Parameter identification for PDEs: *from* neural-network-based learning *to* discretized inverse problems

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Research questions

Evolution system

$$\frac{d}{dt}u - \nabla\cdot(a\nabla u) + cu + f(u) = \varphi \qquad \text{in } (0,T)\times\Omega$$
$$u(t=0) = u_0 \qquad \text{on } \Omega$$
$$Mu = y \qquad \text{mesured data.}$$

Parameter identification: find *parameters* $\lambda := (a, c, \varphi, u_0)$ & *state* u.

- Question: What if f is unknown?
- Attempt: $f(u) \approx \mathcal{N}_\theta(u)$.
- Question: Find $f \approx \mathcal{N}_\theta$ first or simultaneously with λ, u.
- Attempt: Find all at once.

Learning-informed Minimization

Abstract setting

$$\dot{u} = F(\lambda, u) + f(u) \qquad \text{PDE model}$$
$$Mu = y \qquad \text{data measurement.}$$

For $f \approx \mathcal{N}_\theta$, the all-at-once formulation is:

$$\min_{\lambda,u,\theta} \; \mu \underbrace{\|\dot{u} - F(\lambda,u) - \mathcal{N}_\theta(u)\|^2}_{\text{PDE residual}} + \mu \underbrace{\|Mu - y^\delta\|^2}_{\text{data mismatch}} + \mathcal{R}(\lambda, u, \theta).$$

Bypass solution map $(\lambda, \theta) \mapsto u$.

Results

- **Existence** of minimizer
- **Stability** and **convergence**
- Unique existence for the **learning-informed PDE**
- **Solution method** (differentiabilty, *tangential cone condition*, adjoints).

Numerics

Figure 1. $f(u) = u^2 - 1$. *Find* f, u. $\mathcal{N}_\theta = [2,4,2]$, *tanh activation.*

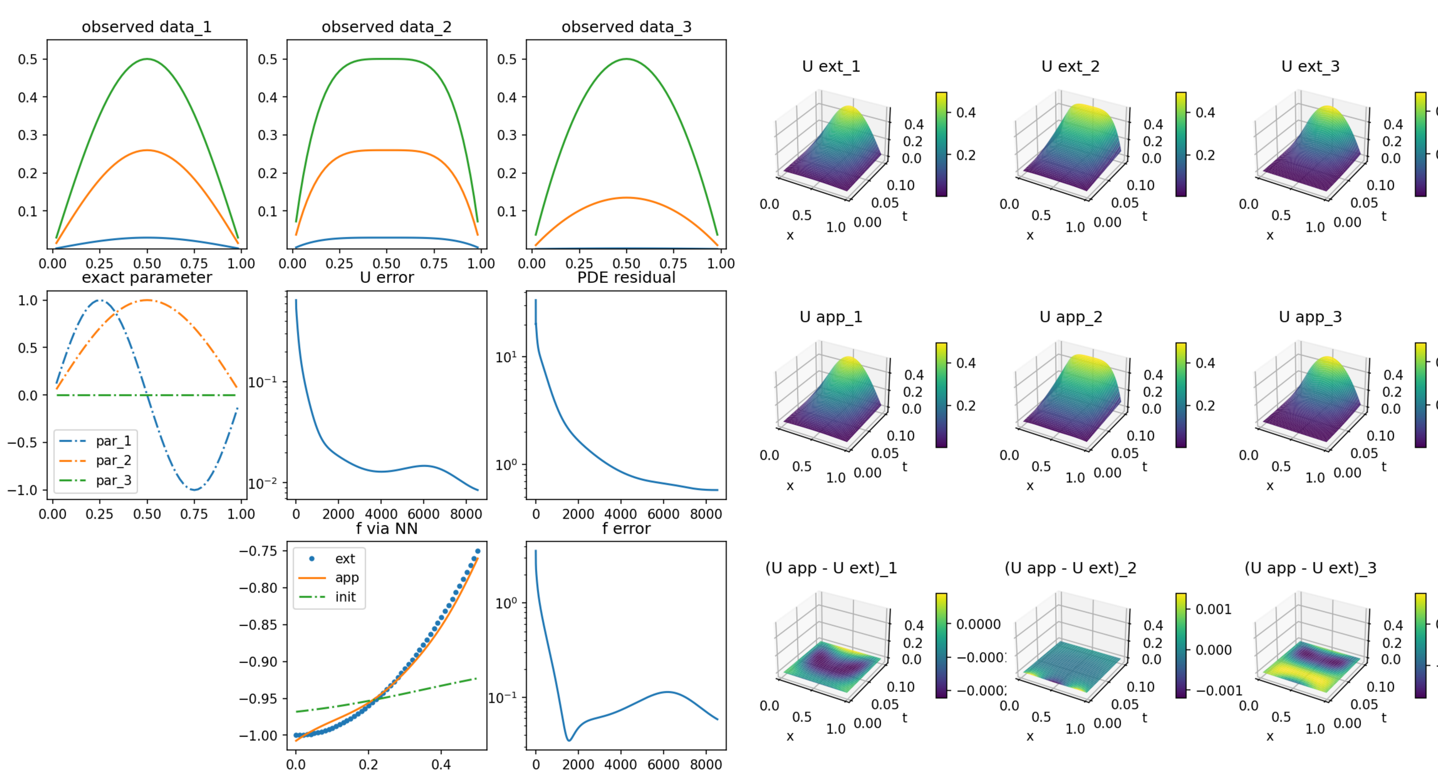
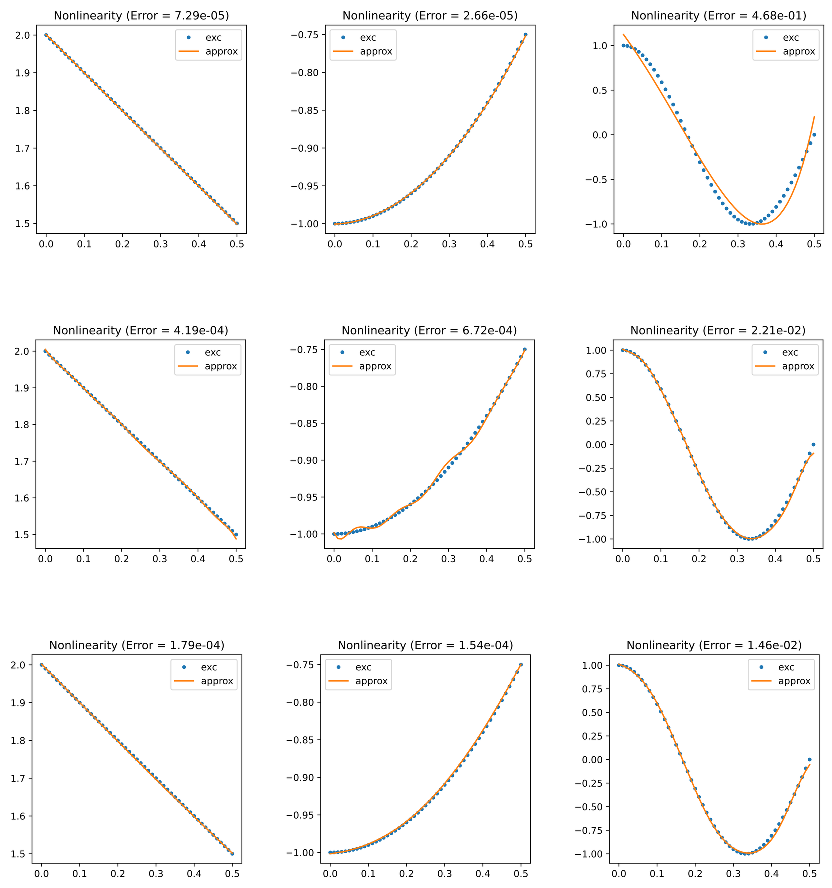


Figure 2. *Different approximation methods.*



Discretization of inverse problems

Motivation:

$$f \approx \mathcal{N}_\theta \quad \Longrightarrow \quad f \approx f_N \in \mathbb{X}_N \subset \mathbb{X}$$

Solve

$$\mathbb{F}(\mathbf{x}) = y.$$

Landweber iteration in $\mathbb{X}_N$:

$$\mathbf{x}^\delta_{N,k+1} = \mathbf{x}^\delta_{N,k} - \mathbb{F}'_N(\mathbf{x}^\delta_{N,k})^*(\mathbb{F}_N(\mathbf{x}^\delta_{N,k}) - \mathbf{y}^\delta).$$

Result: choice of $N(\delta), k_*(\delta)$ such that

$$\mathbf{x}^\delta_{N(\delta),k_*(\delta)} \to \mathbf{x} \text{ exact, as } \delta \to 0.$$

More in [arXiv:2108.10618]:

- Discretized and *projected* Landweber
- Tikhonov regularization.

References

Aarset, Holler, Nguyen, *Learning-informed parameter identification in nonlinear time-dependent PDEs* [arXiv:2202.10915]

Kaltenbacher, Nguyen, *Discretization of parameter identification in PDEs using Neural Networks* [arXiv:2108.10618]