)_t \cap \mathsf{T}(R)^\bullet = I_t$.
2. *If $I \subseteq R^\bullet$, then* $(IR)_t \cap R^\bullet = I_t$.
3. $(IR)_t = (I_t R)_t$.

Proof. 1. By Lemma 1.1,

$$(IR)_t = \bigcup \{(JR)_v \mid J \text{ is a finitely generated fractional subideal of } I\}.$$

Thus, [12, Lemma 3.4] ensures that

$$(IR)_t \cap \mathsf{T}(R)^\bullet = \Big(\bigcup (JR)_v\Big) \cap \mathsf{T}(R)^\bullet = \bigcup \Big((JR)_v \cap \mathsf{T}(R)^\bullet\Big) = \bigcup J_v = I_t.$$

2. $(IR)_t \subseteq R$ by assumption, and so $(IR)_t \cap R^\bullet = (IR)_t \cap \mathsf{T}(R)^\bullet = I_t$ by 1.
3. By 1., $I_t \subseteq (IR)_t$, and hence $(I_t R)_t \subseteq (IR)_t$. Thus, $(IR)_t = (I_t R)_t$. □

Let t-spec(R) be the set of regular prime t-ideals of a ring R and t-spec(H) be the set of non-empty prime t-ideals of a monoid H.

Theorem 2.7. *Let R be a t-Marot ring.*

1. *If $P \in t$-spec(R), then $P^\bullet \in t$-spec$(R^\bullet)$.*
2. *If I is a prime t-ideal of $R^\bullet$, then $(IR)_t$ is a prime t-ideal of R.*
3. *Let $\varphi : t$-spec$(R) \to t$-spec$(R^\bullet)$ be a map defined by $\varphi(P) = P^\bullet$. Then φ is an order-preserving bijection.*

Proof. 1. Clearly, $P^\bullet$ is a prime ideal of $R^\bullet$. Let I be a finitely generated non-empty subideal of $P^\bullet$. Then IR is a finitely generated regular subideal of P, and hence $(IR)_v \subseteq P$. Thus, by Lemma 2.6, $I_v = (IR)_v \cap R^\bullet \subseteq P \cap R^\bullet = P^\bullet$, whence $(P^\bullet)_t = P^\bullet$.

2. Let I be a prime t-ideal of $R^\bullet$ and $P = (IR)_t$. If $P = R$, then by Lemma 2.6, $R^\bullet = (IR)_t \cap R^\bullet = I_t = I \subsetneq R^\bullet$, a contradiction. Thus, it remains to show that P is a prime ideal of R. Let $x, y \in R$ be such that $xy \in P$, and choose $z \in I$. Then

$$(x,z)(y,z) = (xy, xz, yz, z^2) \subseteq P.$$

Let $E = (x,z)_v \cap R^\bullet$ and $F = (y,z)_v \cap R^\bullet$. Since R is t-Marot, then $(ER)_t = (x,z)_v$ and $(FR)_t = (y,z)_v$, whence

$$\begin{aligned} EF &\subseteq \big((EF)R\big)_t = \big((ER)_t(FR)_t\big)_t \\ &= \big((x,z)_v(y,z)_v\big)_t = \big((x,z)(y,z)\big)_t \\ &\subseteq P. \end{aligned}$$

Thus, $EF \subseteq P \cap R^\bullet = I$, and since I is a prime ideal of $R^\bullet$, either $E \subseteq I$ or $F \subseteq I$. Therefore, $x \in P$ or $y \in P$.

3. This follows from 1., 2., and Lemma 2.6. □

Let R be a ring and $p \in R^\bullet$ be a nonunit. Clearly, if p is a prime element of R, then p is a prime element of $R^\bullet$. However, if $p = t$ in Example 4.2, then p is a prime element of $R^\bullet$ but not a prime element of R.

Corollary 2.8. *Let R be a t-Marot ring and $p \in R^\bullet$ be a nonunit. Then p is a prime element of R if and only if p is a prime element of $R^\bullet$.*

Proof. This is an immediate consequence of Theorem 2.7(2). □

Let R (resp., H) be a ring (resp., monoid). We say that R (resp., H) is *of finite t-character* if each regular element of R (resp., each element of H) is contained in only finitely many maximal t-ideals of R (resp., H). For example, if R (resp., H) is a Krull ring (resp., Krull monoid), then R (resp., H) is of finite t-character.

Corollary 2.9. *Let R be a t-Marot ring. Then R is of finite t-character if and only if $R^\bullet$ is of finite t-character.*

Proof. Let t-max(R) (resp., t-max$(R^\bullet)$) be the set of regular maximal t-ideals of R (resp., non-empty maximal t-ideals of $R^\bullet$). Then, by Theorem 2.7, $t\text{-max}(R^\bullet) = \{P \cap R^\bullet \mid P \in t\text{-max}(R)\}$, and for $P_1, P_2 \in t$-max(R), we have that $P_1 \cap R^\bullet = P_2 \cap R^\bullet$ if and only if $P_1 = P_2$. Thus, R is of finite t-character if and only if $R^\bullet$ is of finite t-character. □

Let R be a Marot ring and P be a prime ideal of R. Then $R_{[P]} = R_{(P)}$ and $[P]R_{[P]} = PR_{(P)}$ [16, Theorem 7.6]. The next result is a t-Marot ring analog.

Proposition 2.10. *Let R be a t-Marot ring and P be a regular prime t-ideal of R.*

1. $R_{[P]} = R_{(P)}$.
2. $[P]R_{[P]} = PR_{(P)}$.

Proof. 1. Clearly, $R_{(P)} \subseteq R_{[P]}$. For the reverse containment, let $x \in R_{[P]}$. Then there exists $s \in R \setminus P$ such that $sx \in R$. Hence, if $A = (R :_R x)$, then A is a regular v-ideal of R and $A \not\subseteq P$. Thus, $A^\bullet \not\subseteq P$, because R is t-Marot, and hence there exists $a \in A^\bullet \setminus P$. Therefore, $x \in R_{(P)}$.

2. Let $x \in [P]R_{[P]}$. Then there exists $a \in R \setminus P$ such that $ax \in P$. Note that $x \in R_{[P]}$; so $x \in R_{(P)}$ by 1., whence $x = \frac{c}{b}$ for some $c \in R$ and $b \in (R \setminus P)^\bullet$. Hence, $ac \in bP \subseteq P$, and since $a \notin P$, we have $c \in P$. Thus, $x \in PR_{(P)}$. The reverse containment is clear. □

Proposition 2.11. *Let R be a t-Marot ring with $T = \mathsf{T}(R)$ such that $R \neq T$. Then the following statements are equivalent.*

(1) *R is a rank-one DVR.*
(2) *$R^\bullet$ is a rank-one DVM.*
(3) *R has a principal regular-height-one prime ideal which contains all nonunit regular elements of R.*
(4) *$|t\text{-spec}(R)| = 1$ and (R, P) is a rank-one discrete valuation pair of T for $P \in t$-spec(R).*

Proof. (1) ⇒ (2) Let v be a rank-one discrete valuation on T such that $R = \{x \in T \mid v(x) \geq 0\}$, and set $Q = \{x \in T \mid v(x) > 0\}$. We first assert that $0 < v(a) < \infty$ for a nonunit $a \in R^\bullet$. Let $a \in R^\bullet$ be a nonunit. Since $0 = v(1) = v(aa^{-1}) = v(a) + v(a^{-1})$, we infer that $v(a) < \infty$. If $v(a) = 0$, then $v(a^{-1}) = 0$, and thus $a^{-1} \in R$, whence a is a unit in R, a contradiction. Consequently, $0 < v(d) < \infty$ for each nonunit regular element $d \in R$, and it follows that $Q^\bullet$ is the set of nonunit regular element of R. Since $R \neq T$, we can choose a nonunit regular element $b \in R$. Then $0 < v(b)$, and hence $b \in Q^\bullet$. Thus we infer that Q is a regular prime ideal of

R. Next we assert that reg-ht$Q = 1$. Suppose that Q' is a regular prime ideal of R such that $Q' \subseteq Q$. Let $z \in Q$ and $c \in Q'$ be a regular element with $v(c) = n > 0$. Then $v(z^n c^{-1}) = nv(z) - v(c) \geq 0$, so that $z^n c^{-1} \in R$. It follows that $z^n \in cR \subseteq Q'$, and since Q' is prime, we have that $z \in Q'$. Thus $Q' = Q$, and this shows that Q is a regular-height-one prime ideal of R. Consequently, $Q^\bullet$ is a height-one prime ideal of $R^\bullet$ by Theorem 2.7. Now, let $x \in Q^\bullet$ be such that $v(x) \leq v(b)$ for all $b \in Q^\bullet$. Then $Q^\bullet = xR^\bullet$ which implies that $R^\bullet$ is a rank-one DVM.

(2) $\Rightarrow$ (3) Let $R^\bullet$ be a rank-one DVM with maximal ideal I. Then $I = aR^\bullet$ for some $a \in R^\bullet$. Moreover, I is the set of nonunit regular elements of R and I is a height-one prime ideal of $R^\bullet$. Let $P = (IR)_t$. Then $I \subseteq P = aR$ and P is a regular-height-one prime ideal of R by Theorem 2.7.

(3) $\Rightarrow$ (4) Let P be a principal regular-height-one prime ideal of R such that P contains every nonunit regular element of R. Clearly, P is a regular prime t-ideal of R. Let Q be a regular prime t-ideal of R. Then $Q^\bullet$ is contained in the set of nonunit regular elements of R, and hence $Q^\bullet \subseteq P$. Thus $Q = (Q^\bullet R)_t \subseteq P$, and since reg-ht$P = 1$, we infer that $Q = P$. Note that $(R \setminus P)^\bullet$ is the set of units of R; hence $R_{[P]} = R_{(P)} = R$ and $[P]R_{[P]} = P$ by Proposition 2.10. Thus (R, P) is a rank-one discrete valuation pair of T (see, [6, Theorem 1] or [7, Theorem 2.3]).

(4) $\Rightarrow$ (1) This is obvious. □

Corollary 2.12. *Let R be a t-Marot ring with $T = \mathsf{T}(R)$ and P be a regular prime t-ideal of R. Then*

1. $R_{[P]} \cap T^\bullet = R^\bullet_{P^\bullet}$.
2. $R_{[P]}$ *is a rank-one DVR if and only if* $R^\bullet_{P^\bullet}$ *is a rank-one DVM.*

Proof. 1. Clearly, $R^\bullet_{P^\bullet} \subseteq R_{[P]} \cap T^\bullet$. For the reverse containment, let $x \in R_{[P]} \cap T^\bullet$. Note that $R_{[P]} = R_{(P)}$ by Proposition 2.10. Hence, there exists $s \in (R \setminus P)^\bullet$ such that $sx \in R$, whence $s \in R^\bullet \setminus P^\bullet$ and $xs \in R^\bullet$. Thus $x \in R^\bullet_{P^\bullet}$.

2. By Propositions 2.5(1) and 2.10, $R_{[P]}$ is a t-Marot ring. If $b \in P^\bullet$, then b is a nonunit regular element of $R_{[P]}$, and hence $R_{[P]} \neq T$. Thus the assertion follows from 1. and Proposition 2.11. □

Remark 2.13. Let R be a t-Marot ring and P be a regular prime ideal of R.

1. If R is Marot, then $PR_{[P]}$ is a unique regular maximal ideal of $R_{[P]}$. However, note that if $P = (Y)(+)M$ in Example 2.2, then reg-ht$P = 1$, P is not maximal, and $R_{[P]} = R$; hence $PR_{[P]}$ is not a regular maximal ideal of $R_{[P]}$.
2. Assume that reg-ht$P = 1$, and every nonunit regular element of R is contained in P. Then Proposition 2.11 shows that R is a rank-one DVR if and only if P is principal.

It is known that if R is a Mori ring, then $R^\bullet$ is a Mori monoid, and if R is v-Marot, then the converse holds [12, Theorem 5.3(3)]. By Example 2.3, if $R^\bullet$ is a Mori monoid, then R is t-Marot if and only if R is v-Marot. Hence, the second result of the next proposition recovers [12, Theorem 5.3(3)].

Proposition 2.14. *Let R be a t-Marot ring and A be a regular t-ideal of R.*

1. *A is of finite type if and only if $A^\bullet$ is a t-ideal of finite type.*
2. *R is a Mori ring if and only if $R^\bullet$ is a Mori monoid.*

Proof. 1. ($\Rightarrow$) Since A is of finite type, there is a finitely generated ideal J of $R^\bullet$ such that $A = (JR)_t$ by Lemma 1.1. Hence, by Lemma 2.6,

$$(A^\bullet)_t = (A^\bullet R)_t \cap R^\bullet = A \cap R^\bullet = (JR)_t \cap R^\bullet = J_t,$$

and $(A^\bullet)_t = A^\bullet$. Thus, $A^\bullet$ is a t-ideal of finite type. ($\Leftarrow$) Assume that $A^\bullet = J_t$ for some finitely generated ideal J of $R^\bullet$. Then, by Lemma 2.6,

$$A = (A^\bullet R)_t = (J_t R)_t = (JR)_t.$$

Thus, A is of finite type.

2. This is an immediate consequence of 1. □

3. The class groups of a ring R and $R^\bullet$

Let R be a t-Marot ring. In this section, we compare the t-invertibility of regular ideals of R and that of ideals of $R^\bullet$, and we show that $Cl(R) \cong Cl(R^\bullet)$.

Lemma 3.1. *Let R be a v-Marot ring and I be a t-invertible fractional ideal of $R^\bullet$. Then*

1. *IR is t-invertible,*
2. *$(IR)^{-1} = (I^{-1}R)_t$, and*
3. *$(IR)_t = (IR)_v = (I_v R)_t = (I_t R)_t$.*

Proof. 1. Note that $I_t = J_v$ and $I^{-1} = L_v$ for some finitely generated fractional ideals J and L of $R^\bullet$ with $J \subseteq I$ and $L \subseteq I^{-1}$ [15, Theorem 12.1]. Hence, $R^\bullet = (II^{-1})_t = (J_v L_v)_t = (JL)_t$, $JL \subseteq II^{-1}$, and $\big((JL)R\big)_t = \big((JL)R\big)_v$.

Let $A = JL$, $T = \mathsf{T}(R)$, and $q \in (AR)^{-1} \cap T^\bullet$. Then $qA \subseteq qAR \subseteq R$, and since q is regular, it follows that $qA \subseteq R^\bullet$, whence $q \in A^{-1} = R^\bullet \subseteq R$. Note that R is a v-Marot ring and $(AR)^{-1}$ is a regular fractional v-ideal. Thus, $(AR)^{-1} = R$, whence $(AR)_v = R$. Therefore,

$$R \supseteq \big((II^{-1})R\big)_t \supseteq (AR)_t = (AR)_v = R,$$

and hence $\big((II^{-1})R\big)_t = R$. Thus, $\big((IR)(I^{-1}R)\big)_t = \big((II^{-1})R\big)_t = R$.

2. $R = \big((IR)(I^{-1}R)\big)_t$ implies that $(IR)^{-1} = (I^{-1}R)_t$.

3. Since I is t-invertible, I^{-1} is also t-invertible, and thus 2. ensures that $(IR)_v = ((IR)^{-1})^{-1} = ((I^{-1}R)_t)^{-1} = (I^{-1}R)^{-1} = (I_v R)_t$. □

Proposition 3.2. *Let R be a t-Marot ring and I be a fractional ideal of $R^\bullet$. Then I is t-invertible if and only if IR is t-invertible.*

Proof. ($\Rightarrow$) A t-Marot ring is a v-Marot ring, and thus the assertion follows from Lemma 3.1. ($\Leftarrow$) Assume that IR is t-invertible. Then $(IR)^{-1}$ is a t-invertible regular fractional t-ideal of R. Hence, since R is a t-Marot ring, it follows that $(IR)^{-1} = (JR)_t$ for some fractional ideal J of $R^\bullet$. Thus,

$$R = \big((IR)(IR)^{-1}\big)_t = \big((IR)(JR)_t\big)_t = \big((IR)(JR)\big)_t = \big((IJ)R\big)_t.$$

Therefore, $(IJ)_t = R^\bullet$ by Lemma 2.6. □

It is known that if R is a v-Marot ring, then R is a Krull ring if and only if $R^\bullet$ is a Krull monoid [12, Theorem 3.5(4)]. Note that if R is a v-Marot ring and $R^\bullet$ is a Krull monoid, then R is a t-Marot ring by Example 2.3. Hence, the next result recovers the result of [12, Theorem 3.5(4)].

Corollary 3.3. *Let R be a t-Marot ring. Then R is a Krull ring if and only if $R^\bullet$ is a Krull monoid.*

Proof. Note that R is a Krull ring if and only if every regular ideal of R is t-invertible [7, Theorem 3.5] and $R^\bullet$ is Krull if and only if every non-empty ideal of $R^\bullet$ is t-invertible [15, Theorem 22.8]. Thus, the result follows directly from Proposition 3.2. □

We say that R is a *Prüfer v-multiplication ring* (PvMR) if each finitely generated regular ideal of R is t-invertible. Similarly, a monoid H is a *Prüfer v-multiplication monoid* (PvMM) if each non-empty finitely generated ideal of H is t-invertible. It is clear that R (resp., H) is a PvMR (resp., PvMM) if and only if each regular t-ideal (resp., each non-empty t-ideal) of finite type is t-invertible.

Corollary 3.4. *Let R be a t-Marot ring. Then R is a PvMR if and only if $R^\bullet$ is a PvMM.*

Proof. ($\Rightarrow$) Let I be a non-empty finitely generated ideal of $R^\bullet$. Then IR is a finitely generated regular ideal of R, and since R is a PvMR, IR is t-invertible. Thus, I is t-invertible by Proposition 3.2.

($\Leftarrow$) Let A be a finite type regular t-ideal of R, and let $A \cap R^\bullet = I$. Then $A = (IR)_t$, and since A is of finite type, I is also of finite type by Proposition 2.14. Hence, I is t-invertible, and thus A is t-invertible by Proposition 3.2. □

Corollary 3.5. *Let R be a t-Marot ring. Then R is a PvMR of finite t-character if and only if $R^\bullet$ is a PvMM of finite t-character.*

Proof. This follows directly from Corollaries 2.9 and 3.4. □

In general, $Cl(R) \ncong Cl(R^\bullet)$. For example, if R is the Krull ring of Example 4.2 with $n \geq 2$, then $Cl(R) \cong \mathbb{Z}_n \neq \{0\} = Cl(R^\bullet)$.

Theorem 3.6. *Let R be a v-Marot ring.*

1. *$Cl(R^\bullet) \hookrightarrow Cl(R)$.*
2. *If R is t-Marot, then $Cl(R^\bullet) \cong Cl(R)$.*

Proof. 1. Let $\varphi : Tinv(R^\bullet) \to Tinv(R)$ be a map defined by $\varphi(I_t) = (IR)_t$. Then, by Lemma 3.1, φ is well-defined. Moreover, if I, J are two t-invertible fractional ideals of $R^\bullet$, then

$$\begin{aligned} \varphi(I_t \cdot_t J_t) &= \varphi\big((IJ)_t\big) = \big((IJ)R\big)_t = \big((IR)(JR)\big)_t \\ &= \big((IR)_t(JR)_t\big)_t = (IR)_t \cdot_t (JR)_t \\ &= \varphi(I_t) \cdot_t \varphi(J_t)\,, \end{aligned}$$

whence φ is a group homomorphism. Clearly, $\varphi\big(Prin(R^\bullet)\big) = Prin(R)$, and thus $\widetilde{\varphi} : Cl(R^\bullet) \to Cl(R)$, given by $\widetilde{\varphi}\big([I_t]\big) = \big[(IR)_t\big]$, is a well-defined group homomorphism.

Next, let I, J be two t-invertible fractional ideals of $R^\bullet$ such that $\widetilde{\varphi}([I_t]) = \widetilde{\varphi}([J_t])$. Then $(IR)_t = q(JR)_t = (qJR)_t$ for some $q \in \mathsf{T}(R)^\bullet$, and hence Lemma 2.6 ensures that

$$I_t = (IR)_t \cap \mathsf{T}(R)^\bullet = (qJR)_t \cap \mathsf{T}(R)^\bullet = qJ_t\,.$$

So, $[I_t] = [J_t]$, and thus $\widetilde{\varphi}$ must be injective.

2. By 1., it suffices to show that φ is surjective. Let A be a t-invertible regular fractional t-ideal of R and $I = A^\bullet$. Then $A = (IR)_t$, and since A is t-invertible, I is t-invertible by Proposition 3.2. Thus, $I_t \in Tinv(R^\bullet)$ and $\varphi(I_t) = A$. □

We will say that R is a *factorial ring* if every nonunit regular element of R is a product of finitely many regular prime elements of R. Then R is a factorial ring if and only if R is a Krull ring with $Cl(R) = \{0\}$, if and only if every regular prime ideal of R contains a regular prime element [2, Theorem]. Clearly, if R is a factorial ring, then R is a t-Marot ring and $R^\bullet$ is a factorial monoid. However, R need not be a factorial ring nor a Krull ring even though $R^\bullet$ is a factorial monoid (see, for example, [3, Example 5.2]).

Corollary 3.7. [8, Proposition 2.9.22] *Let R be a t-Marot ring. Then R is a factorial ring if and only if $R^\bullet$ is a factorial monoid.*

Proof. It is clear that if R is factorial, then $R^\bullet$ is a factorial monoid. Conversely, assume that $R^\bullet$ is a factorial monoid. Then $R^\bullet$ is a Krull monoid with $Cl(R^\bullet) = \{0\}$ by [11, Corollary 2.3.13]. Hence, R is a Krull ring with $Cl(R) = \{0\}$ by Corollary 3.3 and Theorem 3.6. Thus, R is a factorial ring. □

Corollary 3.8. *The following statements are equivalent for a t-Marot ring R.*

(1) *R is a PvMR and $Cl(R) = \{0\}$.*
(2) *$R^\bullet$ is a PvMM and $Cl(R^\bullet) = \{0\}$.*
(3) *$R^\bullet$ is a GCD-monoid.*

Proof. (1) ⇔ (2) This follows by Theorem 3.6 and Corollary 3.4. (2) ⇔ (3) [15, p. 188]. □

Remark 3.9. Let R be a ring.

1. In [1, 2], Anderson and Markanda called R a factorial ring if $R^\bullet$ is a factorial monoid. In this case, a factorial ring need not be a Krull ring [3, Example 5.2]. This happens because a factorial ring R in [1, 2] is defined by $R^\bullet$ being factorial, while a Krull ring R is not defined by $R^\bullet$ being a Krull monoid. The factorial rings of this paper are just Krull rings with trivial class group (Elliott [8, Definition 2.5.27] called R an *r-UFR* if it is a Krull ring with trivial class group), and fortunately, Corollary 3.7 shows that there is no difference between the two factorial rings in the case of a t-Marot ring.
2. In [8, Definition 2.5.27], Elliott called R an *r-GCD ring* if it is a PvMR and $Cl(R) = \{0\}$. Observe that R need not be an r-GCD ring if $R^\bullet$ is a GCD-monoid. If R is a Krull ring such that $R^\bullet$ is a factorial monoid but $Cl(R) \cong \mathbb{Z}_n \neq \{0\}$ [3, Example 5.4], then $R^\bullet$ is a GCD-monoid but R is not an r-GCD ring. However, Corollary 3.8 (or [8, Corollary 2.8.20]) ensures that an r-GCD ring R can be defined by $R^\bullet$ being a GCD-monoid in the t-Marot case.

4. Krull rings and monoids

A monoid homomorphism $\varphi : H \to F$ is said to be a *divisor homomorphism* if, for $a, b \in H$, $\varphi(a) \mid \varphi(b)$ implies that $a \mid b$, and a *divisor theory* if F is free abelian, φ is a divisor homomorphism, and for all $p \in F$, there exists a finite subset $X \subseteq H$ such that $p = \gcd\big(\varphi(X)\big)$.

Theorem 4.1. *Let H be a monoid and $X^1(H)$ be the set of height-one prime ideals of H. Then the following statements are equivalent.*

(1) *H is a Krull monoid.*
(2) (i) $H = \bigcap_{P \in X^1(H)} H_P$, (ii) *$H_P$ is a rank-one DVM for all $P \in X^1(H)$, and* (iii) *each element of H is contained in only finitely many prime ideals in $X^1(H)$.*
(3) *Every proper principal ideal aH of H is a t-product of prime ideals; i.e., $aH = (P_1 \cdots P_n)_t$ for some prime ideals $P_1, \ldots, P_n$ of H.*
(4) *H has a divisor theory.*
(5) *There exists a divisor homomorphism from H to a free abelian monoid.*

Proof. (1) ⇔ (2) [9, Theorem 3.4]. (1) ⇔ (3) [15, Theorem 22.8]. (1) ⇔ (4) ⇔ (5) [11, Theorem 2.4.8]. □

We next give an example of Krull rings which shows that the divisor homomorphism of Theorem 4.1(5) is not unique.

Example 4.2. Let D be a Dedekind domain with maximal ideal P that is not principal, but some power of P is principal. Let

$$A = \bigoplus \{D/Q \mid Q \neq P \text{ is a maximal ideal of } D\},$$

and set $R = D(+)A$ be the idealization of A in D and $M = P(+)A$. Then R is a Krull ring with unique regular-height-one prime ideal M such that

- M is invertible,
- there exists the least integer $n > 1$ such that $M^n = tR$ for some $t \in R$ for some $t \in R$, and
- if $a \in R^\bullet$, then $aR = t^k R$ for some integer $k \geq 0$.

Hence, R is a Krull ring with $Cl(R) \cong \mathbb{Z}_n$ and $Cl(R^\bullet) = \{0\}$ [3, Example 5.4]. Moreover, R is not a t-Marot ring by Theorem 3.6, and we have two divisor homomorphisms from $R^\bullet$ into free abelian monoids.

(1) Let F_1 be a free abelian monoid with basis $\{M\}$, and define

$$\varphi_1 : R^\bullet \to F_1\,, \quad \text{by} \quad \varphi_1(a) = M^k,$$

where $aR = M^k$ for some integer k. Then φ_1 is a divisor homomorphism, but not a divisor theory. In this case, $\mathcal{C}(\varphi_1) = \mathsf{q}(F_1)/\mathsf{q}\big(\varphi_1(R^\bullet)\big) \cong \mathbb{Z}_n \cong Cl(R)$ (see, [11, Definition 2.4.1]).

(2) Let F_2 be a free abelian monoid with basis $\{t\}$, and define

$$\varphi_2 : R^\bullet \to F_2\,, \quad \text{by} \quad \varphi_2(a) = t^k,$$

where $aR = t^k R$ for some integer $k \geq 1$. Then φ_2 is a divisor theory and $\mathcal{C}(\varphi_2) = \mathsf{q}(F_2)/\mathsf{q}\big(\varphi_2(R^\bullet)\big) \cong Cl(R^\bullet) = \{0\}$ (see, [11, Theorem 2.4.7]).

A monoid H is said to be *primary* if $H \neq H^\times$ and every $q \in H \setminus H^\times$ is primary, or equivalently, for $a, b \in H \setminus H^\times$, there is an integer $n \geq 1$ such that $a|b^n$. A Krull monoid H is called an *almost factorial monoid* if $Cl(H)$ is torsion. Clearly, a Krull monoid H is almost factorial if and only if for each $a \in H$, there is an integer $m = m(a) \geq 1$ such that a^m can be written as a finite product of primary elements (cf. [15, Exercise 4 on p. 258]).

Proposition 4.3. *Let $\varphi : H \to F$ be a divisor homomorphism of monoids.*

1. *If F is primary, then H is primary, and converse holds if φ is surjective.*
2. *If φ is a divisor theory, then F is a root extension of $\varphi(H)$ if and only if H is an almost factorial monoid.*

Proof. 1. Let $a, b \in H \setminus H^\times$. Since φ is a divisor homomorphism, we have $\varphi^{-1}(F^\times) = H^\times$, and hence $\varphi(a), \varphi(b) \in F \setminus F^\times$. Since F is primary, there exists an integer $n \geq 1$ such that $\varphi(a) \,|\, \varphi(b)^n = \varphi(b^n)$, which implies that $a \,|\, b^n$. Thus H is primary. Suppose now that φ is surjective. If $a, b \in F \setminus F^\times$, then there exist $x, y \in H$ such that $\varphi(x) = a$ and $\varphi(y) = b$. Since $a, b \notin F^\times$, it follows that $x, y \notin H^\times$. Since H is primary, we may assume that $x \mid y^n$ for some $n \in \mathbb{N}$, and it follows that $a \mid b^n$. Therefore F is primary.

2. Since φ is a divisor homomorphism, we have $\varphi(H) = \mathsf{q}\big(\varphi(H)\big) \cap F$. Since φ is a divisor theory, [11, Proposition 2.4.6 and Corollary 2.4.3] ensures that $F/\varphi(H) = \{a\mathsf{q}\big(\varphi(H)\big) \mid a \in F\} = \mathsf{q}(F)/\mathsf{q}\big(\varphi(H)\big)$. Thus the assertion follows from $Cl(H) \cong \mathsf{q}(F)/\mathsf{q}\big(\varphi(H)\big)$. □

We say that a ring R is a *weakly Krull ring* if $R = \bigcap_{P \in X_r^1(R)} R_{[P]}$ and each regular element of R is contained in only finitely many regular-height-one prime ideals of

R. For example, every Krull ring is a weakly Krull ring. As in [15], a monoid H is called a *weakly Krull monoid* if $H = \bigcap_{Q\in X^1(H)} H_Q$ and each element of H is contained in only finitely many height-one prime ideals of H.

Theorem 4.4. *Let R be a t-Marot ring. Then R is a weakly Krull ring if and only if $R^\bullet$ is a weakly Krull monoid. In this case, $X_r^1(R) = \{(PR)_t \mid P \in X^1(R^\bullet)\}$ and $X^1(R^\bullet) = \{P^\bullet \mid P \in X_r^1(R)\}$.*

Proof. ($\Rightarrow$) By definition, R has the following two properties; (i) $R = \bigcap_{P\in X_r^1(R)} R_{[P]}$ and (ii) each regular element of R is contained in only finitely many regular-height-one prime ideals of R. Then

$$R^\bullet = \bigcap_{P\in X_r^1(R)} \left(R_{[P]} \cap \mathsf{T}(R)^\bullet\right) = \bigcap_{P\in X_r^1(R)} R^\bullet_{P^\bullet}$$

by Corollary 2.12(1) and $X^1(R^\bullet) = \{P^\bullet \mid P \in X_r^1(R)\}$ by Theorem 2.7, so each element of $R^\bullet$ is contained in only finitely many height-one prime ideals of $R^\bullet$. Thus, $R^\bullet$ is a weakly Krull monoid.

($\Leftarrow$) Assume that $R^\bullet$ is a weakly Krull monoid, so (i) $R^\bullet = \bigcap_{Q\in X^1(R^\bullet)} R^\bullet_Q$ and (ii) each element of $R^\bullet$ is contained in only finitely many height-one prime ideals of $R^\bullet$. Then $X_r^1(R) = \{(QR)_t \mid P \in X^1(R^\bullet)\}$ by Theorem 2.7 and each regular element of R is contained in only finitely many prime ideals in $\{(QR)_t \mid Q \in X^1(R^\bullet)\}$. Hence, it suffices to show that

$$R = \bigcap_{Q\in X^1(R^\bullet)} R_{[(QR)_t]}.$$

Clearly, $R \subseteq \bigcap_{Q\in X^1(R^\bullet)} R_{[(QR)_t]}$. For the reverse containment, let $x \in \bigcap_{Q\in X^1(R^\bullet)} R_{[(QR)_t]}$, and let $A = (R :_R x)$. Then A is a regular v-ideal of R and $A \nsubseteq (QR)_t$ for all $Q \in X^1(R^\bullet)$. Moreover, since R is t-Marot, $A^\bullet \nsubseteq (QR)_t^\bullet = Q$ for all $Q \in X^1(R^\bullet)$. Hence, $A^\bullet = R^\bullet$ (see, [15, Theorem 22.5]), and thus $A = (A^\bullet R)_t = R$ by Theorem 2.7. Therefore, $x \in R$. □

We next give a new proof of Corollary 3.3 that a t-Marot ring R is a Krull ring if and only if $R^\bullet$ is a Krull monoid. The proof shows the exact relationship of the t-ideal structures of R and $R^\bullet$, which is why we prove it again.

Corollary 4.5. *Let R be a t-Marot ring. Then R is a Krull ring if and only if $R^\bullet$ is a Krull monoid.*

Proof. Let R be a Krull ring. Then, for each $P \in X_r^1(R)$, $R_{[P]}$ is a rank-one DVR by [7, Theorem 3.5], and hence $R^\bullet_{P^\bullet}$ is a rank-one DVM by Corollary 2.12(2). Thus, by Theorems 4.1 and 4.4, $R^\bullet$ is a Krull monoid.

Conversely, assume that $R^\bullet$ is a Krull monoid and $Q \in X^1(R^\bullet)$. Then $R^\bullet_Q$ is a rank-one DVM by Theorem 4.1, $(QR)_t \in X_r^1(R)$ by Theorem 2.7, and $(QR)_t \cap R^\bullet = Q$ by Lemma 2.6, whence $R_{[(QR)_t]}$ is a rank-one DVR by Corollary 2.12(2). Moreover, $[(QR)_t]R_{[(QR)_t]}$ is a regular-height-one prime ideal of $R_{[(QR)_t]}$ (cf. Proposition 2.10). Thus, by Proposition 2.11, $(R_{[(QR)_t]}, [(QR)_t]R_{[(QR)_t]})$ is a rank-one discrete valuation pair of $\mathsf{T}(R)$. Therefore, R is a Krull ring by Theorem 4.4 and [7, Theorem 3.5]. □

A nonunit regular element $q \in R$ is said to be primary if qR is a primary ideal, so q is primary in R if and only if $q|ab$ for $a, b \in R$ implies that either $q|a$ or $q|b^n$ for some integer $n \geq 1$. Clearly, a regular prime element is primary. Moreover, if q is primary, then $\sqrt{qR}$ is a maximal t-ideal of R. (Proof. Let Q be a prime ideal of

R such that $\sqrt{qR} \subsetneq Q$. Choose $z \in Q \setminus \sqrt{qR}$, and let $w \in (q,z)^{-1}$. Then $qw \in R$, $z \notin \sqrt{qR}$, and $(qw)z = q(wz) \in qR$, so $qw \in qR$, and since q is regular, $w \in R$. Hence, $(q,z)^{-1} = R$, and thus $R = (q,z)_v \subseteq Q_t$. Thus, $Q_t = R$.)

Lemma 4.6. *Let R be a t-Marot ring and $q \in R^\bullet$ be a nonunit. Then q is a primary element of R if and only if q is a primary element of $R^\bullet$.*

Proof. ($\Rightarrow$) Let $a,b \in R^\bullet$ be such that $ab \in qR^\bullet$ and $a \notin qR^\bullet$. Then $ab \in qR$ and $a \notin qR$. Since q is primary, it follows that $b^n \in qR$ for some integer $n \geq 1$. Thus $b^n = qc$ for some $c \in R$, and since $b^n \in R^\bullet$, we infer that $c \in R^\bullet$, whence $b^n \in qR^\bullet$. Therefore, q is a primary element of $R^\bullet$.

($\Leftarrow$) Let q be a primary element of $R^\bullet$ and $a,b \in R$ be such that $ab \in qR$ and $a \notin qR$. Set $I = (a,q)_t$ and $J = (b,q)_t$. Observe that I and J are regular t-ideals of R. Moreover, $I^\bullet J^\bullet \subseteq (I^\bullet J^\bullet R)_t = \big((I^\bullet R)_t (J^\bullet R)_t\big)_t = (IJ)_t = (ab, aq, bq, q^2)_t \subseteq qR$ and $I^\bullet \nsubseteq qR$, for if $I^\bullet \subseteq qR$, then $a \in I = (I^\bullet R)_t \subseteq qR$, a contradiction. Consequently, there exists some $y \in I^\bullet \setminus qR$. Since $yJ^\bullet \subseteq I^\bullet J^\bullet \subseteq qR \cap R^\bullet = qR^\bullet$ and q is a primary element of $R^\bullet$, it follows that $J^\bullet \subseteq \{r \in R^\bullet \mid r^n \in qR^\bullet$ for some integer $n \geq 1\} \subseteq \sqrt{qR}$. Note that qR is a t-ideal of R, so that every minimal prime ideal of qR is a t-ideal of R. Since $\sqrt{qR}$ is the intersection of all minimal prime ideals of qR, it follows that $\sqrt{qR}$ is a t-ideal, and so $b \in J = (J^\bullet R)_t \subseteq (\sqrt{qR})_t = \sqrt{qR}$. Thus, q is a primary element of R. □

As in [5], we say that R is a *weakly factorial ring* if every nonunit regular element of R is a product of finitely many regular primary elements of R. Since a regular prime element is primary, a factorial ring is a weakly factorial ring. It is known that a weakly factorial ring is a weakly Krull ring [5, Corollary 2.3]. A monoid H is weakly factorial if and only if H is a weakly Krull monoid and $Cl(H) = \{0\}$ [15, Exercise 5 on p. 258].

Proposition 4.7. *The following statements are equivalent for a t-Marot ring R.*
(1) *R is a weakly factorial ring.*
(2) *$R^\bullet$ is a weakly factorial monoid.*
(3) *$R^\bullet$ is a weakly Krull monoid and $Cl(R^\bullet) = \{0\}$.*
(4) *R is a weakly Krull ring and $Cl(R) = \{0\}$.*

Proof. (1) $\Leftrightarrow$ (2) This follows from Lemma 4.6.
(2) $\Leftrightarrow$ (3) [15, p. 258].
(3) $\Leftrightarrow$ (4) This follows from Theorems 4.4 and 3.6. □

Now, let $q \in R$ be a nonunit regular element of R. It is easy to see that if q is primary as an element of R, then q is also primary as an element of $R^\bullet$ (see the proof of Lemma 4.6). But, the next example shows that (i) q need not be primary as an element of R even though q is a prime element of $R^\bullet$, and (ii) $R^\bullet$ is a factorial monoid but R is not a weakly factorial ring.

Example 4.8. Let A be a Dedekind domain with $\mathrm{Pic}(A) \cong \mathbb{Z}$ (see [10, Theorem 14.10] for such a Dedekind domain), $\{X_1, X_2, \ldots\}$ be a countably infinite set of indeterminates over A, and $B = A[\{X_1, X_2, \ldots\}]$ be the polynomial ring over A. Then B is a Krull domain, $Cl(B) \cong \mathbb{Z}$, and every divisor class of B contains a height-one prime ideal (cf. [10, Theorem 14.3]). Let S be the multiplicative set of B generated by all prime elements (cf. the proof of [10, Theorem 14.2]). Then $D := B_S$ is a Dedekind domain, $\mathrm{Pic}(D) \cong \mathbb{Z}$, and every divisor class of D contains a maximal ideal. Then there are non-principal prime ideals P_1 and P_2, so that $P_1P_2 = qD$ for some $q \in D$. Let $A = \bigoplus\{D/M \mid M$ is a maximal ideal of D and $M \neq P_1, P_2\}$ and $R = D(+)A$ be the idealization of A in D. Then

$R^{\bullet} = \{(uq^n, m) \mid u \in D$ is a unit, $n \geq 0$ is an integer, and $m \in A\}$, whence $R^{\bullet}$ is a factorial monoid with a unique prime element $(q, 0)$ (up to associates). But, $\sqrt{(q,0)R} = (P_1(+)A) \cap (P_2(+)A)$, so $\sqrt{(q,0)R}$ is not a prime ideal of R. Thus, $(q, 0)$ is not a primary element of R. Since q is irreducible, $(q, 0)$ cannot be written as a finite product of primary elements. Thus, R is not a weakly factorial ring.

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