Boundary and Finite Element Methods for Dirichlet Boundary Control Problems

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Subproject: Fast Finite and Boundary Element Methods
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Outline

Dirichlet boundary control problem

Dirichlet boundary control problem to minimize

$$J(u,z) = \frac{1}{2}\int_\Omega [u(x) - \overline{u}(x)]^2\,dx + \frac{1}{2}\alpha\|z\|_A^2 \quad \text{for} \quad (u,z) \in H^1(\Omega) \times H^{1/2}(\Gamma)$$

subject to

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega, \qquad u(x) = z(x) \quad \text{for } x \in \Gamma = \partial\Omega$$

where the operator A

$$A : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma), \qquad \|z\|_A^2 \simeq \langle Az, z\rangle_\Gamma \simeq \|z\|^2_{H^{1/2}(\Gamma)}$$

Example: $A = \widetilde{D}$ modified hypersingular operator

$$\langle \widetilde{D}u, v\rangle_\Gamma := \langle Du, v\rangle_\Gamma + \langle u, 1\rangle_\Gamma \langle v, 1\rangle_\Gamma$$

Dirichlet boundary control problem

Primal Dirichlet boundary value problem

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega, \quad u(x) = z(x) \quad \text{for } x \in \Gamma$$

Adjoint Dirichlet boundary value problem

$$-\Delta p(x) = u(x) - \overline{u}(x) \quad \text{for } x \in \Omega, \quad p(x) = 0 \quad \text{for } x \in \Gamma$$

Optimality condition

$$\alpha A z = \frac{\partial}{\partial n} p(x) \quad \text{for } x \in \Gamma$$

Laplace fundamental solution

$$U^*(x,y) = \begin{cases} -\dfrac{1}{2\pi} \log |x-y| & \text{for } d = 2 \\ \dfrac{1}{4\pi} \dfrac{1}{|x-y|} & \text{for } d = 3 \end{cases}$$

Boundary integral equation for primal problem

Primal Dirichlet boundary value problem

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega, \quad u(x) = z(x) \quad \text{for } x \in \Gamma$$

Boundary integral equation

$$Vt = \left(\frac{1}{2}I + K\right) z - N_0 f, \quad t = \frac{\partial}{\partial n} u \in H^{-1/2}(\Gamma)$$

where

$$(Vt)(x) := \int_\Gamma U^*(x,y) t(y)\, ds_y \qquad \text{for } x \in \Gamma,$$

$$(Kz)(x) := \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) z(y)\, ds_y \qquad \text{for } x \in \Gamma,$$

$$(N_0 f)(x) := \int_\Omega U^*(x,y) f(y)\, dy \qquad \text{for } x \in \Gamma.$$

Boundary integral equation for adjoint problem

Adjoint Dirichlet boundary value problem

$$-\Delta p(x) = u(x) - \overline{u}(x) \quad \text{for} \quad x \in \Omega, \qquad p(x) = 0 \quad \text{for} \quad x \in \Gamma$$

Representation formula for $x \in \Omega$

$$p(x) = \int\limits_{\Gamma} U^*(x,y) \frac{\partial}{\partial n} p(y)\, ds_y + \int\limits_{\Omega} U^*(x,y)[u(y) - \overline{u}(y)]\, dy$$

Boundary integral equation

$$Vq = \left(\frac{1}{2} I + K \right) p - N_0(u - \overline{u})$$

where

$$q = \frac{\partial}{\partial n} p \in H^{-1/2}(\Gamma)$$

Boundary integral equation for adjoint problem

Due to $p = 0$ on Γ, we obtain

$$\int_\Gamma U^*(x,y)q(y)\,ds_y = -\int_\Omega U^*(x,y)[u(y) - \overline{u}(y)]\,dy \quad \text{for} \quad x \in \Gamma$$

$$U^*(x,y) = \Delta_x V^*(x,y) = \Delta_y V^*(x,y)$$

Bi-Laplace fundamental solution $V^*(x,y)$

$$V^*(x,y) = \begin{cases} -\dfrac{1}{8\pi}|x-y|^2\left(\log|x-y| - 1\right) & \text{for } d = 2 \\ \dfrac{1}{8\pi}|x-y| & \text{for } d = 3 \end{cases}$$

Boundary integral equation for adjoint problem

Newton potential

$$\int_\Omega U^*(x,y)u(y)\,dy = \int_\Omega \Delta_y V^*(x,y)u(y)\,dy$$

Boundary integral equation for adjoint problem

Newton potential

$$\int_\Omega U^*(x,y)u(y)\,dy = \int_\Omega \Delta_y V^*(x,y)u(y)\,dy$$

$$= \int_\Gamma \frac{\partial}{\partial n_y} V^*(x,y)u(y)\,ds_y - \int_\Gamma V^*(x,y)\frac{\partial}{\partial n}u(y)\,ds_y + \int_\Omega V^*(x,y)\Delta u(y)\,dy$$

Boundary integral equation for adjoint problem

Newton potential

$$\int_\Omega U^*(x,y)u(y)\,dy = \int_\Omega \Delta_y V^*(x,y)u(y)\,dy$$

$$= \int_\Gamma \frac{\partial}{\partial n_y} V^*(x,y)u(y)\,ds_y - \int_\Gamma V^*(x,y)\frac{\partial}{\partial n}u(y)\,ds_y + \int_\Omega V^*(x,y)\Delta u(y)\,dy$$

$$= \int_\Gamma \frac{\partial}{\partial n_y} V^*(x,y)z(y)\,ds_y - \int_\Gamma V^*(x,y)t(y)\,ds_y - \int_\Omega V^*(x,y)f(y)\,dy$$

Boundary integral equation for adjoint problem

Representation formula for $x \in \Omega$

$$p(x) = \int\limits_{\Gamma} U^*(x,y)q(y)\,ds_y + \int\limits_{\Gamma} \frac{\partial}{\partial n_y} V^*(x,y)z(y)\,ds_y$$

$$- \int\limits_{\Gamma} V^*(x,y)t(y)\,ds_y - \int\limits_{\Omega} V^*(x,y)f(y)\,dy - \int\limits_{\Omega} U^*(x,y)\overline{u}(y)\,dy$$

Boundary integral equation for $x \in \Gamma$

$$Vq = V_1 t - K_1 z + N_0 \overline{u} + M_0 f$$

Boundary integral equation for adjoint problem

Representation formula for $x \in \Omega$

$$p(x) = \int_\Gamma U^*(x,y)q(y)\,ds_y + \int_\Gamma \frac{\partial}{\partial n_y} V^*(x,y)z(y)\,ds_y$$

$$-\int_\Gamma V^*(x,y)t(y)\,ds_y - \int_\Omega V^*(x,y)f(y)\,dy - \int_\Omega U^*(x,y)\overline{u}(y)\,dy$$

Boundary integral equation for $x \in \Gamma$

$$Vq = V_1 t - K_1 z + N_0 \overline{u} + M_0 f$$

Normal derivative for $x \in \Gamma$

$$q = \left(\frac{1}{2}I + K'\right)q - D_1 z - K_1' t - N_1 \overline{u} - M_1 f = \alpha A z$$

Boundary integral equations

We end up with a system (Nonsymmetric formulation)

$$\begin{cases} Vt = \left(\frac{1}{2}I + K\right) z - N_0 f \\ Vq = V_1 t - K_1 z + N_0 \overline{u} + M_0 f \\ \alpha A z = q \end{cases}$$

where

$$t, q \in H^{-1/2}(\Gamma), \quad z \in H^{1/2}(\Gamma).$$

The Schur complement system

$$T_\alpha z = V^{-1} N_0 \overline{u} + V^{-1} M_0 f - V^{-1} V_1 V^{-1} N_0 f$$

where the operator T_α is defined by

$$T_\alpha := V^{-1} K_1 - V^{-1} V_1 V^{-1} (\frac{1}{2} I + K) + \alpha A$$

Bi-Laplace boundary integral operators

Consider Bi-Laplace equation

$$-\Delta^2 u(x) = 0 \quad \text{for} \quad x \in \Omega$$

Representation formula for $\widetilde{x} \in \Omega$

$$\begin{aligned} u(\widetilde{x}) &= \int_\Gamma U^*(\widetilde{x}, y) \frac{\partial}{\partial n} u(y)\, ds_y - \int_\Gamma \frac{\partial}{\partial n_y} U^*(\widetilde{x}, y) u(y)\, ds_y \\ &\quad + \int_\Gamma V^*(\widetilde{x}, y) \frac{\partial}{\partial n} \Delta u(y)\, ds_y - \int_\Gamma \frac{\partial}{\partial n_y} V^*(\widetilde{x}, y) \Delta u(y)\, ds_y \end{aligned}$$

Boundary integral equations

Taking the limit $\Omega \ni \widetilde{x} \to x \in \Gamma$ and using the following notations

$$v(x) = \gamma_0^{int} u(x), \quad w(x) = \gamma_1^{int} u(x),$$

$$r(x) = \gamma_0^{int} \Delta u(x), \quad s(x) = \gamma_1^{int} \Delta u(x)$$

we obtain a system, including the so-called Calderón projection C,

$$\begin{bmatrix} v \\ w \\ r \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2}I - K & V & -K_1 & V_1 \\ D & \frac{1}{2}I + K' & D_1 & K_1' \\ 0 & 0 & \frac{1}{2}I - K & V \\ 0 & 0 & D & \frac{1}{2}I + K' \end{bmatrix} \begin{bmatrix} v \\ w \\ r \\ s \end{bmatrix}$$

Mapping properties

Lemma

The Calderón operator C is a projection, i.e., $C = C^2$.

Lemma

For all boundary integral operators there hold the relations

$$VK' = KV, \quad K'D = DK, \quad VD = \frac{1}{4}I - K^2, \quad DV = \frac{1}{4}I - K'^2,$$

$$K_1V - V_1K' = VK_1' - KV_1, \quad K'D_1 - D_1K = DK_1 - K_1'D,$$

$$DV_1 + D_1V + K'K_1' + K_1'K' = 0, \quad VD_1 + V_1D + KK_1 + K_1K = 0.$$

Mapping properties

Lemma

For any $q \in H^{-1/2}(\Gamma)$ there holds the equality

$$\|\widetilde{V}q\|^2_{L^2(\Omega)} = \langle K_1 V q, q\rangle_\Gamma - \langle V_1(\frac{1}{2} + K')q, q\rangle_\Gamma .$$

Theorem

The boundary integral operator

$$T_\alpha := V^{-1}K_1 - V^{-1}V_1V^{-1}(\frac{1}{2}I + K) + \alpha A : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is self-adjoint, bounded and $H^{1/2}(\Gamma)$-elliptic.

The Schur complement system

$$T_\alpha z = V^{-1}N_0\overline{u} + V^{-1}M_0 f - V^{-1}V_1V^{-1}N_0 f$$

admits a unique solution $z \in H^{1/2}(\Gamma)$.

Boundary element method

We have to find $t, q \in H^{-1/2}(\Gamma), z \in H^{1/2}(\Gamma)$ such that

$$\begin{cases} -V_1 t + Vq + K_1 z = N_0 \overline{u} + M_0 f \\ Vt - \left(\frac{1}{2} I + K\right) z = -N_0 f \\ q - \alpha A z = 0 \end{cases}$$

Let

$$S_h^0(\Gamma) = span\{\varphi_k^0\}_{k=1}^N \subset H^{-1/2}(\Gamma), \quad S_H^1(\Gamma) = span\{\varphi_k^1\}_{k=1}^M \subset H^{1/2}(\Gamma)$$

be some boundary element spaces of piecewise constant and piecewise linear basis functions.

Galerkin variational formulation

The Galerkin boundary element formulation then reads to find $(t_h, q_h, z_H) \in S_h^0(\Gamma) \times S_h^0(\Gamma) \times S_H^1(\Gamma)$ such that

$$\begin{cases} -\langle V_1 t_h, w_h\rangle_\Gamma + \langle V q_h, w_h\rangle_\Gamma + \langle K_1 z_H, w_h\rangle_\Gamma = \langle N_0 \overline{u} + M_0 f, w_h\rangle_\Gamma, \\ \langle V t_h, \tau_h\rangle_\Gamma - \langle \left(\frac{1}{2} I + K\right) z_H, \tau_h\rangle_\Gamma = -\langle N_0 f, \tau_h\rangle_\Gamma, \\ -\langle q_h, v_H\rangle_\Gamma + \alpha\langle A z_H, v_H\rangle_\Gamma = 0 \end{cases}$$

is satisfied for all $(w_h, \tau_h, v_H) \in S_h^0(\Gamma) \times S_h^0(\Gamma) \times S_H^1(\Gamma)$.
Linear system of algebraic equations,

$$\begin{pmatrix} -V_{1,h} & V_h & K_{1,h} \\ V_h & & -\frac{1}{2} M_h - K_h \\ & -M_h^T & \alpha A_H \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{q} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \underline{0} \end{pmatrix}$$

Unique solvability

The Galerkin matrix V_h is positive definite and therefore invertible. Hence we obtain the Schur complement system

$$\left[\alpha A_H - M_h^T V_h^{-1} V_{1,h} V_h^{-1} (\frac{1}{2} M_h + K_h) + M_h^T V_h^{-1} K_{1,h}\right] \underline{z} = M_h^T V_h^{-1} \left[\underline{f}_1 + V_{1,h} V_h^{-1} \underline{f}_2\right]$$

where

$$\widetilde{T}_{\alpha,H} := \alpha A_H - M_h^T V_h^{-1} V_{1,h} V_h^{-1} (\frac{1}{2} M_h + K_h) + M_h^T V_h^{-1} K_{1,h}$$

defines Galerkin boundary element approximation of the Schur complement boundary integral operator T_α.

Unique solvability

Theorem

The approximate Schur complement $\widetilde{T}_{\alpha,H}$ is positive definite, i.e.

$$(\widetilde{T}_{\alpha,H}\underline{z},\underline{z}) \geqslant \frac{1}{2}c_1^T\|z_H\|^2_{H^{1/2}(\Gamma)}$$

for all $\underline{z} \in \mathbb{R}^M \leftrightarrow z_H \in S^1_H(\Gamma)$, if

$$h \leqslant c_0 H$$

for some constant $c_0 > 0$.

Error estimates

Theorem

Let t, z be the unique solution of system of boundary integral equations minimizing the reduced cost functional $J(u, z)$. Let t_h, z_H be the unique solution of the Galerkin variational problem. When assuming $z \in H^2(\Gamma)$ there hold the error estimates

$$\|z - z_H\|_{H^{1/2}(\Gamma)} \leqslant C\left(H^{3/2}|z|_{H^2(\Gamma)} + h^{3/2}\|z\|_{H^2(\Gamma)} + h^{3/2}\right),$$

$$\|t - t_h\|_{H^{-1/2}(\Gamma)} \leqslant C\left(H^{3/2}|z|_{H^2(\Gamma)} + h^{3/2}\|z\|_{H^2(\Gamma)} + h^{3/2}\right).$$

Remark: In $L^2(\Gamma)$ norm

$$\|z - z_H\|_{L^2(\Gamma)} = \mathcal{O}(h^2 + H^2),$$

$$\|t - t_h\|_{L^2(\Gamma)} = \mathcal{O}(h + H).$$

Symmetric formulation

Boundary integral equations

$$\begin{pmatrix} -V_1 & V & K_1 \\ V & 0 & -\frac{1}{2}I - K \\ -K_1' & \frac{1}{2}I + K' & -(\alpha A + D_1) \end{pmatrix} \begin{pmatrix} t \\ q \\ z \end{pmatrix} = \begin{pmatrix} N_0\overline{u} + M_0 f \\ -N_0 f \\ N_1\overline{u} + M_1 f \end{pmatrix}$$

The Schur complement system

$$\begin{aligned} &\Big[K_1' V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K')V^{-1}K_1 \\ &\quad - (\frac{1}{2}I + K')V^{-1}V_1V^{-1}(\frac{1}{2}I + K) + (\alpha A + D_1)\Big] z \\ =&K_1'V^{-1}N_0 f - (\frac{1}{2}I + K')V^{-1}V_1V^{-1}N_0 f \\ &\quad + (\frac{1}{2}I + K')V^{-1}N_0\overline{u} + (\frac{1}{2}I + K')V^{-1}M_0 f - N_1\overline{u} - M_1 f \end{aligned}$$

Unique solvability for symmetric formulation

Theorem

The boundary integral operator

$$T_\alpha := K_1' V^{-1}(\frac{1}{2}I + K) - (\frac{1}{2}I + K')V^{-1}V_1V^{-1}(\frac{1}{2}I + K)$$
$$+(\frac{1}{2}I + K')V^{-1}K_1 + (\alpha A + D_1) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is self-adjoint, bounded and $H^{1/2}(\Gamma)$-elliptic.

Linear system of algebraic equations,

$$\begin{pmatrix} -V_{1,h} & V_h & K_{1,h} \\ V_h & & -\frac{1}{2}M_h - K_h \\ K_{1,h}^T & -\frac{1}{2}M_h^T - K_h^T & \alpha A_H + D_{1,H} \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{q} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \end{pmatrix}$$

Finite Element Methods

Primal Dirichlet boundary value problem

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega, \qquad u(x) = z(x) \quad \text{for } x \in \Gamma$$

$u = u_0 + \mathcal{E}z, \quad u_0 \in H^1_0(\Omega)$:

$$\int_\Omega \nabla[u_0(x) + \mathcal{E}z(x)]\nabla v(x)\, dx = \int_\Omega f(x)v(x)\, dx \quad \text{for all} \quad v \in H^1_0(\Omega)$$

Adjoint Dirichlet boundary value problem

$$-\Delta p(x) = u(x) - \overline{u}(x) \quad \text{for } x \in \Omega, \qquad p(x) = 0 \quad \text{for } x \in \Gamma$$

$p \in H^1_0(\Omega)$ is the weak solution of the variational problem

$$\int_\Omega \nabla p(x)\nabla v(x)\, dx = \int_\Omega [u(x) - \overline{u}(x)]v(x)\, dx \quad \text{for all} \quad v \in H^1_0(\Omega)$$

Finite Element Methods

Optimality condition

$$\alpha(Az)(x) = q(x) = \frac{\partial}{\partial n} p(x) \quad \text{for } x \in \Gamma$$

$w \in H^{1/2}(\Gamma)$

$$\begin{aligned} \alpha\langle Az, w\rangle_\Gamma &= \int_\Gamma \frac{\partial}{\partial n_x} p(x) w(x)\, ds_x \\ &= \int_\Omega \nabla p(x) \nabla \mathcal{E} w(x)\, dx - \int_\Omega [u(x) - \overline{u}(x)] \mathcal{E} w(x)\, dx \end{aligned}$$

Finite element discretization

$$\begin{pmatrix} -M_{II} & K_{II} & -M_{CI} \\ K_{II} & & K_{CI} \\ M_{IC} & -K_{IC} & (M_{CC} + \alpha A_h) \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{p} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} -\underline{f}_1 \\ \underline{f}_2 \\ \underline{f}_3 \end{pmatrix}$$

Numerical results (nonsymmetric formulation)

In our numerical example, we consider the square $\Omega = (0, 0.5)^2$, $h = H$, and

$$\overline{u}(x) = -\left(4 + \frac{1}{\alpha}\right)[x_1(1-2x_1) + x_2(1-2x_2)], \quad f(x) = -\frac{8}{\alpha}, \quad \alpha = 0.01$$

$$A = \widetilde{D}: \quad \langle \widetilde{D}u, v\rangle_\Gamma := \langle Du, v\rangle_\Gamma + \langle u, 1\rangle_\Gamma \langle v, 1\rangle_\Gamma$$

with non zero control $z(x)$, (see paper of S. May, R. Rannacher, B. Vexler).

Mesh	$\|t_h - t_{h_9}\|_{L^2(\Gamma)}$		$\|z_h - z_{H_9}\|_{L^2(\Gamma)}$	
	Error	Order	Error	Order
16	33.5093		0.45219	
32	24.0337	0.48	0.12773	1.82
64	14.7956	0.70	0.03536	1.85
128	8.4434	0.81	9.026e-03	1.97
256	4.5915	0.88	2.175e-03	2.05
512	2.4033	0.93	5.158e-04	2.08
1024	1.1718	1.04	1.389e-04	1.89
expected		1.00		2.00

Numerical results (symmetric formulation)

We consider the square $\Omega = (0, 0.5)^2$, $h = H$, and

$$\overline{u}(x) = -\left(4 + \frac{1}{\alpha}\right)\left[x_1(1-2x_1) + x_2(1-2x_2)\right], \quad f(x) = -\frac{8}{\alpha}, \quad \alpha = 0.01$$

$$A = \widetilde{D}: \quad \langle \widetilde{D}u, v\rangle_\Gamma := \langle Du, v\rangle_\Gamma + \langle u, 1\rangle_\Gamma \langle v, 1\rangle_\Gamma$$

with non zero control $z(x)$.

Mesh	$\|t_h - t_{h_9}\|_{L^2(\Gamma)}$		$\|z_h - z_{H_9}\|_{L^2(\Gamma)}$	
	Error	Order	Error	Order
16	39.8736		2.24551	
32	28.6702	0.48	0.46550	2.27
64	16.2201	0.82	0.08836	2.39
128	8.8797	0.87	0.01631	2.44
256	4.7272	0.91	3.019e-03	2.43
512	2.4459	0.95	5.731e-04	2.40
1024	1.1853	1.05	1.243e-04	2.20
expected		1.00		2.00

Compare numerical results: BEM and FEM

Consider the square $\Omega = (0, 0.5)^2$, $h = H$ and

$$\overline{u}(x) = -\left(4 + \frac{1}{\alpha}\right)\left[x_1(1-2x_1) + x_2(1-2x_2)\right], \quad f(x) = -\frac{8}{\alpha}, \quad \alpha = 0.01$$

$$A = \widetilde{D}: \quad \langle \widetilde{D}u, v\rangle_\Gamma := \langle Du, v\rangle_\Gamma + \langle u, 1\rangle_\Gamma \langle v, 1\rangle_\Gamma$$

with non zero control $z(x)$.

Level	Symmetric BEM		FEM		
	$\|z_H - z_{H_9}^{BEM}\|_{L^2(\Gamma)}$	order	$\|z_H - z_{H_9}^{FEM}\|_{L^2(\Gamma)}$	order	$\|z_H - z_{H_9}^{BEM}\|_{L^2(\Gamma)}$
2	2.24551		0.38934		0.38936
3	0.46550	2.27	0.10745	1.86	0.10747
4	0.08836	2.39	0.02809	1.93	0.02811
5	0.01630	2.44	7.281e-03	1.95	7.298e-03
6	3.019e-03	2.43	1.872e-03	1.96	1.889e-03
7	5.731e-04	2.40	4.691e-04	2.00	4.883e-04
8	1.243e-04	2.20	1.060e-04	2.15	1.337e-04

Linear systems: BEM and FEM

Finite element system

$$\begin{pmatrix} -M_{II} & K_{II} & -M_{CI} \\ K_{II} & & K_{CI} \\ M_{IC} & -K_{IC} & (M_{CC} + \alpha A_h) \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{p} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} -\underline{f}_1 \\ \underline{f}_2 \\ \underline{f}_3 \end{pmatrix}$$

Boundary element system

$$\begin{pmatrix} -V_{1,h} & V_h & K_{1,h} \\ V_h & & -\frac{1}{2} M_h - K_h \\ K_{1,h}^T & -\frac{1}{2} M_h^T - K_h^T & \alpha A_H + D_{1,H} \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{q} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \end{pmatrix}$$

- Preconditioned iterative system solvers (saddle point structure)
- Construction of preconditioners (BEM and FEM)

Outlook/Plan

- Stability analysis for symmetric formulation: $h = H$
- Mixed boundary control problems
- Box boundary control problems
- Parabolic control problems
- BIE for time dependent problems

Thanks for your attention!