SpezialForschungsBereich F 32

Karl–Franzens Universität Graz
Technische Universität Graz
Medizinische Universität Graz

# Optimal control of a singular PDE modeling transient MEMS with control or state constraints

Christian Clason    Barbara Kaltenbacher

SFB-Report No. 2012–017    September 2012

A–8010 GRAZ, HEINRICHSTRASSE 36, AUSTRIA

Supported by the
Austrian Science Fund (FWF)

SFB sponsors:

- **Austrian Science Fund** (FWF)
- **University of Graz**
- **Graz University of Technology**
- **Medical University of Graz**
- **Government of Styria**
- **City of Graz**

# OPTIMAL CONTROL OF A SINGULAR PDE MODELING TRANSIENT MEMS WITH CONTROL OR STATE CONSTRAINTS

Christian Clason* Barbara Kaltenbacher†

September 5, 2012

A particular feature of certain microelectromechanical systems (MEMS) is the appearance of a so-called "pull-in" instability, corresponding to a singularity in the underlying PDE model. We here consider a transient MEMS model and its optimal control via the dielectric properties of the membrane and/or the applied voltage. In contrast to the static case, the control problem suffers from low dimensionality of the control compared to the state and hence requires different techniques for establishing first order optimality conditions. For this purpose, we here use a relaxation approach combined with a localization technique.

## 1 INTRODUCTION

Consider the initial boundary value problem

$$
\begin{cases}
y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y + \dfrac{b(t)a(x)}{(1+y)^2} = 0 & \text{in } Q = (0,T)\times\Omega, \\
y = \partial_\nu y = 0 & \text{on } (0,T)\times\partial\Omega, \\
y = y^0,\ y_t = y^1 & \text{in } \{0\}\times\Omega,
\end{cases}
\tag{1.1}
$$

for $\Omega \subseteq \mathbb{R}^n$, $n \in \{1,2,3\}$ (typically $n = 2$), which models the deflection of the membrane of a microelectromechanical system (MEMS), where $y$ is the mechanical displacement, $a$

*Institute for Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, A-8010 Graz, Austria (christian.clason@uni-graz.at).

†Institute of Mathematics, Alpen-Adria Universität Klagenfurt, Universitätsstr. 65–67 9020 Klagenfurt, Austria (barbara.kaltenbacher@aau.at).

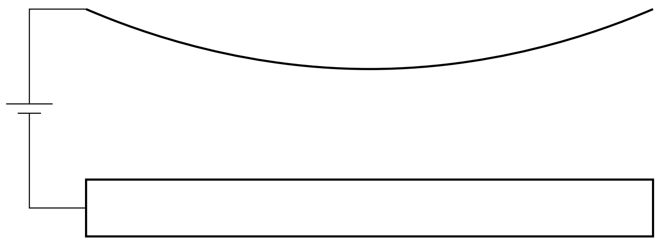


Figure 1: Schematic of a MEMS

is the reciprocal of the dielectric coefficient, and $b$ is a dimensionless number proportional to the applied voltage [Cassani et al. 2009]. The constants $c, d \geqslant 0$, $\rho, \eta > 0$ are material parameters, with the term $cy_t$ modeling possible damping and the term $dy$ potentially taking into account the reset force of a spring in the system. The boundary conditions used here correspond to a clamped setting; for a number of different possible boundary conditions we refer to [Cassani et al. 2009]. Figure 1 shows the schematic of the type of MEMS we are considering here.

The case $y(t, x) = -1$ corresponds to the so-called "pull-in" instability, in which the applied voltage leads to a sufficiently large deflection of the membrane for it to touch the ground plate, possibly damaging the device. This undesirable situation manifests itself in the equation as a potential singularity.

In practical applications, either the dielectric properties, the applied voltage, or both are available as design variables. Consequently, we will consider optimization problems of the form

$$\begin{cases} \min\limits_{y\in\mathcal{Y}, u\in\mathcal{U}} \dfrac{1}{2}\|y - y_d\|^2_{L^2(Q)} + \dfrac{\alpha}{2}\|u\|^2_{\mathcal{U}} =: J(u, y) \\ \text{s. t.} \quad y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y + \dfrac{\beta u}{(1+y)^2} = 0, \\ \qquad\quad y|_{\partial\Omega} = \partial_\nu y|_{\partial\Omega} = 0, \quad y(0) = y^0,\ y_t(0) = y^1. \end{cases}$$

with $\mathcal{Y}$ denoting the state space and the control $u \in \mathcal{U}$ being defined by one of the following three cases:

(i) *control by dielectic properties:* $\mathcal{U} = L^2(\Omega)$ and $\beta = b \in L^2(0, T)$ fixed,

(ii) *control by applied voltage:* $\mathcal{U} = L^2(0, T)$ and $\beta = a \in L^2(\Omega)$ fixed,

(iii) *control by both:* $\mathcal{U}$ is a subspace of $L^2(\Omega) \times L^2(0, T)$ (to be defined below) and $\beta \equiv 1$.

For simplicity of exposition and since it is also of high practical relevance, we restrict ourselves to a tracking type cost function for a prescribed target displacement $y_d$. The existence results below (Theorems 4.1 and 4.6) remain valid for any cost functional $J(u, y)$ that is bounded from below, $\mathcal{U}$-coercive with respect to $u$ (this condition may be omitted in the control constrained case), and weakly lower semi-continuous on $\mathcal{U}$ and $\mathcal{Y}$.

A straightforward approach to prevent instabilities such as the "pull-in" instability at $y = -1$ is to impose control constraints

$$\|u\|_{\mathcal{U}} \leqslant M_u$$

with $M_u$ sufficiently small to indirectly – via the PDE – guarantee that the state never reaches the critical value $y = -1$. However, the singularity can also be prevented by imposing pointwise state constraints

$$-y(t, x) \leqslant M_y < 1.$$

As already demonstrated in [Clason and Kaltenbacher 2011] for the simpler static MEMS model, only the latter approach is able to attain states corresponding to large deflections, which is relevant in applications for achieving a sufficiently large stroke of the device. In the transient situation with state constraints, due to the reduced dimensionality of the control, the approach from [Alibert and Raymond 1998] used in [Clason and Kaltenbacher 2011] is not applicable directly any more. We therefore apply the relaxation approach from [Bonnans and Casas 1989] together with a localization technique as in [Casas and Tröltzsch 2002]. Specifically, we introduce a new independent variable in place of the possibly singular nonlinearity and penalize the deviation from the original minimizers. Taking the limit with respect to the penalty parameter in the corresponding optimality conditions yields the optimality system for the original problem.

This work is organized as follows. In section 2, we provide some necessary results on the (linearized) state and adjoint equations. Section 3 briefly discusses existence of and optimality conditions for each of the three types of controls above in the control constrained case. The corresponding results for the state constrained case, which form the main contribution of this work, are given in section 4.

## 2 STATE EQUATION

We start with a well-posedness result for a linear problem related to (1.1):

$$\begin{cases} y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y + w = 0 & \text{in } Q, \\ y = \partial_\nu y = 0 & \text{on } (0,T) \times \partial\Omega, \\ y = y^0,\ y_t = y^1 & \text{in } \{0\} \times \Omega. \end{cases} \tag{2.1}$$

For this purpose we require the following regularity and compatibility conditions on the initial data:

$$(y^0, y^1) \in H^2(\Omega) \times L^2(\Omega),\ y^0|_{\partial\Omega} = \partial_\nu y^0|_{\partial\Omega} = 0. \tag{2.2}$$

We also introduce for further reference the spaces

$$\begin{aligned} \check{\mathcal{Y}} &:= C(0,T; H_0^2(\Omega)) \cap C^1(0,T; L^2(\Omega)), \\ \tilde{\mathcal{Y}} &:= C(0,T; H_0^2(\Omega)) \cap C^1(0,T; L^2(\Omega)) \cap H^2(0,T; H^{-2}(\Omega)), \\ \check{\check{\mathcal{Y}}} &:= C(0,T; L^2(\Omega)) \cap C^1(0,T; H^{-2}(\Omega)) \cap C^2(0,T; H^{-4}(\Omega)). \end{aligned}$$

**Lemma 2.1.** *For any $w \in \mathcal{M}(0,T;L^2(\Omega))$, and $y^0, y^1$ satisfying (2.2), there exists a unique solution $y \in \breve{\mathcal{Y}}$ to (2.1). Furthermore, there exists a constant $C > 0$ independent of $w, y^0, y^1$ such that*

$$\|y\|_{\breve{\mathcal{Y}}} + c\|y_t\|_{L^2(0,T;L^2(\Omega))} \leqslant C\left(\|y^0\|_{H^2(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|w\|_{\mathcal{M}(0,T;L^2(\Omega))}\right).$$

*If additionally $w \in L^2(0,T;H^{-2}(\Omega))$, then $y \in \tilde{\mathcal{Y}}$ and there exists a $C > 0$ such that for all $w \in \mathcal{M}(0,T;L^2(\Omega)) \cap L^2(0,T;H^{-2}(\Omega))$, there holds*

$$\begin{aligned}\|y\|_{\breve{\mathcal{Y}}} &+ c\|y_t\|_{L^2(0,T;L^2(\Omega))} \\ &\leqslant C(\|y^0\|_{H^2(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|w\|_{\mathcal{M}(0,T;L^2(\Omega))\cap L^2(0,T;H^{-2}(\Omega))}).\end{aligned} \tag{2.3}$$

*Proof.* Existence of a solution can be found, e.g, in [Cassani et al. 2009], and the energy estimate (2.3) can be obtained by means of the multiplier $y_t$ and integration by parts:

$$\begin{aligned}\frac{1}{2}\left[\|y_t\|^2_{L^2(\Omega)} + d\|y\|^2_{L^2(\Omega)} + \rho\|\Delta y\|^2_{L^2(\Omega)} + \eta\|\nabla y\|^2_{L^2(\Omega)}\right]_0^t + c\int_0^t \|y_t\|^2_{L^2(\Omega)} \\ = -\int_0^t\int_\Omega w y_t \, dx\, dt \leqslant \|w\|_{\mathcal{M}(0,T;L^2(\Omega))} \sup_{t\in(0,T)} \|y_t\|_{L^2(\Omega)}.\end{aligned} \tag{2.4}$$

The second order in time estimate in case $w \in L^2(0,T;H^{-2}(\Omega))$ can be easily verified using

$$y_{tt} = -cy_t - dy - \rho\Delta^2 y + \eta\Delta y - w \in L^2(0,T;H^{-2}(\Omega)). \qquad \square$$

Since in the state constrained case, the adjoint equation will have a measure-valued right hand side, the following result will be useful for assessing regularity of the adjoint solution. It is obtained by applying the solution operator $(-\Delta)^{-1}$ of the Laplace equation with homogeneous Dirichlet boundary conditions (which can be continuously extended to an operator from $H^{-2}(\Omega)$ to $L^2(\Omega)$ by duality) to both sides of the equation. The regularity and compatibility conditions on the initial data in this case reduce to

$$(y^0, y^1) \in L^2(\Omega) \times H^{-2}(\Omega), \qquad (-\Delta)^{-1}\partial_\nu y^0|_{\partial\Omega} = 0. \tag{2.5}$$

**Corollary 2.2.** *For any $w \in \mathcal{M}(0,T;\mathcal{M}(\Omega))$ and $y^0, y^1$ satisfying (2.5), there exists a unique solution $y \in C(0,T;L^2(\Omega)) \cap C^1(0,T;H^{-2}(\Omega))$ to (2.1). Furthermore, there exists a constant $C > 0$ independent of $w, y^0, y^1$ such that*

$$\begin{aligned}\sup_{t\in(0,T)} \|(-\Delta)^{-1} y_t(t)\|_{L^2(\Omega)} &+ \sup_{t\in(0,T)} \|y(t)\|_{L^2(\Omega)} \\ &\leqslant C\left(\|y^0\|_{L^2(\Omega)} + \|(-\Delta)^{-1}y^1\|_{L^2(\Omega)} + \|w\|_{\mathcal{M}(0,T;\mathcal{M}(\Omega))}\right).\end{aligned}$$

*Proof.* Since $\frac{2n}{n+2} < \frac{n}{n-1}$ holds for $n \in \{1,2,3\}$, we get from Theorem 7.7 in [Tröltzsch 2010] with $q = \frac{2n}{n+2}$, and Sobolev's embedding $W^{1,q}(\Omega) \to L^2(\Omega)$, that

$$\|(-\Delta)^{-1} g(t)\|_{L^2(\Omega)} \leqslant C_1\|(-\Delta)^{-1} g(t)\|_{W^{1,q}(\Omega)} \leqslant C_2\|g(t)\|_{\mathcal{M}(\Omega)}$$

for any $g \in C^\infty(0,T;\mathcal{M}(\Omega))$. Therefore we can apply Lemma 2.1 together with the density of $C^\infty(0,T;\mathcal{M}(\Omega))$ in $\mathcal{M}(0,T;\mathcal{M}(\Omega))$ to conclude the assertion. $\square$

Similarly we get

**Corollary 2.3.** *For any* $w \in L^2(0,T;H^{-2}(\Omega))$ *and* $y^0, y^1$ *satisfying* (2.5), *there exists a unique solution* $y \in \check{\mathcal{Y}}$ *to* (2.1). *Furthermore, there exists a constant* $C > 0$ *independent of* $w, y^0, y^1$ *such that*

$$\|y\|_{\check{\mathcal{Y}}} \leqslant C\left(\|y^0\|_{L^2(\Omega)} + \|(-\Delta)^{-1}y^1\|_{L^2(\Omega)} + \|w\|_{L^2(0,T;H^{-2}(\Omega))}\right). \tag{2.6}$$

For the derivation of the optimality system in the control constrained case, we will rely on a reduced approach using the control-to-state map. For this purpose we have to show well-posedness of the PDE for all admissible controls. Here and below we will make use of the continuous lifting from $L^2(\Omega)$ to $C_0(\Omega)$ via the inverse Laplacian with Dirichlet conditions, i.e., the existence of a constant $C_L > 0$ such that for all $v \in L^2(\Omega)$,

$$\|(-\Delta)^{-1}v\|_{C(\Omega)} \leqslant C_L\|v\|_{L^2(\Omega)}. \tag{2.7}$$

In order to deal with a linear state space later on (see (4.1) below) we set – without loss of generality – $y^0 = y^1 = 0$ in the following.

**Theorem 2.4.** *There exists a constant* $M > 0$ *such that for all* $f \in \mathcal{M}(0,T;L^2(\Omega))$ *with*

$$\|f\|_{\mathcal{M}(0,T;L^2(\Omega))} \leqslant M,$$

*there exists a unique solution* $y \in \check{\mathcal{Y}}$ *to*

$$\begin{cases} y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y + \dfrac{f}{(1+y)^2} = 0 & \textit{in } Q = (0,T)\times\Omega, \\ y = \partial_\nu y = 0 & \textit{on } (0,T)\times\partial\Omega, \\ y = 0,\ y_t = 0 & \textit{in } \{0\}\times\Omega. \end{cases} \tag{2.8}$$

*Moreover, there exists an* $\overline{m} > 0$ *independent of* $f$ *such that*

$$\|y\|_{C(0,T;C_0(\Omega))} \leqslant \overline{m} < 1. \tag{2.9}$$

*Proof.* Proceeding similarly to the static case in [Clason and Kaltenbacher 2011] (see also the well-posedness proof in [Cassani et al. 2009]), we use Banach's fixed point theorem. Let

$$\mathcal{W} = \left\{ v \in \check{\mathcal{Y}} : \sup_{t\in(0,T)} \|v\|_{C(\Omega)} \leqslant \overline{m}, v(0) = y^0,\ v_t(0) = y^1 \right\}$$

for some $\overline{m} \in (0,1)$ to be chosen below, and consider the fixed point operator $T : \mathcal{W} \to \mathcal{W}$, $v \mapsto Tv := y$, where $y$ solves

$$\begin{cases} y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y = -\dfrac{f}{(1+v)^2} & \text{in } Q, \\ y = \partial_\nu y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y = y^0,\ y_t = y^1 & \text{in } \{0\}\times\Omega. \end{cases}$$

Well-definedness of T follows from Lemma 2.1.

To show that T is a self-mapping, consider an arbitrary $v \in \mathcal{W}$. Then we have by (2.4) and Young's inequality (canceling the $\|y_t\|_{C(0,T;L^2(\Omega))}$ terms) that

$$\sup_{t\in(0,T)} \|(Tv)(t)\|_{C(\Omega)} \leqslant \frac{C_L}{\sqrt{\rho}(1-\overline{m})^2}\|f\|_{\mathcal{M}(0,T;L^2(\Omega))},$$

where $C_L$ is the constant in (2.7), and the right hand side is not larger than $\overline{m}$ if

$$\|f\|_{\mathcal{M}(0,T;L^2(\Omega))} \leqslant \frac{\sqrt{\rho}\overline{m}(1-\overline{m})^2}{C_L}. \tag{2.10}$$

Contractivity of T follows from the fact that for any $v, w \in \mathcal{W}$, there holds

$$\begin{aligned}
&\sup_{t\in(0,T)} \|(Tv - Tw)(t)\|_{C(\Omega)} \\
&\quad\leqslant \frac{C_L}{\sqrt{\rho}}\|f\|_{\mathcal{M}(0,T;L^2(\Omega))} \sup_{t\in(0,T)} \left\| \frac{1}{(1+v)^2} - \frac{1}{(1+w)^2}\right\|_{C(\Omega)} \\
&\quad= \frac{C_L}{\sqrt{\rho}}\|f\|_{\mathcal{M}(0,T;L^2(\Omega))} \sup_{t\in(0,T)} \left\| \left(\frac{1}{(1+v)^2(1+w)} + \frac{1}{(1+w)^2(1+v)}\right)(v-w)(t)\right\|_{C(\Omega)} \\
&\quad\leqslant \frac{2C_L}{\sqrt{\rho}(1-\overline{m})^3}\|f\|_{\mathcal{M}(0,T;L^2(\Omega))} \sup_{t\in(0,T)} \|(v-w)(t)\|_{C(\Omega)}.
\end{aligned}$$

Hence contractivity holds if

$$\|f\|_{\mathcal{M}(0,T;L^2(\Omega))} < \frac{\sqrt{\rho}(1-\overline{m})^3}{2C_L}. \tag{2.11}$$

The maximum over $\overline{m} \in (0,1)$ of the minimum of the two right hand sides in (2.10) and (2.11) can be found by equilibrating $\frac{\sqrt{\rho}\overline{m}(1-\overline{m})^2}{C_L} = \frac{\sqrt{\rho}(1-\overline{m})^3}{2C_L}$, which yields the optimal bound

$$\overline{m} = \tfrac{1}{3}.$$

Therefore, a solution to (1.1) exists if

$$\|f\|_{\mathcal{M}(0,T;L^2(\Omega))} \leqslant M := \frac{4\sqrt{\rho}}{27C_L}.$$

□

Hence, for

$$\mathcal{D}_S \subseteq \mathcal{B}_M^{\mathcal{M}(L^2)}(0) = \left\{f \in \mathcal{M}(0,T;L^2(\Omega)) : \|f\|_{\mathcal{M}(0,T;L^2(\Omega))} \leqslant M\right\}$$

with $M > 0$ sufficiently small, the operator

$$\begin{aligned}
\mathcal{S}: \ \mathcal{D}_S &\to \check{\mathcal{Y}} \\
f &\mapsto y \text{ solving (2.8)},
\end{aligned}$$

from which we will derive the control-to-state mapping S later on, is well-defined. For proving smoothness of this mapping, we will consider the linearization of (2.8).

$$
\text{(2.12)} \qquad \begin{cases} y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y - \dfrac{2\bar{f}}{(1+\bar{y})^3} y = w & \text{in } Q, \\ y = \partial_\nu y = 0 & \text{on } (0,T) \times \partial\Omega, \\ y = 0,\ y_t = 0 & \text{in } \{0\} \times \Omega, \end{cases}
$$

for given $\bar{f} \in \mathcal{D}_S, \bar{y} \in \breve{\mathcal{Y}}, w \in \mathcal{M}(0,T;L^2(\Omega))$.

**Lemma 2.5.** *There exists a constant* $M > 0$ *such that for all* $\bar{f} \in \mathcal{D}_S \subseteq \mathcal{B}_M^{\mathcal{M}(L^2)}(0)$, $\bar{y} = \mathcal{S}(\bar{f})$, *and all* $w \in \mathcal{M}(0,T;L^2(\Omega))$, *the linearized state equation* (2.12) *has a unique solution* $y \in \breve{\mathcal{Y}}$, *which depends continuously on* $w$.

*Proof.* By Theorem 2.4 the coefficient $e = -\frac{2\bar{f}}{(1+\bar{y})^3}$ is contained in $\mathcal{M}(0,T;L^2(\Omega))$. Since $e$ might take negative values – leading to negative contributions on the left hand side of an energy estimate obtained as in (2.4), – it is clear that we have to impose smallness of $e$. This can be obtained again using Theorem 2.4 from smallness of $\bar{f}$. We can thus apply a fixed point argument as in the proof of Theorem 2.4, this time with the (linear) fixed point operator $T_{\text{lin}} : \mathcal{W}_{\text{lin}} \to \mathcal{W}_{\text{lin}}, v \mapsto T_{\text{lin}} v := y$, where $y$ solves

$$
\begin{cases} y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y = \dfrac{2\bar{f}}{(1+\bar{y})^3} v + w & \text{in } Q, \\ y = \partial_\nu y = 0 & \text{on } (0,T) \times \partial\Omega, \\ y = 0,\ y_t = 0 & \text{in } \{0\} \times \Omega. \end{cases}
$$

By Lemma 2.1, the operator T is well-defined, contractive, and a self-mapping on

$$
\mathcal{W}_{\text{lin}} = \left\{ v \in \breve{\mathcal{Y}} : \sup_{t \in (0,T)} \|v\|_{C(\Omega)} \leqslant R, v(0) = 0,\ v_t(0) = 0 \right\}
$$

with R sufficiently large so that

$$
R \frac{C_L}{\sqrt{\rho}} \left\| \frac{2\bar{f}}{(1+\bar{y})^3} \right\|_{\mathcal{M}(0,T;L^2(\Omega))} + \|w\|_{\mathcal{M}(0,T;L^2(\Omega))} \leqslant R.
$$

From this we deduce the assertion. ☐

To show weak continuity of $\mathcal{S}$ as needed for the existence of a minimizer later on, we will use compactness of the embedding of $\breve{\mathcal{Y}}$ in $C(0,T;C_0(\Omega))$. This is established in the following lemma, which is based on the Dubinskii theorem quoted here for the sake of convenience.

**Proposition 2.6** ([Vishik and Fursikov 1988, Theorem 4.1])**.** *Assume that* $E_0, E, E_1$ *are reflexive Banach spaces, with continuous embeddings* $E_0 \hookrightarrow E \hookrightarrow E_1$, *and compact embedding* $E_0 \hookrightarrow E$, *and let* $1 < q, q_1 < \infty$. *Moreover, assume that the set* $M$ *is bounded in* $L^q(0,T;E_0)$ *and consists of functions* $u(t)$ *equicontinuous in* $C(0,T;E_1)$. *Then* $M$ *is relatively compact in* $L^{q_1}(0,T;E)$ *and in* $C(0,T;E_1)$.

**Lemma 2.7.** *For all* $\sigma \in (0, 1-\frac{n}{4})$ *and* $\varepsilon \in (0, 2(1-\frac{n}{4}-\sigma))$ *we have*

$$\breve{\mathcal{Y}} \hookrightarrow W^{\sigma,\infty}(0,T;W^{\varepsilon,\infty}(\Omega)) \hookrightarrow C(0,T;C_0(\Omega)),$$

*where both embeddings are continuous and the latter is compact.*

*Proof.* Continuity of the first embedding follows by continuity of the embeddings $L^2(\Omega) \hookrightarrow W^{-s,r}(\Omega)$ and $H^2(\Omega) \hookrightarrow W^{t,r}(\Omega)$ for $s = \frac{n}{2}$, $t = 2 - \frac{n}{2}$ and any $r \in [1,\infty)$, and by interpolation

$$(W^{1,r}(0,T;W^{-s,r}(\Omega)), L^r(0,T;W^{t,r}(\Omega))_{\theta,r} = W^{(1-\theta),r}(0,T;W^{-(1-\theta)s+\theta t,r}(\Omega)).$$

With $r \geqslant \frac{8}{n}$ and $\theta = 1-\sigma-\frac{2}{r}$, the latter space is continuously embedded in $W^{\sigma,\infty}(0,T;W^{\varepsilon,\infty}(\Omega))$ for $\sigma \in (0, 1-\frac{n}{4})$, $\varepsilon \in (0, 2(1-\frac{n}{4}-\sigma)-\frac{n+4}{r})$, where we may let r tend to infinity. Compactness of the second embedding in (4.1) can be concluded from Proposition 2.6 applied to $E_0 = W^{\varepsilon,2p}(\Omega)$ and $E_1 = W^{\frac{\varepsilon}{2},p}(\Omega)$ with $p > \max\{1, \frac{2n}{\varepsilon}\}$ so that $E_1 \hookrightarrow C_0(\Omega)$. □

Finally, we address differentiability of the control-to-state mapping.

**Theorem 2.8.** *There exists a constant* $M > 0$ *such that* $\mathcal{S} : \mathcal{D}_S \to \breve{\mathcal{Y}}$ *is Lipschitz continuous, weakly continuous, and Fréchet differentiable with respect to the* $L^2(Q)$ *topology in preimage space. The derivative* $\mathcal{S}'(\bar{f})(\tilde{f}-\bar{f})$ *is given by the solution* $y \in \breve{\mathcal{Y}}$ *to*

(2.13)
$$\begin{cases} y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y - \dfrac{2\bar{f}}{(1+\bar{y})^3} y = -\dfrac{1}{(1+\bar{y})^2}(\tilde{f}-\bar{f}) & \textit{in } Q,\\ y = \partial_\nu y = 0 & \textit{on } (0,T)\times\partial\Omega,\\ y = 0,\ y_t = 0 & \textit{in } \{0\}\times\Omega, \end{cases}$$

*where* $\bar{y} = \mathcal{S}(\bar{f})$.

*Proof.* Lipschitz continuity follows by applying a similar argument to the one in Lemma 2.5 to the equation

$$\begin{cases} z_{tt} + cz_t + dz + \rho\Delta^2 z - \eta\Delta z - \dfrac{\tilde{f}(2+\tilde{y}+y)}{(1+y)^2(1+\tilde{y})^2} z = -\dfrac{1}{(1+y)^2}(\tilde{f}-f) & \text{in } Q,\\ z = \partial_\nu z = 0 & \text{on } (0,T)\times\partial\Omega,\\ z = 0,\ z_t = 0 & \text{in } \{0\}\times\Omega, \end{cases}$$

which is satisfied by the difference $z = \tilde{y} - y$ between two solutions $\tilde{y} = \mathcal{S}(\tilde{f})$ and $y = \mathcal{S}(f)$.

Weak continuity of $\mathcal{S}$ can be obtained by a subsequence-subsequence argument using the fact that $\breve{\mathcal{Y}}$ is compactly embedded in $C(0,T;C_0(\Omega))$ (see Lemma 2.7), which allows taking limits in the nonlinear part of the PDE defining $\mathcal{S}$ (see the proof of Theorem 2.2 in [Clason and Kaltenbacher 2011]).

Fréchet differentiability follows from the fact that $y \in \breve{\mathcal{Y}}$ is well-defined by (2.13) due to Lemma 2.1, and $z := \mathcal{S}(\tilde{f}) - \mathcal{S}(\bar{f}) - y = \tilde{y} - \bar{y} - y$ solves

$$\begin{cases} z_{tt} + cz_t + dz + \rho\Delta^2 z - \eta\Delta z - \dfrac{2\bar{f}}{(1+\bar{y})^3} z = \dfrac{2}{(1+y)^3}(\tilde{f} - f)(\tilde{y} - y) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad - \dfrac{\tilde{f}(3 + y - \tilde{y})}{(1+y)^3(1+\tilde{y})^2}(\tilde{y} - y)^2 & \text{in } Q, \\ z = \partial_\nu z = 0 & \text{on } (0,T) \times \partial\Omega, \\ z = 0, \; z_t = 0 & \text{in } \{0\} \times \Omega. \end{cases}$$

The $\mathcal{M}(0,T;L^2(\Omega))$ norm of the right hand side can be estimated by a multiple of $\|\tilde{f} - \bar{f}\|^2_{\mathcal{M}(0,T;L^2(\Omega))}$ due to the Lipschitz continuity shown above, so that Lemma 2.5 yields the desired quadratic estimate of the first order Taylor remainder. □

# 3 CONTROL CONSTRAINED OPTIMAL CONTROL

For comparison, we first consider the control constrained case. We distinguish between separate control by either the dielectric properties (case (i) above) or the applied voltage (case (ii) above) and combined control by both (case (iii) above).

## 3.1 CONTROL BY DIELECTRIC PROPERTIES OR VOLTAGE

Since case (i) and (ii) only differ in the domain on which the control is defined, we can treat both using the same approach. We define the unconstrained control space as $\mathcal{U} := L^2(X)$ with $X = \Omega$ in case (i) and $X = (0,T)$ in case (ii) and consider the the control constrained optimal control problem

$$(\mathcal{P}_{cc}) \qquad \begin{cases} \min\limits_{y \in \breve{\mathcal{Y}}, u \in \mathcal{U}} J(u,y) = \dfrac{1}{2}\|y - y_d\|^2_{L^2(Q)} + \dfrac{\alpha}{2}\|u\|^2_{L^2(X)} \\ \text{s.t.} \quad y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y + \dfrac{\beta u}{(1+y)^2} = 0 \text{ in } Q \\ \qquad\; y = \partial_\nu y = 0 \text{ on } (0,T) \times \partial\Omega, \quad y = y_t = 0 \text{ in } \{0\} \times \Omega, \\ \qquad\; \|u\|_{L^2(\Omega)} \leqslant M_u. \end{cases}$$

The bound $M_u$ is chosen small enough so that $\|y\|_{C(\Omega)} < 1$, which is possible due to (2.9). The corresponding constrained control space is defined as

$$\mathcal{U}_M = \left\{ u \in \mathcal{U} : \|u\|_{L^2(X)} \leqslant M_u \right\}.$$

Since the state equation is well-posed for every $u \in \mathcal{U}_M$ (see Theorem 2.4), we can use the control-to-state mapping $S : \mathcal{U}_M \to \breve{\mathcal{Y}}$, to obtain the reduced problem

$$(\mathcal{P}'_{cc}) \qquad \min_{u \in \mathcal{U}_M} J(u, S(u)).$$

Note that by Theorem 2.8, S is weakly continuous and Fréchet differentiable.

Since $\mathcal{U}_M$ is nonempty and weakly sequentially compact, S is weakly continuous for all $u \in \mathcal{U}_M$ by Theorem 2.8, and J is weakly $\breve{\mathcal{Y}} \times \mathcal{U}$ lower semi-continuous and bounded from below, we obtain the existence of a minimizer $u^* \in \mathcal{U}_M$ by standard arguments; cf., e.g., [Tröltzsch 2010].

Similarly, we obtain first order necessary optimality conditions. For a local minimizer $u^*$ of $(\mathcal{P}'_{cc})$ and $y^* := S(u^*) \in \breve{\mathcal{Y}}$, we can introduce the adjoint state $p^* \in \breve{\mathcal{Y}}$ solving

(3.1)
$$\begin{cases} p^*_{tt} - cp^*_t + dp^* + \rho\Delta^2 p^* - \eta\Delta p^* - \dfrac{2\beta u^*}{(1+y^*)^3} p^* = -(y - y_d) & \text{in } Q, \\ p^* = \partial_\nu p^* = 0 & \text{on } (0,T)\times\partial\Omega, \\ p^* = 0,\ p^*_t = 0 & \text{in } \{T\}\times\Omega, \end{cases}$$

due to Lemma 2.5 with t replaced by $T - t$.

Furthermore, a Slater condition is trivially satisfied for the inequality constraint (take $u = 0$). From, e.g., [Bonnans and Shapiro 2000, Proposition 3.2], we thus deduce the existence of a corresponding Lagrange multiplier $\lambda^* \in \mathbb{R}$ and hence the following optimality system.

**Theorem 3.1.** *Let $u^* \in \mathcal{U}_M$ be a local minimizer of $(\mathcal{P}'_{cc})$, $y^* := S(u^*) \in \breve{\mathcal{Y}}$, and $p^* \in \breve{\mathcal{Y}}$ satisfy (3.1). Then there exists $\lambda^* \in \mathbb{R}$, $\lambda^* \geqslant 0$ satisfying*

$$\begin{cases} (\alpha + 2\lambda^*)u^* = \displaystyle\int_Z \frac{\beta}{(1+y^*)^2} p^*\, ds, \\ \lambda^*(\|u^*\|^2_{L^2(\Omega)} - M_u^2) = 0, \end{cases}$$

*with $Z = (0,T)$ if $X = \Omega$ and $Z = \Omega$ if $X = (0,T)$.*

## 3.2 CONTROL BY DIELECTRIC PROPERTIES AND VOLTAGE

In the case (iii), we take as control variable

$$u(t,x) = \nu(x)(1 + \varphi(t)) \qquad \text{for all } (t,x) \in Q.$$

Expressing the applied voltage as $1 + \varphi(t)$ together with a normalization of $\varphi$ (see (3.3) below) is required to avoid the non-uniqueness of the straightforward representation $u(t,x) = \nu(x)\varphi(t)$, with $\nu$ and $\varphi$ elements of linear spaces. Correspondingly, we define the unconstrained control space as

$$\mathcal{U} = \left\{\nu \cdot (\mathbf{1} + \varphi) : \nu \in L^2(\Omega),\ \varphi \in L^2_\diamond(0,T)\right\}, \tag{3.2}$$

where

$$L^2_\diamond(0,T) = \left\{\psi \in L^2(0,T) : \int_0^T \psi\, dt = 0\right\} \tag{3.3}$$

and $\mathbf{1} \in L^\infty(Q)$ is the constant function with value one. Furthermore, we set

$$\|u\|_{\mathcal{U}}^2 := \|v\|_{L^2(\Omega)}^2 + \|\varphi\|_{L^2(0,T)}^2 \tag{3.4}$$

The following lemma allows us to identify $\mathcal{U}$ with a closed affine subspace of $L^2(\Omega) \times L^2(0,T)$, which implies that $\|u\|_{\mathcal{U}}$ is well-defined.

**Lemma 3.2.** *For every* $u \in \mathcal{U}$*, the representation* $u = v \cdot (\mathbf{1} + \varphi)$ *with* $v \in L^2(\Omega)$ *and* $\varphi \in L^2_\diamond(0,T)$ *is unique. Moreover, we have*

$$\|v \cdot (\mathbf{1} + \varphi)\|_{L^2(Q)}^2 = \|v\|_{L^2(\Omega)}^2 \left(T + \|\varphi\|_{L^2(0,T)}^2\right). \tag{3.5}$$

*Proof.* For proving uniqueness of the representation, assume that for $v_1, v_2 \in L^2(\Omega)$, $\varphi_1, \varphi_2 \in L^2(0,T)$ we have

$$v_1(x) \cdot (1 + \varphi_1(t)) = v_2(x) \cdot (1 + \varphi_2(t))$$

for almost all $x \in \Omega$ and $t \in (0,T)$. This implies that there exists a $\lambda \in \mathbb{R}$, $\lambda \neq 0$ such that

$$v_1 = \frac{1}{\lambda} v_2 \qquad \text{and} \qquad \mathbf{1} + \varphi_1 = \lambda(\mathbf{1} + \varphi_2),$$

the latter implying

$$(\lambda - 1)\mathbf{1} + \lambda\varphi_2 - \varphi_1 = 0.$$

Taking the $L^2(0,T)$ inner product with $\mathbf{1}$ and using the normalization condition $\int_0^T \varphi_i \, dt = 0$ for $i = 1, 2$, we easily see that this implies $\lambda - 1 = 0$ and hence $v_1 = v_2$, $\varphi_1 = \varphi_2$. □

We now consider the optimization problem

$$(\mathcal{P}^2_{cc}) \qquad \begin{cases} \displaystyle\min_{y \in \check{\mathcal{Y}}, u \in \mathcal{U}} J(u,y) = \frac{1}{2}\int_0 \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{\mathcal{U}}^2 \\ \text{s.t.} \quad y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y + \dfrac{u(t,x)}{(1+y)^2} = 0 \text{ in } Q \\ \qquad y = \partial_\nu y = 0 \text{ on } (0,T) \times \partial\Omega, \quad y = y_t = 0 \text{ in } \{0\} \times \Omega, \\ \qquad \|u\|_{\mathcal{U}} \leqslant M_u. \end{cases}$$

The corresponding constrained control space is defined by

$$\mathcal{U}_M = \{u \in \mathcal{U} : \|u\|_{\mathcal{U}} \leqslant M_u\}.$$

In the following we will identify $\mathcal{U}$ with $L^2(\Omega) \times (\mathbf{1} + L^2_\diamond(0,T))$, which is a closed (linear) subspace of $L^2(\Omega) \times L^2(0,T)$, and use the topology of $L^2(\Omega) \times L^2(0,T)$ on $\mathcal{U}$, i.e.,

$$\|u\|_{\mathcal{U}} = \|(v,\varphi)\|_{L^2(\Omega) \times L^2(0,T)},$$

cf. (3.4). Any ball

$$\mathcal{B}_r^{\mathcal{U}}(u_0) = \{u = v \cdot (\mathbf{1} + \varphi) \in \mathcal{U} : \|u - u_0\|_{\mathcal{U}} \leqslant r\}$$

with $u_0 = v_0 \cdot (\mathbf{1} + \varphi_0) \in \mathcal{U}$ is then convex, (weakly) closed and weakly compact in $\mathcal{U}$ in the sense that

$$\left\{(v, \varphi) \in L^2(\Omega) \times L^2_\diamond(0,T) : \|(v - v_0, \varphi - \varphi_0)\|_{L^2(\Omega)\times L^2(0,T)} \leqslant r\right\}$$

has these properties. Moreover, we can make use of Theorems 2.4 and 2.8, since by

$$\begin{aligned}\|u_1 - u_2\|_{L^2(Q)} &= \|v_1(\mathbf{1} + \varphi_1) - v_2(\mathbf{1} + \varphi_2)\|_{L^2(Q)} \\ &\leqslant \|v_1\|_{L^2(\Omega)}\|\varphi_1 - \varphi_2\|_{L^2(0,T)} + \|\varphi_2\|_{L^2(0,T)}\|v_1 - v_2\|_{L^2(\Omega)},\end{aligned}$$

smallness of $\|u_1 - u_2\|_{\mathcal{U}}$ implies smallness of $\|u_1 - u_2\|_{L^2(Q)}$.

Using these facts we can proceed as in section 3.1 to obtain existence of a minimizer and first order optimality conditions.

**Theorem 3.3.** *The reduced problem*

$$\min_{u \in \mathcal{U}_M} J(u, S(u))$$

*has a local minimizer* $u^* = v^* \cdot (\mathbf{1} + \varphi^*) \in \mathcal{U}_M$. *Furthermore, for any minimizer* $u^*$ *let* $y^* := S(u^*) \in \breve{\mathcal{Y}}$. *Then there exist* $p^* \in \breve{\mathcal{Y}}$ *satisfying* (3.1) *and* $\lambda^* \in \mathbb{R}$ *with* $\lambda^* \geqslant 0$ *such that*

$$\begin{cases} (\alpha + 2\lambda^*)v^* = \displaystyle\int_0^T \frac{\mathbf{1} + \varphi^*}{(1 + y^*)^2} p^* \, dt, \\ (\alpha + 2\lambda^*)\varphi^* = \displaystyle\int_\Omega \frac{v^*}{(1 + y^*)^2} p^* \, dx, \\ \lambda^*(\|v^*\|^2_{L^2(\Omega)} + \|\varphi^*\|^2_{L^2(0,T)} - M_u^2) = 0. \end{cases}$$

# 4 STATE CONSTRAINED OPTIMAL CONTROL

Here we use the pointwise state constraints

$$-y(t,x) \leqslant M_y < 1 \qquad \text{for all } (t,x) \in Q$$

to prevent the singularity of (1.1) at $y(t,x) = -1$. The unconstrained state space is defined as

$$\mathcal{Y} = \left\{y \in \tilde{\mathcal{Y}} : y(0) = 0, \ y_t(0) = 0\right\} \tag{4.1}$$

where

$$\tilde{\mathcal{Y}} \hookrightarrow W^{\sigma,\infty}(0,T; W^{\varepsilon,\infty}(\Omega)) \hookrightarrow C(0,T; C_0(\Omega)),$$

for $\sigma \in (0, 1 - \frac{n}{4})$ and $\varepsilon \in (0, 2(1 - \frac{n}{4} - \sigma))$, see Lemma 2.7 and note that $\tilde{\mathcal{Y}} \subseteq \breve{\mathcal{Y}}$.

For $M_y < 1$, we define the constrained state space as

$$\mathcal{Y}_M := \{y \in \mathcal{Y} : -y(t,x) \leqslant M_y \text{ for all } (t,x) \in Q\}.$$

We again start with cases (i) and (ii), i.e., $\mathcal{U} = L^2(X)$ with $X = \Omega$ in case (i) and $X = (0,T)$ in case (ii), and consider the state constrained optimal control problem

$$(\mathcal{P}_{sc}) \qquad \begin{cases} \min\limits_{y\in\mathcal{Y},u\in\mathcal{U}} J(u,y) = \frac{1}{2}\int_0 \|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ \text{s.t.} \quad y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y + \dfrac{\beta u}{(1+y)^2} = 0 \text{ in } Q, \\ \qquad y = \partial_\nu y = 0 \text{ on } (0,T)\times\partial\Omega, \quad y = y_t = 0 \text{ in } \{0\}\times\Omega, \\ \qquad -y \leqslant M_y \quad \text{in } Q. \end{cases}$$

We also define

$$G : \mathcal{U} \times \mathcal{Y}_M \to L^2(0,T;H^{-2}(\Omega))$$

by the weak form of the PDE with boundary and initial conditions on $y$ in ($\mathcal{P}_{sc}$).

**Theorem 4.1.** *There exists a minimizer* $(u^*, y^*) \in \mathcal{U} \times \mathcal{Y}_M$ *of* ($\mathcal{P}_{sc}$).

*Proof.* The set of feasible pairs $(u,y)$ satisfying the equality and inequality constraints is non-empty (take $(u,y) = (0,0)$). By the boundedness of J from below and the coercivity of the functional in $u$, we obtain the existence of a minimizing sequence whose $u$ component is bounded in $\mathcal{U}$. The equality constraint $G(u,y) = 0$ together with Lemma 2.1 implies that the $y$ component of the minimizing sequence is uniformly bounded in $\mathcal{Y}$. Hence, there exists a subsequence, denoted by $\{(u_n,y_n)\}_{n\in\mathbb{N}}$, that converges weakly in the reflexive space $\mathcal{U}\times\mathcal{Y}^s \supset \mathcal{U}\times\mathcal{Y}$ to $(u^*,y^*) \in \mathcal{U}\times\mathcal{Y}^s$, where

$$\mathcal{Y}^s := L^s(0,T;H^2_0(\Omega)) \cap W^{1,s}(0,T;L^2(\Omega)) \cap W^{2,s}(0,T;H^{-2}(\Omega))$$

with arbitrarily large $s \in (1,\infty)$. Due to the compact embedding of $\mathcal{Y}$ in $C(0,T;C_0(\Omega))$ and the continuity of $\Psi : r \mapsto \frac{1}{(1+r)^2}$, we have that a subsequence of $\frac{\beta}{(1+y_n)^2}$ converges strongly to $\frac{\beta}{(1+y^*)^2}$, with $\beta$ fixed, and that $y^*$ satisfies the inequality constraint. Thus, we can pass to the limit in (the weak formulation of) $G(u_n,y_n) = 0$ to obtain $G(u^*,y^*) = 0$, and hence by Lemma 2.1, $y^* \in \mathcal{Y}$. ☐

Let again $(u^*,y^*)$ denote a local minimizer of ($\mathcal{P}_{sc}$), and let $r$ denote the radius of the neighborhood in $\mathcal{U}\times\mathcal{Y}$ in which $(u^*,y^*)$ is optimal.

Since the state equation is not well-posed for every $u \in \mathcal{U}$, we cannot use a control-to-state-mapping for deriving optimality conditions. Furthermore, considering the optimization directly, i.e., without control-to-state mapping, poses difficulties since a regular point condition according to, e.g., [Alibert and Raymond 1998] would require existence of $(u_0,y_0) \in \mathcal{U}\times\mathcal{Y}_M$ such that $(y^*+y_0) \in \operatorname{int}\mathcal{Y}_M$ and

$$G_y(u^*,y^*)y_0 + G_u(u^*,y^*)(u_0-u^*) = 0,$$

where $G_y$ denotes the Fréchet derivative of G with respect to y. However, in the cases considered here, $u_0$ is too low-dimensional to guarantee existence of such a $y_0$ in case $y^* \notin \operatorname{int} \mathcal{Y}_M$. Therefore we follow the relaxation approach from [Bonnans and Casas 1989] combined with a localization technique as in [Casas and Tröltzsch 2002]: We introduce the independent variable $w$ in place of the nonlinearity $\frac{\beta u}{(1+y)^2}$ and consider the family of control problems

$$(\mathcal{P}_{sc,\varepsilon}) \qquad \begin{cases} \min\limits_{u\in\mathcal{U}, y\in\mathcal{Y}, w\in L^2(Q)} & J_\varepsilon(u, w, y) \\ \qquad\qquad \text{s. t.} & y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y + w = 0 \text{ in } Q \\ & y = \partial_\nu y = 0 \text{ on } (0, T) \times \partial\Omega, \quad y = y_t = 0 \text{ in } \{0\} \times \Omega, \\ & -y \leqslant M_y \quad \text{ in } Q, \end{cases}$$

where

$$J_\varepsilon(u, w, y) = J(u, y) + \frac{1}{2\varepsilon}\left\| w - \frac{\beta u}{(1+y)^2} \right\|^2_{L^2(Q)} + \frac{1}{2\delta}\|u - u^*\|^2_{L^2(\Omega)} + \frac{1}{2}\left\| w - \frac{\beta u^*}{(1+y^*)^2} \right\|^2_{L^2(Q)},$$

with

$$\delta < \frac{r^2}{2\max\{1, C\}^2 J(u^*, y^*)} \tag{4.2}$$

and C as in (2.3).

We now define

$$G_{\text{lin}} : L^2(Q) \times \mathcal{Y}_M \to L^2(0, T; H^{-2}(\Omega))$$

by the weak form of the PDE with boundary and initial conditions on y in $(\mathcal{P}_{sc,\varepsilon})$. Similarly to Theorem 4.1 one can then show existence of a minimizer $(u^*_\varepsilon, w^*_\varepsilon, y^*_\varepsilon)$.

**Theorem 4.2.** *There exists a minimizer* $(u^*_\varepsilon, w^*_\varepsilon, y^*_\varepsilon) \in \mathcal{U} \times L^2(Q) \times \mathcal{Y}_M$ *of* $(\mathcal{P}_{sc,\varepsilon})$.

Optimality conditions for the minimizers follow from a regular point condition. In the following, let

$$\mathcal{P} := C(0, T; L^2(\Omega)) \cap C^1(0, T; H^{-2}(\Omega))$$

and set again $Z = (0, T)$ in case $X = \Omega$ and $Z = \Omega$ in case $X = (0, T)$.

**Lemma 4.3.** *Let* $(u^*_\varepsilon, w^*_\varepsilon, y^*_\varepsilon) \in \mathcal{U} \times L^2(Q) \times \mathcal{Y}_M$ *be a local minimizer of* $(\mathcal{P}_{sc,\varepsilon})$. *Then there* *exist* $\mu_\varepsilon^* \in \mathcal{M}(0,T;\mathcal{M}(\Omega))$ *and* $p_\varepsilon^* \in \mathcal{P}$ *satisfying*

$$
(4.3)\qquad \begin{cases} p_{\varepsilon tt}^* - cp_{\varepsilon t}^* + dp_\varepsilon^* + \rho\Delta^2 p_\varepsilon^* - \eta\Delta p_\varepsilon^* = -(y_\varepsilon^* - y_d) - \mu_\varepsilon^* \\ \qquad\qquad + \dfrac{1}{\varepsilon}\dfrac{2\beta u_\varepsilon^*}{(1+y_\varepsilon^*)^3}\left(w_\varepsilon^* - \dfrac{\beta u_\varepsilon^*}{(1+y_\varepsilon^*)^2}\right), \\ \alpha u_\varepsilon^* + \dfrac{1}{\delta}(u_\varepsilon^* - u^*) = \dfrac{1}{\varepsilon}\displaystyle\int_Z \dfrac{\beta}{(1+y_\varepsilon^*)^2}\left(w_\varepsilon^* - \dfrac{\beta u_\varepsilon^*}{(1+y_\varepsilon^*)^2}\right)ds, \\ \dfrac{1}{\varepsilon}\left(w_\varepsilon^* - \dfrac{\beta u_\varepsilon^*}{(1+y_\varepsilon^*)^2}\right) = -\left[w_\varepsilon^* - \dfrac{\beta u^*}{(1+y^*)^2}\right] + p_\varepsilon^*, \\ \langle \mu_\varepsilon^*, y - y_\varepsilon^*\rangle_{\mathcal{M},C} \leqslant 0 \quad \textit{for all } y \in \mathcal{Y}_M. \end{cases}
$$

*Proof.* We wish to apply [Alibert and Raymond 1998, Theorem 2.1]. Due to linearity of $G_{\text{lin}}$, the regular point condition

$$G_{\text{lin},y}(w_\varepsilon^*, y_\varepsilon^*)y_0 + G_{\text{lin},w}(w_\varepsilon^*, y_\varepsilon^*)(w_0 - w_\varepsilon^*) = 0$$

with $(w_0, y_0) \in L^2(Q) \times \mathcal{Y}_M$ such that $(y_\varepsilon^* + y_0) \in \operatorname{int} \mathcal{Y}_M$ can be satisfied by setting

$$y_0 = -\tilde{C}y_\varepsilon^* \in \mathcal{Y}_M, \qquad w_0 = (1-\tilde{C})w_\varepsilon^* \in L^2(Q),$$

with

$$\tilde{C} = \frac{M_y}{2\sup_{(t,x)\in Q} y_\varepsilon^*(t,x)}.$$

It remains to show surjectivity of the linearized equality constraint. An inspection of Case 3 in the proof of Theorem 2.2 and of Lemma 2.3 a) in [Alibert and Raymond 1998] shows that by linearity of $G_{\text{lin},y}$, it suffices to assume surjectivity of $G_{\text{lin},y} : \hat{\mathcal{Y}} \to \hat{\mathcal{V}}$, where $\hat{\mathcal{Y}}$ is dense in a superset of $\mathcal{Y}$. Setting

$$\hat{\mathcal{Y}} := \left\{ y \in \mathcal{Y} : G_{\text{lin},y}(w_\varepsilon^*, y_\varepsilon^*) \in L^2(Q) \right\},$$

we have by definition that $G_{\text{lin},y}(w_\varepsilon^*, y_\varepsilon^*) : \hat{\mathcal{Y}} \to \hat{\mathcal{V}} := L^2(Q)$ is surjective. Furthermore, by density of $L^2(Q)$ in $L^2(0,T;H^{-2}(\Omega))$ and the stability estimate (2.6) of Corollary 2.3, $\hat{\mathcal{Y}}$ is dense in $\check{\mathcal{Y}} \supseteq \mathcal{Y}$. We thus obtain existence of an adjoint state in $\hat{\mathcal{V}}^* = L^2(Q)$. Together with Corollary 2.2, this yields the desired first-order optimality conditions. □

Similarly to [Bonnans and Casas 1989, Lemma 3], one can show convergence of the regularized solutions.

**Lemma 4.4.** *Let* $(u^*, y^*)$ *be a local minimizer of* ($\mathcal{P}_{sc}$)*, and let* $\{(u_\varepsilon^*, w_\varepsilon^*, y_\varepsilon^*)\}_{\varepsilon>0}$ *be a family of global minimizers of* ($\mathcal{P}_{sc,\varepsilon}$)*. Then as* $\varepsilon \to 0$*,*

$$u_\varepsilon^* \to u^* \text{ in } \mathcal{U}, \qquad w_\varepsilon^* \to \frac{\beta u^*}{(1+y^*)^2} \text{ in } L^2(Q), \qquad y_\varepsilon^* \to y^* \text{ in } \mathcal{Y}.$$

*Proof.* By minimality of $(u_\varepsilon^*, w_\varepsilon^*, y_\varepsilon^*)$ we have

$$J_\varepsilon(u_\varepsilon^*, w_\varepsilon^*, y_\varepsilon^*) \leqslant J_\varepsilon\left(u^*, \frac{\beta u^*}{(1+y^*)^2}, y^*\right) = J(u^*, y^*). \tag{4.4}$$

Hence $\{(u_\varepsilon^*, w_\varepsilon^*)\}_{\varepsilon>0}$ is bounded in $\mathcal{U} \times L^2(Q)$; and thus by $G_{\text{lin}}(w_\varepsilon^*, y_\varepsilon^*) = 0$ and Lemma 2.1, $\{(u_\varepsilon^*, w_\varepsilon^*, y_\varepsilon^*)\}_{\varepsilon>0}$ is bounded in $\mathcal{U} \times L^2(Q) \times \mathcal{Y}$. Moreover, due to the penalty term $\frac{1}{2\delta}\|u - u^*\|_{\mathcal{U}}^2$ and our choice of $\delta$ according to (4.2) with (2.3), we have $(u_\varepsilon^*, y_\varepsilon^*) \in \mathcal{B}_r^{\mathcal{U}}(u^*) \times \mathcal{B}_r^{\mathcal{Y}}(y^*)$ for all $\varepsilon > 0$. By compactness of the embedding $\mathcal{Y} \hookrightarrow C(0, T; C_0(\Omega))$, we can therefore extract from any subsequence of $\{(u_\varepsilon^*, w_\varepsilon^*, y_\varepsilon^*)\}_{\varepsilon>0}$ a further subsequence (denoted by the same symbols) such that as $\varepsilon \to 0$,

$$\begin{aligned} &u_\varepsilon^* \rightharpoonup \hat{u} \text{ in } \mathcal{U}, \qquad && w_\varepsilon^* \rightharpoonup \hat{w} \text{ in } L^2(Q), \\ &y_\varepsilon^* \rightharpoonup \hat{y} \text{ in } \mathcal{Y}, \qquad && y_\varepsilon^* \to \hat{y} \text{ in } C(0, T; C_0(\Omega)) \end{aligned} \tag{4.5}$$

for some $(\hat{u}, \hat{w}, \hat{y}) \in \mathcal{B}_r^{\mathcal{U}}(u^*) \times L^2(Q) \times (\mathcal{B}_r^{\mathcal{Y}}(y^*) \cap \mathcal{Y}_M)$. Here we have used weak closedness of $\mathcal{B}_r^{\mathcal{U}}(u^*)$ and $\mathcal{B}_r^{\mathcal{Y}}(y^*) \cap \mathcal{Y}_M$.

Taking now the weak limit in the PDE, we see that $G_{\text{lin}}(\hat{w}, \hat{y}) = 0$. The uniform boundedness (4.4) and the penalty term with factor $\frac{1}{\varepsilon}$ in $J_\varepsilon$ imply that

$$\left\| w_\varepsilon^* - \frac{\beta u_\varepsilon^*}{(1+y_\varepsilon^*)^2} \right\|_{L^2(Q)}^2 \to 0,$$

hence we deduce from (4.5) and the weak lower semicontinuity of the norm that

$$\hat{w} - \frac{\beta \hat{u}}{(1+\hat{y})^2} = 0. \tag{4.6}$$

Together with $G_{\text{lin}}(\hat{w}, \hat{y}) = 0$, this implies that $G(\hat{u}, \hat{y}) = 0$. Thus, $(\hat{u}, \hat{y})$ is admissible for $(\mathcal{P}_{sc})$ and contained in $\mathcal{B}_r^{\mathcal{U}}(u^*) \times (\mathcal{B}_r^{\mathcal{Y}}(y^*) \cap \mathcal{Y}_M)$. We can now use minimality of $(u^*, y^*)$ in $\mathcal{B}_r^{\mathcal{U}}(u^*) \times \mathcal{B}_r^{\mathcal{Y}}(y^*)$ with respect to $(\mathcal{P}_{sc})$, relation (4.4), and weak lower semicontinuity of J and the norms to conclude

(4.7)

$$\begin{aligned} J(\hat{u}, \hat{y}) \geqslant J(u^*, y^*) &\geqslant \limsup_{\varepsilon\to0} J_\varepsilon(u_\varepsilon^*, w_\varepsilon^*, y_\varepsilon^*) \\ &\geqslant \limsup_{\varepsilon\to0}\Big( J(u_\varepsilon^*, y_\varepsilon^*) + \frac{1}{2\delta}\|u_\varepsilon^* - u^*\|_{\mathcal{U}}^2 + \frac{1}{2}\left\| w_\varepsilon^* - \frac{\beta u^*}{(1+y^*)^2}\right\|_{L^2(Q)}^2 \Big) \\ &\geqslant \liminf_{\varepsilon\to0} J(u_\varepsilon^*, y_\varepsilon^*) \\ &\quad + \liminf_{\varepsilon\to0}\Big( \frac{1}{2\delta}\|u_\varepsilon^* - u^*\|_{\mathcal{U}}^2 + \frac{1}{2}\left\| w_\varepsilon^* - \frac{\beta u^*}{(1+y^*)^2}\right\|_{L^2(Q)}^2 \Big) \\ &\geqslant J(\hat{u}, \hat{y}) + \frac{1}{2\delta}\|\hat{u} - u^*\|_{\mathcal{U}}^2 + \frac{1}{2}\left\| \hat{w} - \frac{\beta u^*}{(1+y^*)^2}\right\|_{L^2(Q)}^2 . \end{aligned}$$

This implies that $\hat{u} = u^*$ and – by (4.6) – that $\frac{\beta\hat{u}}{(1+\hat{y})^2} = \frac{\beta u^*}{(1+y^*)^2}$. The uniqueness claim in Lemma 2.1 then yields $\hat{y} = y^*$.

Strong convergence of the $u$ and $w$ components along a subsequence immediately follows from (4.7), and hence the stability estimate of Lemma 2.1 yields strong convergence of the $y$ part. Using a subsequence-subsequence argument, we arrive at the assertion. □

As in [Bonnans and Casas 1989, Theorem 2], we can thus take the limit as $\varepsilon \to 0$ in (4.3) to derive necessary optimality conditions for ($\mathcal{P}_{sc}$).

**Theorem 4.5.** *Let* $(u^*, y^*) \in \mathcal{U} \times \mathcal{Y}_M$ *be a local minimizer of* ($\mathcal{P}_{sc}$). *Then there exist* $\mu^* \in \mathcal{M}(0,T;\mathcal{M}(\Omega))$ *and* $p^* \in \mathcal{P}$ *satisfying*

$$
(4.8)\qquad \begin{cases} p^*_{tt} - cp^*_t + dp^* + \rho\Delta^2 p^* - \eta\Delta p^* - \dfrac{2\beta u^*}{(1+y^*)^3}p^* = -\gamma(y^* - y_d) - \mu^* \\ \gamma\alpha u^* = \displaystyle\int_Z \frac{\beta}{(1+y^*)^2}p^*\,ds, \\ \langle \mu^*, y - y^*\rangle_{\mathcal{M},C} \leqslant 0 \quad \text{for all } y \in \mathcal{Y}_M, \end{cases}
$$

*where* $\gamma \in \{0,1\}$ *and* $\gamma + \|p^*\|_{\mathcal{P}} > 0$.

*Proof.* Eliminating the term $\frac{1}{\varepsilon}\left(w^*_\varepsilon - \frac{\beta u^*}{(1+y^*)^2}\right)$ by means of the last equation in (4.3), we obtain

$$
(4.9)\qquad \begin{cases} p^*_{\varepsilon tt} - cp^*_{\varepsilon t} + dp^*_\varepsilon + \rho\Delta^2 p^*_\varepsilon - \eta\Delta p^*_\varepsilon = -(y^*_\varepsilon - y_d) - \mu^*_\varepsilon \\ \qquad + \dfrac{2\beta u^*_\varepsilon}{(1+y^*_\varepsilon)^3}\left(-\left[w^*_\varepsilon - \dfrac{\beta u^*}{(1+y^*)^2}\right] + p^*_\varepsilon\right), \\ \alpha u^*_\varepsilon + \dfrac{1}{\delta}(u^*_\varepsilon - u^*) = \displaystyle\int_Z \frac{\beta}{(1+y^*_\varepsilon)^2}\left(-\left[w^*_\varepsilon - \frac{\beta u^*}{(1+y^*)^2}\right] + p^*_\varepsilon\right)\,ds, \\ \langle \mu^*_\varepsilon, y - y^*_\varepsilon\rangle_{\mathcal{M},C} \leqslant 0 \quad \text{for all } y \in \mathcal{Y}_M. \end{cases}
$$

If $\{p^*_\varepsilon, \mu^*_\varepsilon\}_{\varepsilon>0}$ has a bounded subsequence in $\mathcal{P} \times \mathcal{M}(0,T;\mathcal{M}(\Omega))$, we can take (weak) limits in (4.9) to obtain (4.8) with $\gamma = 1$.

Otherwise,

$$
\{r_\varepsilon\}_{\varepsilon>0} := \{\|p^*_\varepsilon\|_{\mathcal{P}} + \|\mu^*_\varepsilon\|_{\mathcal{M}(0,T;\mathcal{M}(\Omega))}\}_{\varepsilon>0} \to \infty
$$

as $\varepsilon \to 0$, and we can divide $\{p^*_\varepsilon, \mu^*_\varepsilon\}_{\varepsilon>0}$ by $r_\varepsilon$ to get a bounded sequence $\{\underline{p}^*_\varepsilon, \underline{\mu}^*_\varepsilon\}_{\varepsilon>0} := \{\frac{p^*_\varepsilon}{r_\varepsilon}, \frac{\mu^*_\varepsilon}{r_\varepsilon}\}_{\varepsilon>0}$. Inserting this into (4.9) yields

$$
\begin{cases} \underline{p}^*_{\varepsilon tt} - c\underline{p}^*_{\varepsilon t} + d\underline{p}^*_\varepsilon + \rho\Delta^2 \underline{p}^*_\varepsilon - \eta\Delta \underline{p}^*_\varepsilon = -\dfrac{1}{r_\varepsilon}(y^*_\varepsilon - y_d) - \underline{\mu}^*_\varepsilon \\ \qquad + \dfrac{2\beta u^*_\varepsilon}{(1+y^*_\varepsilon)^3}\left(-\dfrac{1}{r_\varepsilon}\left[w^*_\varepsilon - \dfrac{\beta u^*}{(1+y^*)^2}\right] + \underline{p}^*_\varepsilon\right), \\ \dfrac{1}{r_\varepsilon}(\alpha u^*_\varepsilon + \dfrac{1}{\delta}(u^*_\varepsilon - u^*)) = \displaystyle\int_Z \frac{\beta}{(1+y^*_\varepsilon)^2}\left(-\frac{1}{r_\varepsilon}\left[w^*_\varepsilon - \frac{\beta u^*}{(1+y^*)^2}\right] + \underline{p}^*_\varepsilon\right)\,ds, \\ \left\langle \underline{\mu}^*_\varepsilon, y - y^*_\varepsilon\right\rangle_{\mathcal{M},C} \leqslant 0 \quad \text{for all } y \in \mathcal{Y}_M. \end{cases}
$$

Taking (weak) limits, we then obtain (4.8) with $\gamma = 0$. To show that in this case $p^* \neq 0$, we assume that $p^* = 0$ and note that the first line of (4.8) then yields $\underline{\mu}_\varepsilon \rightharpoonup \mu^* = 0$. By [Bonnans and Casas 1989, Lemma 1], we therefore have $\|\underline{\mu}_\varepsilon\|_{\mathcal{M}(0,T;\mathcal{M}(\Omega))} \to 0$, which implies

$$\|\underline{p}_\varepsilon\|_{\mathcal{P}} = \frac{\|p^*_\varepsilon\|_{\mathcal{P}}}{\|p^*_\varepsilon\|_{\mathcal{P}} + \|\mu^*_\varepsilon\|_{\mathcal{M}(0,T;\mathcal{M}(\Omega))}} = \frac{\|\underline{p}^*_\varepsilon\|_{\mathcal{P}}}{\|\underline{p}^*_\varepsilon\|_{\mathcal{P}} + \|\underline{\mu}^*_\varepsilon\|_{\mathcal{M}(0,T;\mathcal{M}(\Omega))}} \to 1,$$

in contradiction with $\underline{p}_\varepsilon \to p^* = 0$. □

## 4.2 CONTROL BY DIELECTRIC PROPERTIES AND VOLTAGE

Finally, we consider case (iii), i.e., we use the control space $\mathcal{U}$ defined by (3.2) and $\|u\|_{\mathcal{U}}$ defined by (3.4). The corresponding state constrained control problem is

$$(\mathcal{P}^2_{sc}) \qquad \begin{cases} \min_{y\in\mathcal{Y},u\in\mathcal{U}} J(u,y) = \frac{1}{2}\int_0 \|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{\mathcal{U}} \\ \text{s. t.} \quad y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y + \frac{u(t,x)}{(1+y)^2} = 0 \text{ in } Q \\ \qquad y = \partial_\nu y = 0 \text{ on } (0,T)\times\partial\Omega, \quad y = y_t = 0 \text{ in } \{0\}\times\Omega, \\ \qquad -y \leqslant M_y \text{ in } Q. \end{cases}$$

Proceeding as in section 4.1, we can obtain existence of a minimizer and first order optimality conditions.

**Theorem 4.6.** *There exists a local minimizer* $(u^*, y^*) \in \mathcal{U}\times\mathcal{Y}_M$ *of* $(\mathcal{P}^2_{sc})$ *with* $u^* = \nu^*\cdot(\mathbf{1}+\varphi^*)$. *Furthermore, there exist* $\mu^* \in \mathcal{M}(0,T;\mathcal{M}(\Omega))$ *and* $p^* \in \mathcal{P}$ *satisfying*

$$(4.10) \qquad \begin{cases} p^*_{tt} - cp^*_t + dp^* + \rho\Delta^2 p^* - \eta\Delta p^* - \frac{2\beta u^*}{(1+y^*)^3}p^* = -\gamma(y^* - y_d) - \mu^* \\ \gamma\alpha\nu^* = \int_0^T \frac{\mathbf{1}+\varphi^*}{(1+y^*)^2}p^*\,dt, \\ \gamma\alpha\varphi^* = \int_\Omega \frac{\nu^*}{(1+y^*)^2}p^*\,dx, \\ \langle\mu^*, y - y^*\rangle_{\mathcal{M},C} \leqslant 0 \quad \textit{for all } y \in \mathcal{Y}_M, \end{cases}$$

*where* $\gamma \in \{0,1\}$ *and* $\gamma + \|p^*\|_{\mathcal{P}} > 0$.

The proof of this theorem closely follows the proofs in section 4.1, additionally making use of the results on $\mathcal{U}$ derived in section 3.2 (especially the weak closedness and weak compactness of balls $\mathcal{B}^{\mathcal{U}}_r(u^*)$ with respect to $\|\cdot\|_{\mathcal{U}}$). Note that first order optimality conditions in qualified form, i.e., with $\gamma = 1$ in (4.10), would require using $\|\cdot\|_{L^2(Q)}$ as a penalty term. However, the term $\|\cdot\|_{\mathcal{U}}$ is essential for proving existence of a minimizer, since $\|\cdot\|_{L^2(Q)}$ would not yield a $\mathcal{U}$-coercive cost functional, cf. (3.5). (For the control constrained case, $\mathcal{U}$-coercivity was not required, but the low dimensionality of the control still prevents optimality conditions in qualified form.)

# 5 CONCLUSIONS

Using techniques from [Bonnans and Casas 1989] and [Casas and Tröltzsch 2002], we have extended our results on control and state constrained optimal control of static MEMS [Clason and Kaltenbacher 2011] to the time-dependent case, considering as practically relevant controls the dielectric properties of the membrane and/or the applied voltage. The approach followed in this work is able to deal with the relatively low dimensionality of the control and thus promises to be useful for boundary control of singular PDEs (such as the Westervelt equation) arising from high intensity ultrasound applications [Clason, Kaltenbacher, and Veljovic 2009]. Future work in the context of MEMS applications is concerned with the design and control of devices containing magnetostrictive layers, which leads to optimal control problems for PDEs with hysteresis properties.

## ACKNOWLEDGMENTS

The authors wish to thank Ira Neitzel, Christian Meyer, and Daniel Wachsmuth for pointing them to the localization technique from [Casas and Tröltzsch 2002].

The work of the first-named author was supported by the Austrian Science Fund (FWF) under grant SFB F32 (SFB "Mathematical Optimization and Applications in Biomedical Sciences"). The second-named author would like to thank the Austrian Science Fund (FWF) for financial support within the grant P24970 ("Mathematics of Nonlinear Acoustics: Analysis, Numerics, and Optimization").

## REFERENCES

Alibert, J.-J. and Raymond, J.-P. (1998). *A Lagrange multiplier theorem for control problems with state constraints*. Numer. Funct. Anal. Optim. 19.7-8, pp. 697–704. DOI: 10.1080/01630569808816854.

Bonnans, J. F. and Casas, E. (1989). *Optimal control of semilinear multistate systems with state constraints*. SIAM J. Contr. Opt. 27.2, pp. 446–455. DOI: 10.1137/0327023.

Bonnans, J. F. and Shapiro, A. (2000). *Perturbation analysis of optimization problems*. Springer-Verlag, New York.

Casas, E. and Tröltzsch, F. (2002). *Error estimates for the finite-element approximation of a semilinear elliptic control problem*. Control and Cybernetics 31, pp. 695–712.

Cassani, D., Kaltenbacher, B., and Lorenzi, A. (2009). *Direct and inverse problems related to MEMS devices*. Inverse Problems 25. DOI: 10.1088/0266-5611/25/10/105002.

Clason, C., Kaltenbacher, B., and Veljovic, S. (2009). *Boundary optimal control of the Westervelt and the Kuznetsov equation*. Journal of Mathematical Analysis and Applications 356, pp. 738–751. DOI: 10.1016/j.jmaa.2009.03.043.

Clason, C. and Kaltenbacher, B. (2011). *On the use of state constraints in optimal control of singular PDEs*. Tech. rep. 2011–030. SFB “Mathematical Optimization and Applications in Biomedical Sciences”. URL: http://math.uni-graz.at/mobis/publications/SFB-Report-2011-030.pdf.

Tröltzsch, F. (2010). *Optimal control of partial differential equations*. Vol. 112. Theory, methods and applications, Translated from the 2005 German original by Jürgen Sprekels. American Mathematical Society, Providence, RI.

Vishik, M. and Fursikov, A. (1988). *Mathematical Problems in Statistical Hydromechanics*. Kluwer, Boston.