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POD A-POSTERIORI ERROR BASED INEXACT SQP METHOD FOR BILINEAR ELLIPTIC OPTIMAL CONTROL PROBLEMS

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M. KAHLBACHER AND S. VOLKWEIN

Abstract. An optimal control problem governed by a bilinear elliptic equation is considered. This problem is solved by the sequential quadratic programming (SQP) method in an infinite-dimensional framework, where in each level of the iterative method the solution of a linear-quadratic subproblem is computed. For numerical realization this subproblem is discretized by a Galerkin projection using proper orthogonal decomposition (POD). Thus, an approximate (inexact) solution of the subproblem is computed. Based on a POD a-posteriori error estimator developed by Tröltzsch and Volkwein (Comput. Optim. Appl., 44:83-115, 2009) the difference of the suboptimal to the (unknown) optimal solution of the linear-quadratic subproblem is estimated. Hence, the inexctness of the discrete solution is controlled in such a way that locally superlinear or even quadratic rate of convergence of the SQP is ensured. Numerical examples illustrate the efficiency for the proposed approach.

1. Introduction

Optimal control problems governed by partial differential equations (PDEs) can often be formulated as an infinite-dimensional optimization problem in the following form (see, e.g., in [16, 23]):

$$\min_{x\in X} J(x) \quad \text{subject to (s.t.)} \quad e(x) = 0. \tag{1.1}$$

The mapping $J : X \to \mathbb{R}$ denotes the cost functional with a Banach space X. The operator $e : X \to Y'$ describes the partial differential equations with a Banach space Y and its dual Y'. The Lagrangian for (1.1) is given by

$$L(x, p) = J(x) + \langle e(x), p\rangle_{Y',Y} \quad \text{for } (x, p) \in X \times Y,$$

where $\langle \cdot\,, \cdot\rangle_{Y',Y}$ denotes the dual pairing between Y' and Y. If J and e are twice continuously Fréchet-differentiable, second-order methods can be applied to solve (1.1) numerically. One favorite method is the sequential quadratic programming (SQP) method, where in each level of the iteration the linear-quadratic programming problem

$$\begin{cases} \min\limits_{x\in X} L_x(x^k, p^k)x + \dfrac{1}{2}\, L_{xx}(x^k, p^k)(x, x) \\ \text{s.t. } e(x^k) + e'(x^k)x = 0 \end{cases} \tag{1.2}$$

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is solved. The solution $\bar{x}$ to (1.2) is given by the solution to the Karush-Kuhn-Tucker (KKT) system

$$A_k \bar{z} = b_k \quad \text{in } X' \times Y' \tag{1.3}$$

with

$$A_k = \begin{pmatrix} L_{xx}(x^k, p^k) & e'(x^k)^* \\ e'(x^k) & 0 \end{pmatrix}, \; \bar{z} = \begin{pmatrix} \bar{x} \\ \bar{p} \end{pmatrix}, \; b_k = -\begin{pmatrix} L_x(x^k, p^k) \\ e(x^k) \end{pmatrix}.$$

Here, $X' \times Y'$ is identified with the dual of of $X \times Y$, $e'(x^k)^* : Y \to X'$ is the dual operator of the Fréchet derivative $e'(x^k) : X \to Y'$ and L_x (L_{xx}) stands for the first (second) Fréchet derivative of the Lagrangian with respect to x.

In the context of PDE constrained optimization (1.3) has to be discretized. Often that leads to very large scale linear systems. Therefore, different techniques of model order reduction methods have been developed to approximate (1.3) by smaller ones that are tractable with less effort. We apply the method of proper orthogonal decomposition (POD), which is based on projecting the system onto subspaces consisting of $\ell \geq 1$ POD basis elements that contain characteristics of the expected solution; see, e.g., [18, 35] and [2, 5, 6, 27, 31]. This is in contrast to, e.g., finite element techniques, where the elements of the subspaces are uncorrelated to the physical properties of the system that they approximate. The discretization of (1.3) leads to a discrete solution which solves (1.3) inexactly. Thus, we obtain an inexact version of the SQP method; see, e.g., [19]. Utilizing the convergence theory for inexact Newton methods (see, e.g., [9]) the inexactness can be controlled in such a way that a local superlinear or even local quadratic rate of convergence can be ensured.

Utilizing ℓ POD basis functions for the Galerkin projection of (1.3) we arrive at a finite- and low-dimensional linear system

$$A_k^\ell \bar{z}^\ell = b_k^\ell \quad \text{in } \mathbb{R}^n \tag{1.4}$$

with an integer $n = n(\ell)$ depending on the number ℓ of POD basis functions. We prolongate the solution $\bar{z}^\ell$ to (1.4) into the space $X \times Y$ by applying a linear operator $\mathcal{I} : \mathbb{R}^n \to X \times Y$. Convergence of the SQP method can be ensured provided the starting value (x^0, p^0) is appropriately chosen and

$$\|A_k(\mathcal{I}\bar{z}^\ell) - b_k\|_{X' \times Y'} = O\big(\|L'(x^k, p^k)\|^q_{X' \times Y'}\big) \tag{1.5}$$

with $q \in [1, 2]$. Here, L' denotes the Fréchet derivative of the Lagrangian with respect to (x, p). If $q = 1$ holds, then the iterates converge linearly, if $q \in (1, 2)$ is satisfied, the rate of convergence is superlinear, and for $q = 2$ we obtain quadratic rate of convergence. To achieve (1.5) we apply a POD a-posteriori error estimator (see [37]) which is derived for linear-quadratic programming problems. Utilizing the quadratic convergence of the SQP method in function spaces we ensure convergence of the iterates — computed by the POD suboptimal control approach — to the solution of the nonlinear optimization problem (1.1)

For the POD method (and also for other model reduction methods [3] like the reduced-basis method [14, 20, 30] and balanced truncation [7, 26, 33, 42]) no reliable a-priori error analysis for nonlinear optimal control problems. Unless its snapshots are generating a sufficiently rich state space, it is not a-priorily clear how far the optimal solution of the POD problem is from the exact one. A-priori error estimates for POD Galerkin approximations of linear-quadratic optimal control problems were derived in [17], where the POD basis was computed with the *knowledge* of the optimal solution. In [37] the main focus was on a POD a-posteriori analysis for linear-quadratic optimal control problems. It was deduced how far the suboptimal control, computed on the basis of the POD model, is from the (*unknown*) exact one. We use this idea for nonlinear optimal control problems so that we are able to compensate for the lack of a priori analysis for POD methods.

In our work we apply the technique developed in [37] to control the discretization error of the POD Galerkin approximation in each level of the SQP method. The approach is illustrated for an optimal control problem governed by a bilinear elliptic partial differential equation. Within the inexact SQP method we tune the number ℓ of basis functions for the POD Galerkin approximation to ensure the locally fast convergence of the algorithm. Thus, in contrast to [25] the POD basis will be fixed during the numerical algorithm. Only the number of the utilized POD ansatz functions is increased, if necessary. We refer to the papers [39, 22], where also bilinear optimal control problems are considered. Let us mention that the presented approach can also be used for nonlinear parabolic equations as well as for for reduced-basis approximations; see [36].

In this paper we also derive a-priori error estimates for POD Galerkin approximations of the bilinear state equation. In contrast to [21] the parameter dependence is not in the differential operator, but in the right-hand side.

The paper is organized in the following manner: In Section 2 the optimal control problem is introduced and optiality conditions are discussed. The SQP method is formulated in Section 3. In Section 4 we turm to the POD discretization of the linear-quadratic subproblem. Two numerical examples are presented in Section 5. Finally, most of the proofs are given in the Appendix.

2. Optimal control of the bilinear equation

In this section we introduce the optimal control problem. In Section 2.1 we discuss the underlying state equation. The optimal control problem is investigated in Section 2.2 and optimality conditions are presented in Section 2.3.

2.1. **The state equation.** Throughout we suppose that $\Omega \subset \mathbb{R}^d$ is a bounded, open and connected set with $d \in \{1, 2, 3\}$ satisfying a uniform cone condition (that is, there exists a fixed cone K_Ω such that each $s \in \partial\Omega = \Gamma$ on the boundary is the vertex of a cone $K_\Omega(s) \subset \overline{\Omega}$ congruent to K_Ω). Let $L^2(\Omega)$ denote the Lebesgue space of all measurable and square integrable functions on Ω. For brevity, we set $V = H^1(\Omega)$ and refer to [12], for instance, for more details on Lebesgue and Sobolev spaces.

The bilinear elliptic equation is given by

$$-\nabla \cdot \big(\kappa(\mathbf{x})\nabla y(\mathbf{x})\big) + u(\mathbf{x})y(\mathbf{x}) = f(\mathbf{x}) \qquad \text{f.a.a. } \mathbf{x} \in \Omega, \tag{2.1a}$$

$$\kappa(\mathbf{s})\,\frac{\partial y}{\partial n}(\mathbf{s}) + \eta y(\mathbf{s}) = g(\mathbf{s}) \qquad \text{f.a.a. } \mathbf{s} \in \Gamma, \tag{2.1b}$$

where κ denotes the diffusion parameter, η is a positive scalar η and 'f.a.a.' stands for 'for almost all'. The right-hand side satisfies $f \in L^2(\Omega)$ and the boundary data g belongs to $L^2(\Gamma)$. Let Ω be divided in finitely many subdomains Ω_i, $i = 1, \ldots, n_\Omega$, satisfying

$$\overline{\Omega} = \bigcup_{i=1}^{n_\Omega} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j. \tag{2.2}$$

We assume that the diffusion coefficient $\kappa(\mathbf{x}) \geq \underline{\kappa} > 0$ f.a.a. $\mathbf{x} \in \Omega$ to be piecewise constant and the control variable u to be of the form

$$u(\mathbf{x}) = \sum_{i=1}^{N} u_i b_i(\mathbf{x}) \quad \text{f.a.a. } \mathbf{x} \in \Omega,$$

where $b_1, \dots, b_N$ are linearly independent in $L^2(\Omega)$. For instance, the b_i's can be step functions satisfying $b_i \equiv 1$ on Ω_i and $b_i \equiv 0$ on $\Omega \setminus \Omega_i$ for $i = 1, \dots, N$ and $N = n_\Omega$. We define the finite-dimensional control space

$$U = \operatorname{span}\big\{b_1, \dots, b_N\big\} \subset L^2(\Omega)$$

supplied with the topology in $L^2(\Omega)$. Note that $\dim U = N$.

We introduce the Hilbert space

$$X = V \times U$$

endowed with the common product topology. To write the elliptic differential equation (2.1) in a compact form we define the bilinear operator $e : X \to V'$ by

$$\langle e(x), \varphi\rangle_{V',V} = \int_\Omega \kappa \nabla y \cdot \nabla\varphi + \big(uy - f\big)\varphi \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma (y - g)\varphi \,\mathrm{d}\mathbf{s}$$

for $x = (y, u) \in X$ and $\varphi \in V$. Moreover, $\langle \cdot\,, \cdot\rangle_{V',V}$ denotes the dual pairing associated with V and its dual V'. Since Ω satisfies a uniform interior cone condition, V is continuously embedded into $L^6(\Omega)$ for $d \leq 3$; see, e.g., [12, p. 270]. Moreover, $u \in U$ holds. Thus, the operator e and its Fréchet-derivatives are well-defined. In particular, at $x = (u, w) \in X$ we have

$$\begin{aligned}
\langle e'(x)x_\delta, \varphi\rangle_{V',V} &= \int_\Omega \kappa \nabla y_\delta \cdot \nabla\varphi + \big(u_\delta y + u y_\delta\big)\varphi \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma y_\delta \varphi \,\mathrm{d}\mathbf{s},\\
\langle e''(x)(x_\delta, \tilde{x}_\delta), \varphi\rangle_{V',V} &= \int_\Omega \big(u_\delta \tilde{y}_\delta + \tilde{u}_\delta y_\delta\big)\varphi \,\mathrm{d}\mathbf{x}
\end{aligned}$$

in directions $x_\delta = (y_\delta, u_\delta)$, $\tilde{x}_\delta = (\tilde{y}_\delta, \tilde{u}_\delta) \in X$ and for $\varphi \in V$. Due to the bilinear structure of the mapping e the mapping $x \mapsto e''(x)$ does not depend on $x \in X$ so that it is Lipschitz-continuous on X.

The next proposition ensures existence and uniqueness of a weak solution to the state equation for arbitrary non-negative $u \in U$. For a proof we refer to [13, Theorem 2.1].

Proposition 2.1. *Let Ω be a bounded connected open set in $\mathbb{R}^d$ with smooth boundary, i.e., Γ is a variety of dimension $d-1$ of class C^∞ and Ω lies locally on one side of Γ. Then, for every $u \in U$ with $u \geq 0$ almost everywhere (a.e.) in Ω there exists a unique solution $y = y(u) \in V$ of the equation $e(y, u) = 0$. Moreover, y satisfies the estimate*

$$\|y\|_V \leq C\big(1 + \|u\|_{L^2(\Omega)}\big)$$

for some constant $C > 0$ depending on f and g, but not on u.

The following result ensures a standard constraint qualification that is needed to ensure the existence of Lagrange multipliers. For the proof we refer to [13, Theorem 2.2]

Proposition 2.2. *Let Ω satisfy the conditions of Theorem 2.1. Then, for every $x = (y, u) \in X$ with $u \geq 0$ in Ω a.e., the Fréchet derivative $e_y(x) : V \to V'$ of the operator e with respect to y is bijective. In particular, $e'(x)$ is surjective, and there exists a constant $C_{ker} > 0$ such that*

$$\|y_\delta\|_V \leq C_{ker} \|u_\delta\|_{L^2(\Omega)} \quad \textit{for any } (y_\delta, u_\delta) \in \ker e'(x) \subset X.$$

2.2. **The optimal control problem.** Motivated by Propositions 2.1 and 2.2 we define the set of admissible nonnegative control functions by

$$U_{ad} = \big\{ u \in U \,\big|\, u(\mathbf{x}) \geq u_a \text{ f.a.a. } \mathbf{x} \in \Omega \big\} \subset L^2(\Omega),$$

where u_a is a nonnegative real number. We set $X_{ad} = V \times U_{ad}$ and introduce a cost functional $J : X \to \mathbb{R}$ of tracking type

$$J(x) = \frac{1}{2} \|x - y_d\|^2_{L^2(\Omega_m)} + \frac{\sigma}{2} \|u\|^2_{L^2(\Omega)} \quad \text{for } x = (y, u) \in X,$$

where Ω_m is a subset of Ω, $y_d \in L^2(\Omega_m)$ is the desired state, and $\sigma \geq 0$ denotes a regularization parameter. It follows by standard arguments that J is twice continuously Fréchet-differentiable and the mapping $x \mapsto J''(x)$ is Lipschitz-continuous on X. In particular, the first and second derivatives at $x = (y, u) \in X$ are

$$J'(x)x_\delta = \int_\Omega P_m(y - y_d)y_\delta + \sigma u u_\delta \,\mathrm{d}\mathbf{x}, \quad J''(x)(x_\delta, \tilde{x}_\delta) = \int_\Omega P_m y_\delta \tilde{y}_\delta + \sigma u_\delta \tilde{u}_\delta \,\mathrm{d}\mathbf{x}$$

for directions $x_\delta = (y_\delta, u_\delta)$ and $\tilde{x}_\delta = (\tilde{y}_\delta, \tilde{u}_\delta)$. Here, $P_m : L^2(\Omega_m) \to L^2(\Omega)$ is the bounded linear extension operator satisfying $P_m v = v$ in Ω_m a.e. and $P_m v = 0$ in $\Omega \setminus \Omega_m$.

Then, the optimal control problem is given by

$$\min J(x) \quad \text{s.t.} \quad x \in \mathcal{F}(\mathbf{P}), \tag{$\mathbf{P}$}$$

where the feasible set is $\mathcal{F}(\mathbf{P}) = \{x \in X_{ad} \,|\, e(x) = 0 \text{ in } V'\}$. Since $W_{ad} \neq \emptyset$ holds, it follows by standard arguments that there exists at least one optimal solution $x^* = (y^*, u^*)$ to ($\mathbf{P}$).

2.3. **Optimality conditions.** Let us introduce the Lagrange functional $L : X \times V \to \mathbb{R}$ associated with ($\mathbf{P}$):

$$L(x, p) = J(x) + \langle e(x), p \rangle_{V',V} \quad \text{for } (x, p) \in X \times V.$$

It follows from the properties of J and e that the Lagrange functional is twice continuously Fréchet-differentiable and the mapping $(x, p) \mapsto L''(x, p)$ is Lipschitz-continuous on X.

In the following theorem we state first-order necessary optimality conditions for ($\mathbf{P}$). The existence of a unique Lagrange multiplier follows directly from Proposition 2.2 and [29]. For more details we refer to [13, Theorem 2.5]

Theorem 2.3 (First-order necessary optimality conditions)**.** *Suppose that $x^* = (y^*, u^*)$ is a local solution to ($\mathbf{P}$). Then there exists a unique Lagrange multiplier $p^* \in V$ satisfying together with x^* the dual equation*

$$\begin{aligned} -\nabla \cdot \big(\kappa \nabla p^*\big) + u^* p^* &= P_m(y_d - y^*) && \textit{on } \Omega \textit{ a.e.}, \\ \kappa \frac{\partial p^*}{\partial n} + \eta p^* &= 0 && \textit{on } \Gamma \textit{ a.e.} \end{aligned} \tag{2.3}$$

Furthermore, the variational inequality

$$\int_\Omega \Big(\sigma u^* + y^* p^*\Big)(u - u^*)\,\mathrm{d}\mathbf{x} \geq 0 \quad \textit{for all } u \in U_{ad}.$$

For the convergence of the SQP method second-order sufficient optimality conditions are required, at least in a neighborhood of the solution $x^* = (y^*, u^*)$. The second Fréchet-derivative of the Lagrangian at $(x^*, p^*) \in X_{ad} \times V$ with respect to x in the direction $x = (y, u) \in X$ is

$$L_{xx}(x^*, p^*)(x, x) = \int_\Omega P_m(y^2) + \sigma u^2 + 2uyp^*\,\mathrm{d}\mathbf{x} \geq \sigma \|u\|^2_{L^2(\Omega)} + 2\int_\Omega uyp^*\,\mathrm{d}\mathbf{x}.$$

Since V is continuously embedded into $L^4(\Omega)$ there exists a constant $C_{emb} > 0$ such that

$$\|\varphi\|_{L^4(\Omega)} \leq C_{emb}\|\varphi\|_V \quad \text{for all } \varphi \in V. \tag{2.4}$$

Due to Proposition 2.2 we also have $\|y\|_V \leq C_{ker}\|u\|_{L^2(\Omega)}$ for all $(y, u) \in \ker e'(x^*)$ with a constant $C_{ker} > 0$. We set $C = C_{emb}C_{ker}$ and derive

$$\begin{aligned}
&L_{xx}(x^*, p^*)(x, x)\\
&\geq \frac{\sigma}{2}\|u\|^2_{L^2(\Omega)} + \frac{\sigma}{2}\|u\|^2_{L^2(\Omega)} - 2\|u\|_{L^2(\Omega)}\|y\|_{L^4(\Omega)}\|p^*\|_{L^4(\Omega)}\\
&\geq \frac{\sigma}{2}\|u\|^2_{L^2(\Omega)} + \frac{\sigma}{2C_{ker}}\|y\|^2_V - 2C\|u\|^2_{L^2(\Omega)}\|p^*\|_{L^4(\Omega)}\\
&\geq \min\left(\frac{\sigma}{4}, \frac{\sigma}{2C_{ker}}\right)\|x\|^2_X + \frac{1}{4}\Big(\sigma - 8C\|p^*\|_{L^4(\Omega)}\Big)\|u\|^2_{L^2(\Omega)}
\end{aligned}$$

for all $x = (y, u) \in \ker e'(x^*)$. Thus, we have proved the following result.

Theorem 2.4 (Second-order sufficient optimality conditions)**.** *Suppose that $x^* = (y^*, u^*)$ is a local solution to* (**P**) *and $p^* \in V$ is the associated unique Lagrange multiplier. Let the constants C_{emb} and C_{ker} be given by* (2.4) *and Proposition* 2.2*, respectively. If*

$$\sigma - 8C_{emb}C_{ker}\|p^*\|_{L^4(\Omega)} \geq 0 \tag{2.5}$$

holds, the second-order sufficient optimality condition is satisfied at (x^*, p^*), i.e., there exists a $\kappa > 0$ so that

$$L_{xx}(x^*, p^*)(x_\delta, x_\delta) \geq \kappa\|x\|^2_X \quad \textit{for all } x = (y, u) \in \ker e'(x^*).$$

Note that (2.5) can be ensured provided the Lagrange multiplier satisfies

$$\|p^*\|_{L^4(\Omega)} \leq \frac{\sigma}{8C_{emb}C_{ker}}. \tag{2.6}$$

A sufficient condition is presented in the next result which is proved in the Appendix.

Corollary 2.5. *Let all hypotheses of Theorem* 2.4 *be satisfied. If the residuum $\|y^* - y_d\|_{L^2(\Omega_m)}$ is sufficiently small, the second-order sufficient optimality conditon holds at (x^*, p^*).*

3. The inexact SQP method

In this section we formulate the SQP method for (**P**). Moreover, the a-posteriori error estimator for the linear–quadratic subproblems are introduced.

3.1. **The SQP method.** To solve (**P**) numerically, we apply the SQP method. The principal idea is to replace J and e by a quadratic approximation of the Lagrangian and a linearization of the constraint. For the readers convenience we recall the SQP method in Algorithm 1.

Algorithm 1 (Lagrange-SQP method)

Choose $x^0 = (y^0, u^0) \in X_{ad}$, $p^0 \in V$, and set $k = 1$.
repeat
  Compute $J'(x^k)$, $L_{xx}(x^k, p^k)$, $e(x^k)$, and $e'(x^k)$.
  Solve the linear-quadratic minimization problem

$$\begin{aligned} &\min_{x \in X} J^k(x) = J'(x^k)x + \frac{1}{2} L_{xx}(x^k, p^k)(x, x) \\ &\text{s.t.} \quad e'(x^k)x + e(x^k) = 0 \text{ and } x^k + x \in X_{ad}. \end{aligned} \tag{$\mathbf{P}^k$}$$

  Determine a step length parameter $t_k \in (0, 1]$ by an Armijo backtracking line search (see, e.g., [15]).
  Set $x^{k+1} = x^k + t_k x \in X_{ad}$ and $k = k + 1$.
  Choose a new estimate p^k for the Lagrange multiplier.
until a given stopping criterium is satisfied.

Remark 3.1. a) The linear-quadratic minimization problem ($\mathbf{P}^k$) is well-defined provided the operator $L_{xx}(x^k, p^k)$ is coercive on $\ker e'(x^k)$ and $e'(x^k)$ is surjective. Thus, Algorithm 1 is not globally defined.

b) By Proposition 2.2 the operator $e'(x)$ is surjective for all $x \in X_{ad}$. To ensure that ($\mathbf{P}^k$) has a unique solution we modify $L_{xx}(x^k, p^k)$ in the case if coercivity does not hold. For $\beta \in [0, 1]$ let the bilinear operator $B_k^\beta : X \times X \to \mathbb{R}$ be given by

$$B_k^\beta(x, \tilde{x}) = J''(x^k)(x, \tilde{x}) + \beta \, \langle e''(x^k)(x, \tilde{x}), p^k \rangle_{V',V} \quad \text{for } x, \tilde{x} \in X.$$

Then, $B_k^1 = L_{xx}(x^k, p^k)$ and $B_k^0 = J''(x^k)$. Due to Proposition 2.2 we have

$$B_k^0(x, x) \geq \frac{\sigma}{2} \min\left(\frac{1}{C_{ker}}, 1\right) \|x\|_X^2 \quad \text{for all } x \in \ker e'(x^k),$$

i.e., B_k^0 is positive definite. Thus, in the case if coercivity does not hold, we replace ($\mathbf{P}^k$) by

$$\begin{aligned} &\min_{x \in X} J^{k,\beta}(x) = J'(x^k)x + \frac{1}{2} B_k^\beta(x, x) \\ &\text{s.t.} \quad e'(x^k)x + e(x^k) = 0 \text{ and } x^k + x \in X_{ad} \end{aligned} \tag{$\mathbf{P}^{k,\beta}$}$$

with a coercive operator B_k^β (e.g., with $\beta = 0$). ◊.

Next we derive the optimality conditions for the linear-quadratic subproblem ($\mathbf{P}^{k,\beta}$). Throughout we suppose that the parameter $\beta \in [0, 1]$ is chosen in such a way that B_k^β is coercive on $X \times X$, i.e., ($\mathbf{P}^{k,\beta}$) has a unique solution. For our problem the cost J^k in ($\mathbf{P}^{k,\beta}$) has the form

$$J^{k,\beta}(x) = \int_\Omega P_m(y^k - y_d)y + \sigma u^k u + \frac{1}{2} \left(P_m(y^2) + 2\beta u y p^k + \sigma u^2\right) \mathrm{d}\mathbf{x}$$

for $x = (y, u) \in X$. The equation $e'(x^k)x + e(x^k) = 0$ is equivalent with the fact that $x = (y, u)$ satisfies the linearized state equation

$$\int_\Omega \kappa \nabla y \cdot \nabla \varphi + \big(u^k y + u y^k\big)\varphi \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma y\varphi \,\mathrm{d}\mathbf{s} = -\langle e(x^k), \varphi\rangle_{V',V}$$

for all $\varphi \in V$. To obtain $x^k + x \in X_{ad}$ we have to ensure that $u^k + u \in U_{ad}$. Setting $u_a^k = u_a - u^k$ we require

$$u \in U_{ad}^k = \big\{\tilde u \,\big|\, \tilde u \geq u_a^k \text{ in } \Omega \text{ a.e.}\big\}.$$

Remark 3.2. We introduce the linear operator $\mathcal{S} : U \to V$ as follows: for $u \in U$ the function $y = \mathcal{S}u$ is the unique solution to

$$\int_\Omega \kappa \nabla y \cdot \nabla \varphi + u^k y\varphi \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma y\varphi \,\mathrm{d}\mathbf{s} = -\int_\Omega u y^k \varphi \,\mathrm{d}\mathbf{x} \quad \text{for all } \varphi \in V. \tag{3.7}$$

Since $u^k \in U_{ad}$ holds, it follows from the Lax-Milgram lemma that $\mathcal{S}$ is well-defined and bounded. Moreover, $\hat y_k \in V$ is the unique solution to

$$\int_\Omega \kappa \nabla \hat y_k \cdot \nabla \varphi + u^k \hat y_k \varphi \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma \hat y_k \varphi \,\mathrm{d}\mathbf{s} = -\langle e(x^k), \varphi\rangle_{V',V}.$$

Then, $x = (y, u)$ with $y = \hat y_k + \mathcal{S}u$ solves $e'(x^k)x + e(x^k) = 0$. ◊

The computation of the adjoint $\mathcal{S}^\star : V' \to U'$ of $\mathcal{S}$ is described in the following lemma, which is proved in the Appendix.

Lemma 3.3. *Let the operator $\mathcal{S}$ be given as in Remark* 3.2. *Then, its adjoint operator $S^\star : V' \to U'$ is given as follows: for arbitrary $r \in V'$ compute the solution $v \in V$ to the variational problem*

$$\int_\Omega \kappa \nabla v \cdot \nabla \varphi + u^k v\varphi \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma v\varphi \,\mathrm{d}\mathbf{s} = \langle r, \varphi\rangle_{V',V} \quad \textit{for all } \varphi \in V \tag{3.8}$$

and set $\mathcal{S}^\star r = -y^k v$. In particular, $\mathcal{S}^\star r \in L^2(\Omega)$.

Suppose that there is a unique solution $\bar x = (\bar y, \bar u)$ to $(\mathbf{P}^k)$. To derive the optimality conditions, we define the Lagrangian functional $L^{k,\beta} : X \times V \to \mathbb{R}$ associated with $(\mathbf{P}^k)$ by

$$L^{k,\beta}(x, p) = J^{k,\beta}(x) + \langle e'(x^k)x + e(x^k), p\rangle_{V',V} \quad \text{for } (x, p) \in X \times V.$$

From $L_p^{k,\beta}(\bar x, \bar p)p = 0$ for all $p \in V$ we infer that the pair $(\bar y, \bar u)$ solves (3.9a). The equation $L_y^{k,\beta}(\bar x, \bar p)y = 0$ for all $y \in V$ implies

$$\int_\Omega \kappa \nabla y \cdot \nabla \bar p + u^k y \bar p \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma y\bar p \,\mathrm{d}\mathbf{s} = -\int_\Omega \big(P_m(y^k + \bar y - y_d) + \beta \bar u p^k\big) y \,\mathrm{d}\mathbf{x}.$$

Thus, $\bar p$ satisfies the dual problem

$$\begin{aligned} -\nabla \cdot (\kappa \nabla \bar p) + u^k \bar p &= P_m(y_d - y^k - \bar y) - \beta \bar u p^k && \text{in } \Omega \text{ a.e.},\\ \kappa \frac{\partial \bar p}{\partial n} + \eta \bar p &= 0 && \text{on } \Gamma \text{ a.e.} \end{aligned}$$

Finally, the optimality condition $L_u^{k,\beta}(\bar x, \bar p)(u - \bar u) \geq 0$ for all $u \in U_{ad}^k$ implies:

$$\int_\Omega \big(\beta \bar y p^k + \sigma(u^k + \bar u) + y^k \bar p\big)(u - \bar u) \,\mathrm{d}\mathbf{x} \geq 0.$$

Summarizing, the solution $\bar x = (\bar y, \bar u)$ to $(\mathbf{P}^{k,\beta})$ satisfies together with the Lagrange multiplier $\bar p \in V$ the following optimality system:

1) The (linearized) state equation

$$\int_\Omega \kappa\nabla y\cdot\nabla\varphi + \big(u^k y + u y^k\big)\varphi\,\mathrm{d}\mathbf{x} + \eta\int_\Gamma y\varphi\,\mathrm{d}\mathbf{s} = -\langle e(x^k),\varphi\rangle_{V',V} \tag{3.9a}$$

for all $\varphi\in V$,

2) the (linearized) dual equation

$$\int_\Omega \kappa\nabla p\cdot\nabla\varphi + u^k p\varphi\,\mathrm{d}\mathbf{x} + \eta\int_\Gamma p\varphi\,\mathrm{d}\mathbf{s} = \int_\Omega \big(P_m(y_d - y^k - y) - \beta u p^k\big)\varphi\,\mathrm{d}\mathbf{x} \tag{3.9b}$$

for all $\varphi\in V$, and

3) the (linearized) variational inequality

$$\int_\Omega \big(\beta y p^k + \sigma(u^k+u) + y^k p\big)(\tilde u - u)\,\mathrm{d}\mathbf{x}\ge 0 \tag{3.9c}$$

for all $\tilde u\in U_{ad}^k$.

Recall that $\beta\in[0,1]$ is chosen in such a way that (3.9) has a unique solution $(\bar y,\bar u,\bar p)\in V\times U_{ad}^k\times V$.

3.2. **A-posteriori error analysis for $(\mathbf{P}^{k,\beta})$.** Utilizing an a-posteriori error analysis we can ensure that $(\mathbf{P}^{k,\beta})$ is solved with a given tolerance. Therefore, we consider an inexact version of Algorithm 1, where the inexactness arises due to the inexact solution of the optimality system (3.9). Within the SQP method we control the error tolerance for the POD discretization to guarante the overall convergence of the optimization method. The presented approach is not limited to POD model reduction, but can easily be applied to other reduced-order techniques, e.g., to the reduced-basis method. We refer to [36] as a first step in this direction.

The used idea of a-posteriori error estimates was used by Malanowski et al. [28] in the context of error estimates for the optimal control of ODEs. It was extended later to elliptic optimal control problems in [4] and [8]. Let us explain this basic idea for our application.

Let $u^p=\sum_{i=1}^N u_i^p b_i\in U_{ad}^k$ be chosen arbitrarily. Our goal is to estimate the difference

$$\|\bar u - u^p\|_{L^2(\Omega)}$$

without the knowledge of the optimal solution $(\bar y,\bar u,\bar p)$ to (3.9). If $u^p\neq\bar u$ then u^p does not satisfy the necessary (and by convexity sufficient) optimality conditions (3.9c). However, there exists a function $\zeta\in L^2(\Omega)$ such that

$$\int_\Omega \big(\beta y^p p^k + \sigma(u^k+u^p) + y^k p^p + \zeta\big)(u-u^p)\,\mathrm{d}\mathbf{x}\ge 0\quad\text{for all } u\in U_{ad}^k, \tag{3.10}$$

where p^p solves the dual equation

$$\int_\Omega \kappa\nabla p^p\cdot\nabla\varphi + u^k p^p\varphi\,\mathrm{d}\mathbf{x} + \eta\int_\Gamma p^p\varphi\,\mathrm{d}\mathbf{s} = \int_\Omega \big(P_m(y_d - y^k - y^p) - \beta u^p p^k\big)\varphi\,\mathrm{d}\mathbf{x} \tag{3.11}$$

for all $\varphi\in V$ and y^p solves

$$\int_\Omega \kappa\nabla y^p\cdot\nabla\varphi + (u^k y^p + u^p y^k)\varphi\,\mathrm{d}\mathbf{x} + \eta\int_\Gamma y^p\varphi\,\mathrm{d}\mathbf{s} = -\langle e(x^k),\varphi\rangle_{V',V} \tag{3.12}$$

for all $\varphi\in V$. Replacing $\bar u$ by u^p we observe that (3.9a) coincides with (3.12). Moreover, (3.9b) is (3.11) if we replace $(\bar y,\bar u)$ by (y^p,u^p). Therefore, u^p satisfies the optimality condition of a perturbed elliptic optimal control problem with 'perturbation' ζ:

$$\min_{x=(y,u)} J^k(x) = J'(x^k)x + \frac{1}{2} B_k^\beta(x,x) + \int_\Omega \zeta u \,\mathrm{d}\mathbf{x}$$
$$\text{s.t.} \quad e'(x^k)x + e(x^k) = 0 \text{ and } x^k + x \in X_{ad}.$$

The smaller ζ is, the closer u^p is to $\bar{u}$. The computation of ζ is possible on the basis of the known data u^p, y^p, and p^p.

Next we derive an estimate for the differences $\bar{y} - y^p$ and $\bar{p} - p^p$ in terms of the difference $\bar{u} - u^p$. This result will be used to prove the local convergence of the inexact SQP method in Secton 4.6. For the proof we refer to the Appendix.

Proposition 3.4. *Assume that the sequence $\{(y^k, u^k, p^k)\}_{k\in\mathbb{N}}$ is uniformly bounded in $X \times V$. Let $(\bar{y}, \bar{u}, \bar{p})$ be the solution to* (3.9)*. Suppose that $u^p \in U_{ad}^k$ is chosen arbitrarily. Let y^p and p^p solve* (3.12) *and* (3.11)*, respectively. Then there exists a constant $C_p > 0$ which is independent on (y^k, u^k, p^k) so that*

$$\|(\bar{y}, \bar{p}) - (y^p, p^p)\|_{V\times V} \le C_p \|\bar{u} - u^p\|_U.$$

We proceed by deriving an estimate for $\|\bar{u} - u^p\|_{L^2(\Omega)}$ in terms of $\|\zeta\|_{L^2(\Omega)}$. The proof is given in the Appendix.

Theorem 3.5. *Let $(\bar{y}, \bar{u}, \bar{p})$ be the solution to* (3.9) *and $u^p \in U_{ad}^k$ be chosen arbitrarily. Suppose that y^p and p^p are the solution to* (3.12) *and* (3.11)*, respectively. Then, it follows that*

$$\|\bar{u} - u^p\|_{L^2(\Omega)} \le \frac{1}{\sigma} \|\zeta\|_{L^2(\Omega)},$$

where ζ is chosen such that (3.10) *holds.*

To construct ζ we can proceed as in [37]. Recall that $L^2(\Omega)$ is a Banach lattice with respect to the natural order $u \leqq v$ in $L^2(\Omega)$ iff $u(\mathbf{x}) \leqq v(\mathbf{x})$ f.a.a. $\mathbf{x} \in \Omega$; see, e.g., [43, p. 35]. We introduce $\mathcal{G}^{k,\beta} : X \times V \to L^2(\Omega)$ by

$$\mathcal{G}^{k,\beta}(x,p) = \beta p^k y + y^k p + \sigma\,(u^k + u) \quad \text{for } x = (y,u) \in X \text{ and } p \in V$$

in the pointwise almost everywhere sense. Then, (3.10) can be expressed as

$$\int_\Omega \big(\mathcal{G}^{k,\beta}(x^p, p^p) + \zeta\big)(u - u^p)\,\mathrm{d}\mathbf{x} \ge 0 \quad \text{for all } u \in U_{ad}^k.$$

Proposition 3.6. *Suppose that the hypotheses of Theorem* 3.5 *are satisfied. Define $\zeta \in L^2(\Omega)$ as follows*

$$\zeta(\mathbf{x}) = \begin{cases} \big[\mathcal{G}^{k,\beta}(y,u,p)(\mathbf{x})\big]_- & \text{f.a.a. } \mathbf{x} \in \mathcal{A}^k = \big\{\mathbf{x} \in \Omega \,|\, u^p(\mathbf{x}) = u_a^k(\mathbf{x})\big\}, \\ -\,\mathcal{G}^{k,\beta}(y,u,p)(\mathbf{x}) & \text{f.a.a. } \mathbf{x} \in \Omega \setminus \mathcal{A}^k, \end{cases}$$

where $[s]_- = -\min(0,s)$ for $s \in \mathbb{R}$. Then the estimate

$$\|\bar{u} - u^p\|_{L^2(\Omega)} \le \frac{1}{\sigma} \|\zeta\|_{L^2(\Omega)} \tag{3.13}$$

holds.

Proof. The proof is a variant of the proof of Proposition 3.2 in [37]. □

We call (3.13) an a-posteriori error estimate, since, in the next section, we shall apply it to suboptimal solutions u^p to the optimality system (3.9) that have already be computed by a POD Galerkin method. After having computed u^p, we determine the associated state y^p and adjoint state p^p. Then we can determine ζ and its L^2-norm and (3.13) gives an upper bound for the distance of u^p to $\bar{u}$. In this way, the error caused by the POD method can be estimated a-posteriorily. If the error is too large, then we have to include more POD basis functions in our Galerkin approximation for (3.9).

4. The POD Galerkin discretization of ($\mathbf{P}^{k,\beta}$)

In this section we briefly introduce the POD method and derive the reduced-order model for the optimality system (3.9) of ($\mathbf{P}^{k,\beta}$). Moreover, a-priori error estimates for POD Galerkin schemes for the state as well as for the adjoint equation are shown.

4.1. The POD method.

Let $u \in U$ be given. Then there exists a vector $\mathbf{u} = (u_1, \ldots, u_N)^T \in \mathbb{R}^N$ such that

$$u(\mathbf{x}) = \sum_{i=1}^{N} u^i b_i(\mathbf{x}) \quad \text{f.a.a. } \mathbf{x} \in \Omega. \tag{4.1}$$

Furthermore, we suppose that

$$\mathbf{u} \in \mathcal{D} = \big[\underline{u}_1, \overline{u}_1\big] \times \ldots \big[\underline{u}_N, \overline{u}_N\big] \subset \mathbb{R}^N \quad \text{with} \quad 0 < \underline{u}_i \leq \overline{u}_i \text{ for } i = 1, \ldots, N.$$

By $y = y(\mathbf{u})$ we denote the unique solution to (3.9a), where u is given as in (4.1). The snapshot ensemble is chosen to be

$$\mathcal{V} = \operatorname{span}\big\{y(\mathbf{u})\,|\,\mathbf{u} \in \mathcal{D}\big\} \subset V. \tag{4.2}$$

Then, $d = \dim \mathcal{V} \leq \infty$. Let $\ell < \infty$ satisfy $1 \leq \ell \leq d$. The POD basis $\{\psi_i\}_{i=1}^{\ell}$ of rank ℓ is given by the solution to the following minimization problem:

$$\min_{\{\psi_i\}_{i=1}^{\ell} \subset V} \int_{\mathcal{D}} \Big\| y(\mathbf{u}) - \sum_{i=1}^{\ell} \langle y(\mathbf{u}), \psi_i \rangle_V \psi_i \Big\|_V^2 \,\mathrm{d}\mathbf{u} \quad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle_V = \delta_{ij}, \tag{$\mathbf{P}^\ell$}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. It is well-known that the solution to ($\mathbf{P}^\ell$) can be derived by one of the following procedures [18, 35]:

1) Solve the symmetric eigenvalue problem

$$\mathcal{R}\psi_i = \lambda_i \psi_i \quad \text{for } i = 1, \ldots, \ell$$

in V, where $\mathcal{R} : V \to \mathcal{V} \subset V$, given by

$$\mathcal{R}\psi = \int_{\mathcal{D}} \langle y(\mathbf{u}), \psi \rangle_V y(\mathbf{u}) \,\mathrm{d}\mathbf{u} \quad \text{for } \psi \in V,$$

is a bounded, linear, compact, self-adjoint and nonnegative operator.

2) Solve the symmetric eigenvalue problem

$$\mathcal{K} v_i = \lambda_i v_i \quad \text{for } i = 1, \ldots, \ell$$

in $L^2(\mathcal{D})$, where $\mathcal{K} : L^2(\mathcal{D}) \to L^2(\mathcal{D})$ is given by

$$\big(\mathcal{K} v\big)(\tilde{\mathbf{u}}) = \int_{\mathcal{D}} \langle y(\mathbf{u}), y(\tilde{\mathbf{u}}) \rangle_V v(\mathbf{u}) \,\mathrm{d}\mathbf{u} \quad \text{for } \tilde{\mathbf{u}} \in \mathcal{D} \text{ and } v \in L^2(\mathcal{D}),$$

and set

$$\psi_i = \frac{1}{\sqrt{\lambda_i}} \int_{\mathcal{D}} y(\mathbf{u}) v_i(\mathbf{u}) \, \mathrm{d}\mathbf{u} \quad \text{for } i = 1, \ldots, \ell.$$

From the Hilbert-Schmidt theorem [34, p. 29] it follows that there exists a complete orthogonal basis $\{\psi_i\}_{i=1}^d$ for $\mathcal{V} = \text{range}\,(\mathcal{R})$ and a sequence $\{\lambda_i\}_{i=1}^d$ of real numbers such that

$$\mathcal{R}\psi_i = \lambda_i \psi_i \text{ for } i = 1, \ldots, d \quad \text{and} \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d \geq 0.$$

To obtain a complete orthogonal basis in the separable Hilbert space V we need an orthogonal basis for $(\text{range}\,(\mathcal{R}))^\perp$. This can be done by the Gram-Schmidt procedure. Hence, we suppose in the following that $\{\psi_i\}_{i=1}^\infty$ is a complete orthogonal basis for V. In particular, we have

$$\int_{\mathcal{D}} \Big\| y(\mathbf{u}) - \sum_{i=1}^{\ell} \langle y(\mathbf{u}), \psi_i \rangle_V \psi_i \Big\|_V^2 \, \mathrm{d}\mathbf{u} = \sum_{i=\ell+1}^{\infty} \lambda_i. \tag{4.3}$$

If $1 \leq d = \dim \mathcal{V} < \infty$ holds, it follows that $\lambda_i > 0$ for $1 \leq i \leq d$ and $\mathcal{R}\psi_i = 0$ for all $i > d$.

Remark 4.1. In real computations, we do not have the $y(\mathbf{u})$ for all $\mathbf{u} \in \mathcal{D}$ at hand. For that purpose let $\{\mathbf{u}_i\}_{i=1}^M$ define grid points in $\mathcal{D}$ and $y_j = y(\mathbf{u}_j)$, $j = 1, \ldots, M$, be approximations for u at the grid points $\mathbf{u}_j$. We set

$$\mathcal{V}_M = \text{span}\,\big\{y_1, \ldots, y_M\big\}$$

with $d_M = \dim \mathcal{V}_M \leq M$. Then, for given $\ell \leq d_M$ we consider the minimization problem

$$\min_{\{\psi_i\}_{i=1}^{\ell} \subset V} \sum_{j=1}^{M} \alpha_j \Big\| y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i \rangle_V \psi_i \Big\|_V^2 \quad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle_V = \delta_{ij}, \tag{$\mathbf{P}_M^\ell$}$$

instead of $(\mathbf{P}^\ell)$. In $(\mathbf{P}_M^\ell)$ the α_j's stand for weights in the used quadrature rule. The solution to $(\mathbf{P}_M^\ell)$ is given by the solution to the eigenvalue problem

$$\mathcal{R}_M \psi_i = \sum_{j=1}^{M} \alpha_j \langle y_j, \psi_i \rangle_V \, y_j = \lambda_i \psi_i, \quad i = 1, \ldots, \ell,$$

where $\mathcal{R}_M : V \to \mathcal{V}_M \subset V$ is a bounded, linear, compact, self-adjoint and nonnegative operator. We refer to [24], where the relationship between $(\mathbf{P}^\ell)$ and $(\mathbf{P}_M^\ell)$ is investigated. $\Diamond$

4.2. **POD Galerkin scheme for the state equation.** The error analysis presented in this section shows that there is a real chance to decrease the error by increasing the number of snapshots used by the POD method. First we derive an error estimate for the state equation, where the contol $u \in U_{ad}^k$ is fixed.

Let $y = \hat{y}^k + \mathcal{S}u$ be the state associated with some control $u \in U_{ad}^k$, and let $\mathcal{V}$ by given as in (4.2). We fix ℓ with $\ell \leq \dim \mathcal{V}$ and compute the first ℓ POD basis functions $\psi_1, \ldots, \psi_\ell \in V$ by solving either $\mathcal{R}\psi_i = \lambda_i \psi_i$ or $\mathcal{K}v_i = \lambda_i v_i$ for $i = 1, \ldots, \ell$. Then we define the finite-dimensional linear space

$$V^\ell = \text{span}\,\big\{\psi_1, \ldots, \psi_\ell\big\} \subset V.$$

Endowed with the topology in V it follows that V^ℓ is a Hilbert space. Let $\mathcal{P}^\ell$ denote the orthogonal projection of V onto V^ℓ defined by

$$\mathcal{P}^\ell\psi = \sum_{i=1}^{\ell} \langle \psi, \psi_i\rangle_V \,\psi_i \quad \text{for } \psi \in V. \tag{4.4}$$

Using (4.3) we have

$$\int_{\mathcal{D}} \left\|y(\mathbf{u}) - \mathcal{P}^\ell y(\mathbf{u})\right\|_V^2 \,\mathrm{d}\mathbf{u} = \left\|y - \mathcal{P}^\ell y\right\|^2_{L^2(\mathcal{D};V)} = \sum_{i=\ell+1}^{\infty} \lambda_i$$

The POD Galerkin scheme for the state equation (3.9a) leads to the following linear problem: determine a function $y^\ell = \sum_{i=1}^{\ell} y_i\psi_i$ such that

$$\int_\Omega \kappa\nabla y^\ell\cdot\nabla\psi + u^k y^\ell\psi\,\mathrm{d}\mathbf{x} + \eta\int_\Gamma y^\ell\psi\,\mathrm{d}\mathbf{s} = -\int_\Omega u y^k\psi\,\mathrm{d}\mathbf{x} - \langle e(x^k),\psi\rangle_{V',V} \tag{4.5}$$

for all $\psi \in V^\ell$. It follows from the Lax-Milgram lemma that there exists a unique $y^\ell \in V^\ell \subset V$ solving (4.5). The proof of the following convergence result is given in the Appendix.

Proposition 4.2. *For given $\mathbf{u} = (u_1,\ldots,u_N)^T \in \mathcal{D}$ let $u = \sum_{i=1}^N u_i b_i \in W$. Moreover, $y = y(\mathbf{u})$ and $y^\ell = y^\ell(\mathbf{u})$ are the solutions to* (3.9a) *and* (4.5)*, respectively. Suppose that, for $\ell \le \dim\mathcal{V}$, the elements $\{\psi_i\}_{i=1}^\ell$ solve* ($\mathbf{P}^\ell$)*. Then there exists a constant $C > 0$ independent of ℓ such that*

$$\int_{\mathcal{D}} \left\|y(\mathbf{u}) - y^\ell(\mathbf{u})\right\|_V^2 \,\mathrm{d}\mathbf{u} \le C\sum_{i=\ell+1}^{\infty} \lambda_i.$$

Moreover, $\lim_{\ell\to\infty}\|y - y^\ell\|_{L^2(\mathcal{D};V)} = 0$ *and*

$$\|y(\mathbf{u}) - y^\ell(\mathbf{u})\|_V \le C\,\|y(\mathbf{u}) - \mathcal{P}^\ell y(\mathbf{u})\|_V \quad \textit{for all } \mathbf{u}\in\mathcal{D}.$$

Remark 4.3. Since $\{\psi_i\}_{i=1}^\infty$ is a complete orthonormal basis in the seperable Hilbert space V, we conclude $\lim_{\ell\to\infty}\|y(\mathbf{u}) - y^\ell(\mathbf{u})\|_V = 0$ for all $\mathbf{u}\in\mathcal{D}$, but we do not have any rate of convergence at hand. $\Diamond$

Let $\hat{y}_k^\ell \in V^\ell$ denote the unique solution to

$$\int_\Omega \kappa\nabla\hat{y}_k^\ell\cdot\nabla\psi + u^k\hat{y}_k^\ell\psi\,\mathrm{d}\mathbf{x} + \eta\int_\Gamma \hat{y}_k^\ell\psi\,\mathrm{d}\mathbf{s} = -\langle e(x^k),\psi\rangle_{V',V} \quad \text{for all } \psi\in V^\ell.$$

Analogously to Remark 3.2 we introduce the linear operator $\mathcal{S}^\ell : V \to V^\ell$ for fixed ℓ: For given $u \in U$ the element $\tilde{y}^\ell = \mathcal{S}^\ell u \in V^\ell$ solves

$$\int_\Omega \kappa\nabla\tilde{y}^\ell\cdot\nabla\psi + u^k\tilde{y}^\ell\psi\,\mathrm{d}\mathbf{x} + \eta\int_\Gamma \tilde{y}^\ell\psi\,\mathrm{d}\mathbf{x} = -\int_\Omega u y^k\psi\,\mathrm{d}\mathbf{x} \quad \text{for all } \psi\in V^\ell.$$

Thus, the solution u^ℓ to (4.5) can be decomposed as $y^\ell = \hat{y}_k^\ell + \mathcal{S}^\ell u$. It follows by standard arguments and by the assumptions on κ, u^k, and η that the operator $\mathcal{S}^\ell$ is bounded. Using similar arguments as in Lemma 3.3 we derive that the adjoint operator $(\mathcal{S}^\ell)^\star : (V^\ell)' \to W'$ of $\mathcal{S}^\ell$ can be computed as follows: for given $r^\ell \in (V^\ell)'$ we have

$$(\mathcal{S}^\ell)^\star r^\ell = -y^k v^\ell \subset L^2(\Omega) \subset U'$$

where $v^\ell \in V^\ell$ solves the variational problem

$$\int_\Omega \kappa \nabla v^\ell \cdot \nabla \psi + u^k v^\ell \psi \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma v^\ell \psi \,\mathrm{d}\mathbf{x} = \langle r^\ell, \psi \rangle_{(V^\ell)', V^\ell} \quad \text{for all } \psi \in V^\ell.$$

4.3. **POD Galerkin scheme for the adjoint equation.** The convergence result $y^\ell \to y$ (compare Proposition 4.2) implies an associated one for the adjoint state, i.e., $p^\ell \to p$ as $\ell \to \infty$. This will be proved next. Let p solve (3.9b), $u \in U$ be arbitrary, $\{\psi_i\}_{i=1}^\ell$ be a POD basis of rank ℓ and y^ℓ be the solution to (4.5). Then, $p^\ell = \sum_{i=1}^\ell p_i \psi_i$ satisfies the variational problem

$$\int_\Omega \kappa \nabla p^\ell \cdot \nabla \psi + u^k p^\ell \psi \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma p^\ell \psi \,\mathrm{d}\mathbf{s} = \int_\Omega \big(P_m(y_d - y^k - y^\ell) - \beta u p^k\big)\psi \,\mathrm{d}\mathbf{x} \quad (4.6)$$

for all $\psi \in V^\ell$. Analogously to the arguments for the solvability of (4.5), it follows that for any $u \in U$ and $y^\ell \in V^\ell$ there exists a unique solution $p^\ell \in V^\ell$ to (4.6).

Proposition 4.4. *For given $\mathbf{u} = (u_1, \ldots, u_N)^T \in \mathcal{D}$ let $u = \sum_{i=1}^N u_i b_i \in U$. Moreover, $y(\mathbf{u}) = \hat{y}^k + \mathcal{S}u$ and $y^\ell(\mathbf{u}) = \hat{y}_k^\ell + \mathcal{S}^\ell u$. Suppose that for $\ell \leq \dim \mathcal{V}$ the elements $\{\psi_i\}_{i=1}^\ell$ solve $(\mathbf{P}^\ell)$. Let $p = p(\mathbf{u})$ and $p^\ell(\mathbf{u})$ be the solutions to* (3.9b) *and* (4.6)*, respectively, with $\bar{u} = u$, $\bar{y} = y(\mathbf{u})$, and $y^\ell = y^\ell(\mathbf{u})$. Then there exists a constant $C > 0$ such that*

$$\int_\mathcal{D} \|p(\mathbf{u}) - p^\ell(\mathbf{u})\|_V^2 \,\mathrm{d}\mathbf{u} \leq C\big(\|y - y^\ell\|_{L^2(\mathcal{D};V)} + \|p - \mathcal{P}^\ell p\|_{L^2(\mathcal{D};V)}\big)$$

where the linear projector $\mathcal{P}^\ell : V \to V^\ell$ is given by (4.4)*. Furthermore, we have $\lim_{\ell\to\infty} \|p - p^\ell\|_{L^2(\mathcal{D};V)} = 0$ and*

$$\|p(\mathbf{u}) - p^\ell(\mathbf{u})\|_V \leq C\left(\|y(\mathbf{u}) - \mathcal{P}^\ell y(\mathbf{u})\|_V + \|p(\mathbf{u}) - \mathcal{P}^\ell p(\mathbf{u})\|_V\right) \textit{ for all } \mathbf{u} \in \mathcal{D}.$$

For a proof we refer to the Appendix.

Remark 4.5. Of course, the convergence rate of p^ℓ to p as $\ell \to \infty$ depends on the approximation properties of the POD basis for the adjoint variable; see [10, 17]. ◊

4.4. **Convergence of the suboptimal perfusions.** In this subsection we derive convergence results utilizing a POD basis of rank ℓ that is not necessarily related to the optimal control $\bar{u}$ as an input function for the generation of the snapshots. This is in contrast to the work [17].

The POD Galerkin appoximation of (3.9) yields the following linear system: determine $(y^\ell, u^\ell, p^\ell) \in V^\ell \times U_{ad}^k \times V^\ell$ satifying

1) The (linearized) state equation

$$\int_\Omega \kappa \nabla y^\ell \cdot \nabla \psi + \big(u^k y^\ell + u^\ell y^k\big)\psi \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma y^\ell \psi \,\mathrm{d}\mathbf{s} = -\langle e(x^k), \psi \rangle_{V', V} \quad (4.7\text{a})$$

for all $\psi \in V^\ell$,

2) the (linearized) dual equation

$$\int_\Omega \kappa \nabla p^\ell \cdot \nabla \psi + u^k p^\ell \psi \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma p^\ell \psi \,\mathrm{d}\mathbf{s} = \int_\Omega \big(P_m(y_d - y^k - y^\ell) - \beta u^\ell p^k\big)\psi \,\mathrm{d}\mathbf{x} \quad (4.7\text{b})$$

for all $\psi \in V$, and

3) the (linearized) variational inequality

$$\int_\Omega \big(\beta y^\ell p^k + \sigma(u^k + u^\ell) + y^k p^\ell\big)(u - u^\ell)\,\mathrm{d}\mathbf{x} \geq 0 \tag{4.7c}$$

for all $u \in U_{ad}^k$.

Proposition 4.6. *Suppose that the POD basis of rank ℓ is computed using an arbitrarily chosen $u \in U$. Let $(\bar y, \bar u, \bar p)$ and $(\bar y^\ell, \bar u^\ell, \bar p^\ell)$ be the solutions to* (3.9) *and* (4.7)*, respectively. Let $\tilde y^\ell$ be the POD state solving* (4.5) *with $u = \bar u$ and $\tilde p^\ell$ the POD adjoint state satisfying* (4.6) *with $y^\ell = \tilde y^\ell$ and $u = \bar u$. Then, there exists a constant $C > 0$ such that*

$$\|\bar u - \bar u^\ell\|_{L^2(\Omega)} \leq C\left(\|\bar y(\bar{\mathbf{u}}) - \tilde y^\ell(\bar{\mathbf{u}})\|_{L^3(\Omega)} + \|\bar p(\bar{\mathbf{u}}) - \tilde p^\ell(\bar{\mathbf{u}})\|_{L^3(\Omega)}\right), \tag{4.8}$$

where $\bar{\mathbf{u}} = (\bar u_1, \ldots, \bar u_N)^T$ is given by $\bar u = \sum_{i=1}^N \bar u_i b_i$.

For the proof we refer the reader to the Appendix.

Remark 4.7. Since $\bar y$ and $\tilde y^\ell$ as well as $\bar p$ and $\tilde p^\ell$ are computed from the same control $\bar u$, we are in the position to apply Propositions 4.2 and 4.4. By the continuous embedding from V into $L^3(\Omega)$ we obtain

$$\|\bar u - \bar u^\ell\|_{L^2(\Omega)} \leq C\left(\|\bar y(\bar{\mathbf{u}}) - \mathcal{P}^\ell \bar y(\bar{\mathbf{u}})\|_V + \|\bar p(\bar{\mathbf{u}}) - \mathcal{P}^\ell \bar p(\bar{\mathbf{u}})\|_V\right)$$

for a constant $C > 0$. In particular, $\lim_{\ell\to\infty} \|\bar u - \bar u^\ell\|_{L^2(\Omega)} = 0$.

4.5. **A-posteriori error estimate for the POD approximation.** In this subsection we complete the discusion of the a-posteriori estimate by combining Proposition 3.6 and Remark 4.7. The proposition permits to estimate $\|\bar u - \bar u^\ell\|_{L^2(\Omega)}$ by the norm of an appropriate ζ, while Remark 4.7 will be used to show that ζ tends to zero as $\ell \to \infty$, since it ensures the convergence of $\bar u^\ell$ to the optimal control $\bar u$ for $(\mathbf{P}^{k,\beta})$.

For any ℓ let $\bar u^\ell \in U_{ad}^k$ be the optimal control solving (4.7) together with $\bar y$ and $\bar p$. Then, $\bar u^\ell$ is taken as a suboptimal u^p for $(\mathbf{P}^{k,\beta})$, i.e., in Proposition 3.6 we choose $u^p := \bar u^\ell$.

Theorem 4.8. *Suppose that $(\bar y, \bar u, \bar p) \in V \times U_{ad}^k \times V$ is the solution to* (3.9)*.*

1) *Let $\ell \leq d$ be arbitrarily given and $(\bar y^\ell, \bar u^\ell, \bar p^\ell) \in V \times U_{ad}^k \times V$ be the solution to* (4.7)*. Suppose that $\tilde y$ is the solution to* (3.12) *with $u^p = \bar u^\ell$ and that $\tilde p$ solves* (3.11) *with $(y^p, u^p) = (\tilde y, \bar u^\ell)$. Define, according to Proposition* 3.2*, the function $\zeta^\ell \in L^2(\Omega)$ by*

$$\zeta^\ell(\mathbf{x}) = \begin{cases} \big[\mathcal{G}^{k,\beta}(\tilde y, \bar u^\ell, \tilde p)(\mathbf{x})\big]_- & \text{on } \mathcal{A} = \big\{\mathbf{x} \in \Omega \,|\, \bar w^\ell(\mathbf{x}) = w_a^k(\mathbf{x})\big\}, \\ -\,\mathcal{G}^{k,\beta}(\tilde y, \bar u^\ell, \tilde p)(\mathbf{x}) & \text{on } \Omega \setminus \mathcal{A}, \end{cases}$$

Then,

$$\|\bar u - \bar u^\ell\|_{L^2(\Omega)} \leq \frac{1}{\sigma}\,\|\zeta^\ell\|_{L^2(\Omega)}.$$

2) *If all hypothesis of Propositions* 4.2*,* 4.4*, and* 4.6 *are satisfied, in particular $\{\psi_i\}_{i=1}^\infty$ is a complete orthonormal basis for V, then the sequence $\{\mathcal{G}^{k,\beta}(\tilde y, \bar u^\ell, \tilde p)\}$ converge to $\mathcal{G}^{k,\beta}(\bar y, \bar u, \bar p)$ in $L^2(\Omega)$ as $\ell \to \infty$ and*

$$\|\zeta^\ell\|_{L^2(\Omega)} \to 0 \quad \text{as } \ell \to \infty.$$

The proof is a variant of the proof of Theorem 4.11 in [37].

Remark 4.9. 1) Notice that $\tilde{y}$ and $\tilde{p}$ must be taken as the solutions to the (full) state and adjoint equations.

2) Part 2) of Theorem 4.8 shows that $\|\zeta^\ell\|_{L^2(\Omega)}$ can be expected smaller than any $\varepsilon > 0$ provided that ℓ is taken sufficiently large. Motivated by this result we set up Algorithm 2.

Algorithm 2 (POD method for $(\mathbf{P}^{k,\beta})$ with a-posteriori estimator)

1: Choose a maximal number $\ell_{max} > 0$ of POD basis function, an $\ell < \ell_{max}$, and a stopping criterium $\varepsilon > 0$.
2: Compute a POD basis of rank ℓ by solving $(\mathbf{P}^\ell)$.
3: **repeat**
4: Derive a reduced-order model of rank ℓ for $(\mathbf{P}^{k,\beta})$.
5: Calculate the suboptimal control $\bar{u}^\ell$ to $(\mathbf{P}^{k,\beta})$.
6: Evalute $\tilde{y}(\bar{u}^\ell) = \hat{y}^k + \mathcal{S}\bar{u}^\ell$ and compute the solution $\tilde{p}(\bar{u}^\ell)$ as well as ζ^ℓ.
7: **if** $\|\zeta^\ell\|_{L^2(\Omega)} \geq \varepsilon$ **then**
8: Set $\ell = \ell + 1$.
9: **end if**
10: **until** $\|\zeta^\ell\|_{L^2(\Omega)} < \varepsilon$ **or** $\ell > \ell_{max}$
11: Return ℓ and suboptimal control $\bar{u}^\ell$.

Remark 4.10. Of, course, step 8 can be replaced by
8: Set $\ell = \ell + L$.
with any natural number L. ◊

4.6. **Convergence of the inexact SQP method.** Let us assume that the solution $\bar{z} = (\bar{y}, \bar{u}, \bar{p})$ to (3.9) satisfies $\bar{u} > u_a^k$ (inactive control). Then, the variational inequality (3.9c) can be replaced by the equation

$$\int_\Omega \big(\beta y p^k + \sigma(u^k + u) + y^k p\big) b_i \,\mathrm{d}\mathbf{x} = 0 \quad \text{for } 1 \leq i \leq N. \tag{3.9c'}$$

Thus, (3.12), (3.11), and (3.9c') leads to a linear operator equation in $X' \times V'$ of the form (1.3) for the variable $\bar{z}$. Since the mapping $(x, p) \mapsto L^{k,\beta}(x, p)$ is twice continuously Fréchet-differentiable, it can be shown that there exists a constant $C > 0$ independent of the iteration level k so that $\|A_k\|_{L(X\times V, X'\times V')} \leq C$, where $L(X \times V, X' \times V')$ denotes the Banach space of all bounded linear operators from $X \times V$ to $X' \times V'$ endowed with the common operator norm.

Let the solution $\bar{z}^\ell = (\bar{y}^\ell, \bar{u}^\ell, \bar{p}^\ell)$ to (4.7) satisfy $\bar{u}^\ell > u_a$. Then, the variational inequality (4.7c) yields the equation

$$\int_\Omega \big(\beta y^\ell p^k + \sigma(u^k + u^\ell) + y^k p^\ell\big) b_i \,\mathrm{d}\mathbf{x} = 0 \quad \text{for } 1 \leq i \leq N. \tag{4.7c'}$$

so that (4.7a), (4.7b), and (4.7c') can be formulated as a finite-dimensional linear system of the form (1.4) for the variable $\bar{z}^\ell \in \mathbb{R}^n$ with $n = 2\ell + N$ (ℓ coefficients $\bar{y}_i$ for $\bar{y}^\ell$, N coefficients for $\bar{u}^\ell$, and ℓ coefficients $\bar{p}_i$ for $\bar{p}^\ell$). From (1.4) we obtain the coefficients $\bar{u}_i$, $1 \leq i \leq N$, for the suboptimal control $\bar{u}^\ell$. Then, we define the bounded operator $\mathcal{I}:\mathbb{R}^n \to X \times Y$ as follows:

$$\mathbb{R}^n \ni \bar{z}^\ell = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_\ell \\ \bar{u}_1 \\ \vdots \\ \bar{u}_N \\ \bar{p}_1 \\ \vdots \\ \bar{p}_\ell \end{pmatrix} \mapsto \mathcal{I}\bar{z}^\ell = \begin{pmatrix} \tilde{y}^\ell \\ \sum\limits_{i=1}^N \bar{u}_i b_i \\ \tilde{p}^\ell \end{pmatrix} \in V \times U \times V$$

where $\tilde{y}^\ell$ and $\tilde{p}^\ell$ solve (3.12) and (3.11), respectively, with $(y^p, u^p, p^p) = (\tilde{y}^\ell, \bar{u}^\ell, \tilde{p}^\ell)$. From (1.3) and Proposition 3.4 it follows

$$\begin{aligned} \|A_k\mathcal{I}\bar{z} - b_k\|_{X'\times V'} &= \|A_k(\mathcal{I}\bar{z}^\ell - \bar{z})\|_{X'\times V'} \leq \|A_k\|_{L(X\times V, X'\times V')} \|\mathcal{I}\bar{z}^\ell - \bar{z}\|_{X\times V} \\ &\leq C\sqrt{\|\bar{y} - \tilde{y}^\ell\|_V^2 + \|\bar{u} - \bar{u}^\ell\|_{L^2(\Omega)}^2 + \|\bar{p} - \tilde{p}^\ell\|_V^2} \\ &\leq \tilde{C}\,\|\bar{u} - \bar{u}^\ell\|_{L^2(\Omega)} \end{aligned}$$

with $\tilde{C} = C\sqrt{1 + C_p^2}$. Consequently, Theorem 4.8 implies that

$$\|A_k\mathcal{I}\bar{z} - b_k\|_{X'\times V'} \leq \frac{\tilde{C}}{\sigma}\,\|\zeta^\ell\|_{L^2(\Omega)} \overset{\ell\to\infty}{\longrightarrow} 0.$$

Therefore, we combine Algorithms 1 and 2 to arrive at POD a-posteriori error based inexact SQP method for the bilinear optimal control problem; see Algorithm 3. By L' we denote the Fréchet derivative of the Lagrangian with respect to (x, p).

Algorithm 3 (POD a-posteriori error based inexact SQP method)

Choose $x^0 = (y^0, u^0) \in X_{ad}$, $p^0 \in V$, tol > 0, and set $k = 1$.
while $\|L'(x^k, p^k)\| \geq$ tol **do**
  Choose $\ell_{max} > 0$, $\ell < \ell_{max}$, and $\varepsilon_k = \min(0.5, 0.5\,\|L'(x^k, p^k)\|^q)$, $q \in (1, 2]$.
  Compute a POD basis of rank ℓ by solving ($\mathbf{P}^\ell$).
  repeat
    Derive a reduced-order model of rank ℓ for ($\mathbf{P}^{k,\beta}$).
    Calculate the suboptimal control $\bar{u}^\ell$, $\tilde{y}^\ell$, $\tilde{p}^\ell$ and ζ^ℓ.
    if $\|\zeta^\ell\|_{L^2(\mathfrak{D})} \geq \varepsilon_k$ **then**
      Set $\ell = \ell + 1$.
    end if
  until $\|\zeta^\ell\|_{L^2(\mathfrak{D})} < \varepsilon_k$ **or** $\ell > \ell_{max}$
  if $\ell > \ell_{max}$ **then**
    STOP and restart the algorithm (e.g., with a larger $\ell_{\max}$).
  end if
  Determine $t_k \in (0, 1]$ by a line search.
  Set $(x^{k+1}, p^{k+1}) = (x^k, p^k) + t_k(\tilde{y}^\ell, \bar{u}^\ell, \tilde{p}^\ell)$ and $k = k + 1$.
end while

Remark 4.11. In our numerical experiments it is more efficient to compute the POD basis of rank $\ell_{\max}$ only once at the beginning of the SQP method. As snapshots $y = y(\mathbf{u})$, $\mathbf{u} = (u_1, \ldots, u_N)$, we take solutions from the bilinear problem (2.1) for different controls $\mathbf{u} \in \mathcal{D}$. Then we apply Algorithm 3 without the step 4. $\Diamond$

For Algorithm 3 we have proved the next convergence theorem.

Theorem 4.12. *Let $x^* \in X_{ad}$ be a local solution to $(\mathbf{P})$, p^* the associated Lagrange multiplier, and $z^* = (x^*, p^*)$. Suppose that*

- **(A1)** *the starting value (x^0, p^0) of Algorithm 3 is sufficiently close to (x^*, p^*),*
- **(A2)** *the optimality system (3.9) admits a (unique) solution $(\bar{y}, \bar{u}, \bar{p})$ so that $\bar{u} > u_a^k$ in Ω a.e. (inactive $\bar{u}$),*
- **(A3)** *for sufficiently large $\ell \le \ell_{\max}$ the optimality system (4.7) admits a (unique) solution $(\bar{y}^\ell, \bar{u}^\ell, \bar{p}^\ell)$ so that $\bar{u}^\ell > u_a^k$ in Ω a.e. (inactive $\bar{u}^\ell$)*

Let the iterates $\{z^k\}_{k\in\mathbb{N}}$, $z^k = (x^k, p^k)$, be generated by Algorithm 3. Then, $\lim_{k\to\infty} z^k = z^*$ in $X \times V$. In particular, we obtain superlinear and quadratic rate of convergence:

$$\|z^{k+1} - z^*\|_{X\times V} \le c_k \|z^k - z^*\|_{X\times V} \qquad \text{for all } k \text{ if } q \in (0,1),$$
$$\|z^{k+1} - z^*\|_{X\times V} \le c \|z^k - z^*\|^2_{X\times V} \qquad \text{for all } k \text{ if } q = 1,$$

where c_k satisfies $\lim_{k\to\infty} c_k = 0$ and c is a positive constant independent of k.

Remark 4.13.
1) Assumption **(A1)** ensures that the iterates (x^k, p^k) belong to a neighborhood of (x^*, p^*), where the convergence of the SQP method is ensured without any globalization strategy. In particular, at each level k of the SQP method the linear-quadratic optimal control problem $(\mathbf{P}^{k,\beta})$ admits a unique solution for $\beta = 1$ and we can choose $t_k = 1$.
2) If Assumption **(A2)** and **(A3)** do not hold, we have to deal with the variational inequalities (3.9c) and (4.7c), respectively. Thus, (3.9) and (4.7) are generalized equations. We have to apply the theory of Newton methods for generalized equations; see [1, 11, 32, 38]. $\Diamond$

5. Numerical experiments

We present two examples concerning a-posteriori error estimates for POD. The numerical tests are executed on a standard 3.0 GHz desktop PC. We are using the Matlab 7.1 package including its integrated PDE Toolbox for the FE discretization.

Run 1. Let the domain Ω be given by

$$\Omega = \Big\{\mathbf{x} = (x_1, x_2) \Big| \frac{x_1^2}{0.8^2} + \frac{x_2^2}{0.7^2} < 1\Big\} \subset \mathbb{R}^2.$$

Moreover, we assume that Ω consists of two disjunct subdomains (Ω_1 and Ω_2), where Ω_2 is given as the quadrilateral with corners $(0.22, -0.28)$, $(0.35, 0.34)$, $(-0.30, 0.41)$, and $(-0.18, -0.32)$, and $\Omega_1 = \Omega \setminus \Omega_2$; see left plot of Figure 5.1. We choose two characteristic functions as shape functions for the control, i.e., $N = 2$ and $b_i = \chi_{\Omega_i}$ for $i = 1, 2$. In Ω_1 let $\kappa = 0.8$ and $f = 2000$, whereas $\kappa = 1.12$ and $f = 10000$ in Ω_2. Moreover, we set $\eta = 50$ and $g = 200$. The domain is discretized by a FE grid that consists of 5788 degrees of freedom. In the context of Remark 4.1 we utilize 260

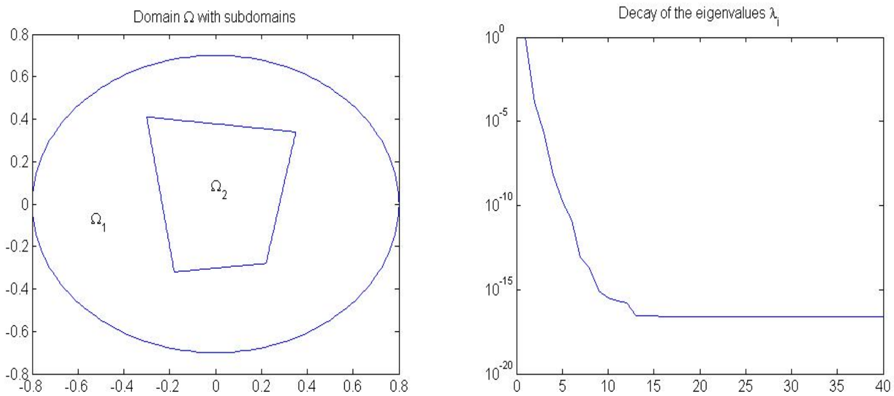

FIGURE 5.1. Run 1: Domain Ω with subdomains (left plot) and decay of the 40 largest eigenvalues.

k	$\varepsilon_k \, \|L'(x^k, p^k)\|_{X\times V}$	$\|\zeta^\ell\|_{L^2(\Omega)}/\sigma$	ℓ
0	0.5000	0.4284	15
1	0.5000	0.2786	15
2	0.5000	0.1848	8
3	0.5000	0.1976	4
4	0.3521	0.0085	6
5	$1.83 \cdot 10^{-5}$	$1.64 \cdot 10^{-5}$	13
6	$6.68 \cdot 10^{-11}$	–	–

TABLE 5.1. Run 1: stopping criterium $\varepsilon_k \, \|L'(x^k, p^k)\|_{X\times V}$, a-posteriori error estimator and numbers ℓ of POD ansatz functions for each SQP iteration k ($\sigma = 0.01$).

snapshots computed on an equidistant grid for $\mathbf{u} = (u_1, u_2) \in [8, 13] \times [0.2, 2] =: \mathcal{D}$ and choose $\ell_{max} = 40$. The computation of the POD basis needs 56 seconds. The 40 largest eigenvalues of the eigenvalue problem in the POD computation are shown in the right plot of Figure 5.1. Notice that the relative error between the finite element solution to (2.1) and the POD solution for $\ell = \ell_{max}$ is about $6.43 \cdot 10^{-9}$.

For the cost functional let $\Omega_m = \Omega_2$, y_d be the solution to (2.1) for $u = 11b_1 + 0.6b_2$, and $\sigma = 0.01$. In Algorithm 3 we choose tol $= 10^{-6}$ (step 1) and $q = 1/2$ (step 3). We initialize the SQP algorithm as follows: $u^0 = 9b_1 + b_2$, y^0 is the solution to (2.1) for $u = u^0$ and p^0 solves the adjoint equation (2.3) with $y^* = y^0$ and $u^* = u^0$. In each SQP iteration, we solve the reduced system (4.7) to obtain $\bar{u}^\ell$. Then, we evaluate $\tilde{y}^\ell$ and $\tilde{p}^\ell$ solving (3.12) and (3.11), respectively, with $(y^p, u^p, p^p) = (\tilde{y}^\ell, \bar{u}^\ell, \tilde{p}^\ell)$. If the a-posteriori error estimator ensures a small error for $\|\bar{u} - \bar{u}^\ell\|_{L^2(\Omega)}$, we set $x^{k+1} = x^k + (\tilde{y}^\ell, \bar{u}^\ell)$ and $p^{k+1} = p^k + \tilde{p}^\ell$.

The SQP method stops after six iterations and requires 72 seconds. In Table 5.1 it is shown how many POD ansatz functions are used in each SQP iteration. Notice that the relative error between the optimal control computed with a FE discretization and the optimal control computed by the POD scheme is lower than 0.001 %.

k	$\varepsilon_k \, \|L'(x^k, p^k)\|_{X\times V}$	$\|\zeta^\ell\|_{L^2(\Omega)}/\sigma$	ℓ
0	0.5000	0.3014	16
1	0.5000	0.3435	15
2	0.5000	0.1796	8
3	0.5000	0.2120	5
4	0.3234	0.0116	6
5	$1.3762 \cdot 10^{-5}$	$1.0324 \cdot 10^{-5}$	11
6	$3.2931 \cdot 10^{-11}$	–	–

TABLE 5.2. Run 1: stopping criterium $\varepsilon_k \, \|L'(x^k, p^k)\|_{X\times V}$, a-posteriori error estimator and numbers ℓ of POD ansatz functions for each SQP iteration k (noisy measurements).

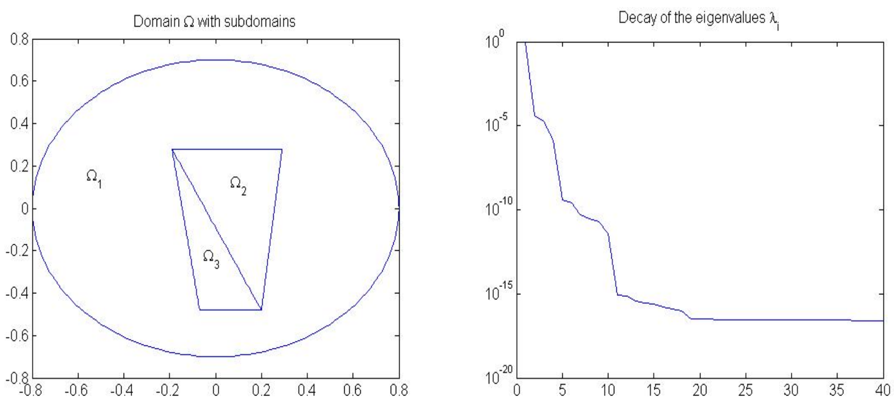

FIGURE 5.2. Run 2: Domain Ω with subdomains (left plot) and decay of the 40 largest eigenvalues.

However, the SQP method with FE discretization takes us 240 seconds for 6 SQP iterations.

Now let us consider noisy measurements y_d (with a random noise of 1%). The SQP method stops again after six SQP iterations. The results are shown in Table 5.2. Again we are very close to the optimal control which is obtained with the FE-based optimization algorithm with a relative error less than 0.001 %. $\Diamond$

Run 2. Let Ω consist of 3 subdomains; see Figure 5.2. We choose two characteristic functions as shape functions for the control, i.e., $N = 3$ and $b_i = \chi_{\Omega_i}$ for $i = 1, 2, 3$. In Ω_1 let $\kappa = 0.642$ and $f = 2000$, in Ω_2 we have $\kappa = 0.936$ and $f = 1000$, whereas $\kappa = 0.84$ and $f = 1000$ in Ω_2. Moreover, we set $\eta = 45$ and $g = 1125$. The domain is discretized by a FE grid that consists of 7099 degrees of freedom. We apply a POD computation (using 336 snapshots computed on an equidistant grid for $\bar{u} = (u_1, u_2, u_3) \in [1.7, 3.1] \times [0.1, 1.1] \times [0.3, 1.5] =: \mathcal{D}$ and choose $\ell_{max} = 40$. The 40 largest eigenvalues of the eigenvalue problem in the POD computation are shown in the right plot of Figure 5.2. The relative error between the POD solution for $\ell = \ell_{max}$ and the FE solution is $2.131 \cdot 10^{-10}$. The computing time for the snapshot computation is 91 seconds, the computation of the 40 POD basis functions costs 4 seconds.

k	$\varepsilon_k \, \|L'(x^k,p^k)\|_{X\times V}$	$\|\zeta^\ell\|_{L^2(\Omega)}/\sigma$	ℓ
0	0.5000	0.0093	10
1	0.5000	0.0212	10
2	0.1999	0.0186	10
3	$1.2373 \cdot 10^{-6}$	$1.1710 \cdot 10^{-6}$	34
4	$1.6669 \cdot 10^{-16}$	–	–

Table 5.3. Run 2: stopping criterium $\varepsilon_k \, \|L'(x^k,p^k)\|_{X\times V}$, a-posteriori error estimator and numbers ℓ of POD ansatz functions for each SQP iteration k.

For the cost functional let $\Omega_m = \Omega_3$, y_d be the solution to (2.1) for $u = 2.5b_1 + 0.54b_2 + 0.8b_3$, and $\sigma = 0.01$. In Algorithm 3 we choose tol $= 10^{-6}$ (step 1) and $q = 1$ (step 3). The SQP methods stops after four SQP iterations and requires 77 seconds. We observe quadratic rate of convergence in the last two iterations. The results are stated in Table 5.3. Again we are very close to the optimal control which is obtained with the FE-based optimization algorithm with a relative error less than 0.001 %. ◊

Appendix

Proof of Corrolary 2.5. From (2.3) we conclude that p^* satisfies the variational equation

$$\int_\Omega \kappa \nabla p^* \cdot \nabla \varphi + u^* p^* \varphi \, \mathrm{d}\mathbf{x} + \eta \int_\Gamma p^* \varphi \, \mathrm{d}\mathbf{x} = \int_{\Omega_m} (y_d - y^*) \varphi \, \mathrm{d}\mathbf{x} \quad \text{for all } \varphi \in V.$$

Choosing $\varphi = p^*$ and using $\kappa \geq \underline{\kappa}$, $u^* \geq u_a$ in Ω a.e. we obtain

$$\min\left(\underline{\kappa}, u_a\right) \|p^*\|_V^2 \leq \|y^* - y_d\|_{L^2(\Omega_m)} \|p^*\|_V.$$

Setting $C_1 = \min(\underline{\kappa}, u_a) > 0$ we find $\|p^*\|_V \leq \|y^* - y_d\|_{L^2(\Omega_m)}/C_1$. Utilizing (2.4) and (2.6) the second-order sufficient optimality conditon holds at (x^*, p^*) if

$$\|y^* - y_d\|_{L^2(\Omega_m)} \leq \varepsilon := \frac{\sigma C_1}{8 C_{ker} C_{emb}^2},$$

where the constants C_{emb} and C_{ker} were introduced in (2.4) and Proposition 2.2, respectively. □

Proof of Lemma 3.3. Let $r \in V'$ and $u \in U$ be chosen arbitrarily. We set $y = \mathcal{S}u \in V$. Recall that $u^k \in U_{ad}$ and $y^k \in V \subset L^6(\Omega)$. By Proposition 2.1 there exists a unique $v \in V \subset L^6(\Omega)$ solving (3.8). Utilizing (3.8) and (3.7) we derive

$$\begin{aligned} \langle \mathcal{S}^\star r, u \rangle_{U',U} = \langle r, \mathcal{S}u \rangle_{V',V} = \langle r, y \rangle_{V',V} &= \int_\Omega \kappa \nabla v \cdot \nabla y + u^k v y \, \mathrm{d}\mathbf{x} + \eta \int_\Gamma v y \, \mathrm{d}\mathbf{x} \\ &= \int_\Omega \kappa \nabla y \cdot \nabla v + u^k y v \, \mathrm{d}\mathbf{x} + \eta \int_\Gamma y v \, \mathrm{d}\mathbf{x} = -\int_\Omega u y^k v \, \mathrm{d}\mathbf{x}. \end{aligned}$$

Hence, $\mathcal{S}^\star r = -y^k v \in L^2(\Omega) \subset U'$. □

Proof of Proposition 3.4. From (3.9a) and (3.12) we conclude that $y_\delta = \bar y - y^p$ satisfies the equation

$$\int_\Omega \kappa \nabla y_\delta \cdot \nabla\varphi + u^k y_\delta \varphi \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma y_\delta \varphi \,\mathrm{d}\mathbf{s} = \int_\Omega y^k (u^p - \bar u)\varphi \,\mathrm{d}\mathbf{x}.$$

By assumption, the sequence $\|(y^k, u^k, p^k)\|_{X\times V}$ is uniformly bounded. Choosing $\varphi = y_\delta$ and using $u^k \geq u_a$ we obtain

$$\|\bar y - y^p\|_V \leq C_1 \|\bar u - u^p\|_U \tag{A.1}$$

with a constant $C_1 > 0$ independent of the level k of the iteration. Analogously, $p_\delta = \bar p - p^p$ satisfies

$$\int_\Omega \kappa \nabla p_\delta \cdot \nabla\varphi + u^k p_\delta \varphi \,\mathrm{d}\mathbf{x} + \eta \int_\Gamma p_\delta \varphi \,\mathrm{d}\mathbf{s} = \int_\Omega \big(P_m(\bar y - y^p) + \beta(\bar u - u^p)p^k\big)\varphi \,\mathrm{d}\mathbf{x}$$

Using (A.1) this implies

$$\|\bar p - p^p\|_V \leq C_2 \|\bar u - u^p\|_U \tag{A.2}$$

where $C_2 > 0$ is independent of k. Thus, (A.1) and (A.2) yields the claim with $C_p = \sqrt{C_1^2 + C_2^2}$. □

Proof of Theorem 3.5. Choosing $u = \bar u$ in (3.9c) and $u = u^p$ in (3.10) we obtain

$$\begin{aligned} 0 &\leq \big\langle p^k \bar y + y^k \bar p + \sigma(u^k + \bar u) - p^k y^p - y^k p^p - \sigma(y^k + y^p) - \zeta, u^p - \bar u\big\rangle_{L^2(\Omega)} \\ &= -\sigma \|\bar u - u^p\|^2_{L^2(\Omega)} + \big\langle p^k(\bar y - y^p) + y^k(\bar p - p^p)) + \zeta, u^p - \bar u\big\rangle_{L^2(\Omega)}. \end{aligned}$$

Let $y = \mathcal{S}(y^p - \bar y)$. Using Lemma 3.3 and (3.9b) we find

$$\begin{aligned} &\big\langle y^k \bar p, u^p - \bar u\big\rangle_{L^2(\Omega)} = \int_\Omega (u^p - \bar u) y^k \bar p \,\mathrm{d}\mathbf{x} \\ &= -\int_\Omega \kappa \nabla y \cdot \nabla \bar p + u^k y \bar p \,\mathrm{d}\mathbf{x} - \eta \int_\Gamma y \bar p \,\mathrm{d}\mathbf{x} = -\int_\Omega \kappa \nabla \bar p \cdot \nabla y + u^k \bar p y \,\mathrm{d}\mathbf{x} - \eta \int_\Gamma \bar p y \,\mathrm{d}\mathbf{x} \\ &= \int_\Omega \big(\bar y p^k + P_m(y^k + \bar y - y_d)\big) y \,\mathrm{d}\mathbf{x} = \big\langle \bar u p^k + P_m(y^k + \bar y - y_d), \mathcal{S}(u^p - \bar u)\big\rangle_{V',V} \\ &= \big\langle \mathcal{S}^\star\big(\bar u p^k + P_m(y^k + \bar y - y_d)\big), u^p - \bar u\big\rangle_{L^2(\Omega)}. \end{aligned}$$

Analogously, we obtain

$$\big\langle y^k p^p, u^p - \bar u\big\rangle_{L^2(\Omega)} = \big\langle \mathcal{S}^\star\big(u^p p^k + P_m(y^k + y^p - y_d)\big), u^p - \bar u\big\rangle_{L^2(\Omega)}.$$

Therefore,

$$\begin{aligned} \big\langle y^k(\bar p - p^p), u^p - \bar u\big\rangle_{L^2(\Omega)} &= \big\langle y^k \bar p, u^p - \bar u\big\rangle_{L^2(\Omega)} - \big\langle y^k p^p, u^p - \bar u\big\rangle_{L^2(\Omega)} \\ &= \big\langle \mathcal{S}^\star\big(P_m(\bar y - y^p) + (\bar u - u^p)p^k\big), u^p - \bar u\big\rangle_{L^2(\Omega)}. \end{aligned}$$

Recall that $\mathcal{S}(u^p - \bar u) = y^p - \bar y$. Hence,

$$\begin{aligned} \big\langle y^k(\bar p - p^p), u^p - \bar u\big\rangle_{L^2(\Omega)} &= -\|\bar y - y^p\|^2_{L^2(\Omega_m)} + \big\langle (\bar u - u^p)p^k, y^p - \bar y\big\rangle_{L^2(\Omega)} \\ &\leq \big\langle p^k(y^p - \bar y), \bar u - u^p\big\rangle_{L^2(\Omega)}. \end{aligned}$$

Summarizing, we obtain

$$\begin{aligned}0 &\le -\sigma \|\bar u - u^p\|^2_{L^2(\Omega)} + \big\langle y^k(\bar p - p^p) - p^k(\bar y - y^p) - \zeta, u^p - \bar u\big\rangle_{L^2(\Omega)}\\ &\le -\sigma \|\bar u - u^p\|^2_{L^2(\Omega)} + \big\langle p^k(y^p - \bar y) - p^k(\bar y - y^p) - \zeta, \bar u - u^p\big\rangle_{L^2(\Omega)}\\ &= -\sigma \|\bar u - u^p\|^2_{L^2(\Omega)} - \langle \zeta, u^p - \bar u\rangle_{L^2(\Omega)}.\end{aligned}$$

Consequently, we have

$$\sigma \|\bar u - u^p\|^2_{L^2(\Omega)} \le -\langle \zeta, u^p - \bar u\rangle_{L^2(\Omega)} \le \|\zeta\|_{L^2(\Omega)}\|u^p - \bar u\|_{L^2(\Omega)},$$

which gives the claim. □

Proof of Proposition 4.2. We make use of the decomposition

$$y^\ell(\mathbf{u}) - y(\mathbf{u}) = y^\ell(\mathbf{u}) - \mathcal{P}^\ell y(\mathbf{u}) + \mathcal{P}^\ell y(\mathbf{u}) - y(\mathbf{u}) = \vartheta^\ell(\mathbf{u}) + \varrho^\ell(\mathbf{u}),$$

where $\vartheta^\ell(\mathbf{u}) = u^\ell(\mathbf{u}) - \mathcal{P}^\ell u(\mathbf{u})$ and $\varrho^\ell(\mathbf{u}) = \mathcal{P}^\ell u(\mathbf{u}) - u(\mathbf{u})$. From (4.3) we derive

$$\int_{\mathcal{D}} \|\varrho^\ell(\mathbf{u})\|^2_V \,\mathrm{d}\mathbf{u} = \sum_{i=\ell+1}^{\infty} \lambda_i. \tag{A.3}$$

Next we estimate $\vartheta^\ell(\mathbf{u}) \in V^\ell$. Using (3.9a) and (4.5) we obtain

$$\begin{aligned}&\int_\Omega \kappa\nabla\vartheta^\ell(\mathbf{u})\cdot\nabla\psi + u^k\vartheta^\ell(\mathbf{u})\psi\,\mathrm{d}\mathbf{x} + \eta\int_\Gamma \vartheta^\ell(\mathbf{u})\psi\,\mathrm{d}\mathbf{x}\\ &\quad= \int_\Omega \kappa\nabla\varrho^\ell(\mathbf{u})\cdot\nabla\psi + u^k\varrho^\ell(\mathbf{u})\psi\,\mathrm{d}\mathbf{x} + \eta\int_\Gamma \varrho^\ell(\mathbf{u})\psi\,\mathrm{d}\mathbf{x} \quad \text{for all } \psi \in V^\ell.\end{aligned}$$

Choosing $\psi = \vartheta^\ell(\mathbf{u}) \in V^\ell$ and using $\kappa \ge \underline{\kappa}$, $u^k \ge u_a$, (2.4) we obtain

$$\min\big(\underline{\kappa}, u_a\big)\|\vartheta^\ell(\mathbf{u})\|^2_V \le \big(\kappa + C_{emb}\,\|u^k\|_{L^2(\Omega)} + \eta C_\Gamma^2\big)\,\|\varrho^\ell(\mathbf{u})\|_V\|\vartheta^\ell(\mathbf{u})\|_V \tag{A.4}$$

where the constant $C_\Gamma > 0$ was introduced in the proof of Proposition 4.2. From the Young's inequality

$$ab \le \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon} \quad \text{for } a, b \in \mathbb{R} \text{ and } \varepsilon > 0 \tag{A.5}$$

and (A.3) we obtain the error estimate. As $\{\psi_i\}_{i\in\mathbb{N}}$ is a complete orthogonal basis in the separable Hilbert space V, we have

$$0 \le \lim_{\ell\to\infty} \|y^\ell - y\|_{L^2(\mathcal{D};V)} \le C \lim_{\ell\to\infty}\sum_{i=\ell+1}^{\infty}\lambda_i = 0$$

which gives the claim. □

Proof of Proposition 4.4. We make the decomposition

$$p^\ell(\mathbf{u}) - p(\mathbf{u}) = p^\ell(\mathbf{u}) - \mathcal{P}^\ell p(\mathbf{u}) + \mathcal{P}^\ell p(\mathbf{u}) - p(\mathbf{u}) = \theta^\ell(\mathbf{u}) + \rho^\ell(\mathbf{u}),$$

where $\theta^\ell(\mathbf{u}) = p^\ell(\mathbf{u}) - \mathcal{P}^\ell p(\mathbf{u})$ and $\rho^\ell(\mathbf{u}) = \mathcal{P}^\ell p(\mathbf{u}) - p(\mathbf{u})$ holds. Then we find

$$\begin{aligned}&\int_\Omega \kappa\nabla\theta^\ell(\mathbf{u})\cdot\nabla\psi + u^k\theta^\ell(\mathbf{u})\psi\,\mathrm{d}\mathbf{x} + \eta\int_\Gamma \theta^\ell(\mathbf{u})\psi\,\mathrm{d}\mathbf{x}\\ &= \int_\Omega P_m\big(u(\mathbf{u}) - u^\ell(\mathbf{u})\big) + \kappa\nabla\rho^\ell(\mathbf{u})\cdot\nabla\psi + u^k\rho^\ell(\mathbf{u})\psi\,\mathrm{d}\mathbf{x} + \eta\int_\Gamma \rho^\ell(\mathbf{u})\psi\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Taking $\psi = \theta^\ell(\mathbf{u}) \in V^\ell$ and setting $C_1 = \min\big(\underline{\kappa}, u_a\big)$ we conclude

$$\begin{aligned}
&C_1 \|\theta^\ell(\mathbf{u})\|_V^2 \\
&\quad \le \|u(\mathbf{u}) - u^\ell(\mathbf{u})\|_{L^2(\Omega_m)} \|\theta^\ell(\mathbf{u})\|_{L^2(\Omega)} + \|\kappa\|_{L^\infty(\Omega)} \|\rho^\ell(\mathbf{u})\|_V \|\theta^\ell(\mathbf{u})\|_V \\
&\qquad + \|u^k\|_{L^2(\Omega)} \|\rho^\ell(\mathbf{u})\|_{L^4(\Omega)} \|\theta^\ell(\mathbf{u})\|_{L^4(\Omega)} + \eta \|\rho^\ell(\mathbf{u})\|_{L^2(\Gamma)} \|\theta^\ell(\mathbf{u})\|_{L^2(\Gamma)}.
\end{aligned}$$

Recall that the constants C_{emb} and C_Γ were introduced in (2.4) and in the proof of Proposition 4.2. Using (A.5) we find

$$\begin{aligned}
C_1 \|\theta^\ell(\mathbf{u})\|_V^2 &\le \big(\|u(\mathbf{u}) - u^\ell(\mathbf{u})\|_V + C_2 \|\rho^\ell(\mathbf{u})\|_V\big) \|\theta^\ell(\mathbf{u})\|_V \\
&\le \frac{1}{2C_1} \big(\|u(\mathbf{u}) - u^\ell(\mathbf{u})\|_V + C_2 \|\rho^\ell(\mathbf{u})\|_V\big)^2 + \frac{C_1}{2} \|\theta^\ell(\mathbf{u})\|_V^2 \\
&\le \frac{1}{4C_1} \Big(\|u(\mathbf{u}) - u^\ell(\mathbf{u})\|_V^2 + C_2^2 \|\rho^\ell(\mathbf{u})\|_V^2\Big) + \frac{C_1}{2} \|\theta^\ell(\mathbf{u})\|_V^2
\end{aligned}$$

with $C_2 = \|\kappa\|_{L^\infty(\Omega)} + C_{emb}^2 \|u^k\|_{L^2(\Omega)} + \eta C_\Gamma^2$. Thus

$$\begin{aligned}
\|\theta^\ell(\mathbf{u})\|_V^2 &\le \frac{1}{2C_1^2} \Big(\|u(\mathbf{u}) - u^\ell(\mathbf{u})\|_V^2 + C_2^2 \|\rho^\ell(\mathbf{u})\|_V^2\Big) \\
&\le C_3 \Big(\|u(\mathbf{u}) - u^\ell(\mathbf{u})\|_V^2 + \|\rho^\ell(\mathbf{u})\|_V^2\Big)
\end{aligned}$$

where $C_3 = \max(1, C_2^2)/(2C_1^2)$. From

$$\|p^\ell - p\|_{L^2(\mathcal{D};V)}^2 = \int_{\mathcal{D}} \|p^\ell - p\|_V^2 \,\mathrm{d}\mathbf{u} \le 2 \int_{\mathcal{D}} \|\theta^\ell(\mathbf{u})\|_V^2 + \|\rho^\ell(\mathbf{u})\|_V^2 \,\mathrm{d}\mathbf{u}$$

and $\rho^\ell(\mathbf{u}) = \mathcal{P}^\ell(\mathbf{u}) - p(\mathbf{u})$ we infer the estimate. Since $\{\psi_i\}_{i\in\mathbb{N}}$ is a complete orthogonal basis in the separable Hilbert space V, we infer from Proposition 4.2 that $\lim_{\ell\to\infty} \|p^\ell - p\|_{L^2(\mathcal{D};V)} = 0$, which gives the claim. □

Proof of Proposition 4.6. From (3.9c) and (4.7c) we find

$$\int_\Omega \mathcal{G}^{k,\beta}(\bar y, \bar u, \bar p)(\bar u^\ell - \bar u)\,\mathrm{d}\mathbf{x} \ge 0 \quad \text{and} \quad \int_\Omega \mathcal{G}^{k,\beta}(\bar y^\ell, \bar u^\ell, \bar p^\ell)(\bar u - \bar u^\ell)\,\mathrm{d}\mathbf{x} \ge 0.$$

Adding both inequalities we deduce

$$\sigma \|\bar u - \bar u^\ell\|_{L^2(\Omega)}^2 \le \int_\Omega \Big(\beta(p^k(\bar y - \bar y^\ell)) + y^k(\bar p - \bar p^\ell)\Big)(\bar u^\ell - \bar u)\,\mathrm{d}\mathbf{x}. \tag{A.6}$$

We estimate

$$
\begin{aligned}
&\int_\Omega y^k(\bar p-\bar p^\ell)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x} = \int_\Omega y^k\bar p(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x} - \int_\Omega y^k\bar p^\ell(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}\\
&= \int_\Omega \mathcal{S}^\star\big(P_m(y^k+\bar y-y_d)+\bar u p^k\big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}\\
&\quad+\int_\Omega (\mathcal{S}^\ell)^\star\big(P_m(y^k+\bar y^\ell-y_d)+\bar u^\ell p^k\big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}\\
&= \int_\Omega \big(\mathcal{S}^\star-(\mathcal{S}^\ell)^\star\big)\big(P_m(y^k-y_d)\big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}\\
&\quad+\int_\Omega \Big(\mathcal{S}^\star\big(P_m\bar y+\bar u p^k\big)-(\mathcal{S}^\ell)^\star\big(P_m\bar y^\ell+\bar u^\ell p^k\big)\Big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}\\
&= \int_\Omega \Big(\big(\mathcal{S}^\star-(\mathcal{S}^\ell)^\star\big)\big(P_m(y^k-y_d)\big)+\mathcal{S}^\star(p^k\bar u)-(\mathcal{S}^\ell)^\star(p^k\bar u^\ell)\Big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}\\
&\quad+\int_\Omega \Big(\mathcal{S}^\star P_m\bar y-(\mathcal{S}^\ell)^\star P_m\bar y^\ell\Big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Recall that $\bar y=\hat y_k+\mathcal{S}\bar u$ and $\bar y^\ell=\hat y_k+\mathcal{S}^\ell\bar u^\ell$. Therefore,

$$
\begin{aligned}
&\int_\Omega \big(\mathcal{S}^\star P_m\bar y-(\mathcal{S}^\ell)^\star P_m\bar y^\ell\big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}\\
&\quad= \int_\Omega \big(\mathcal{S}^\star P_m(\hat y_k+\mathcal{S}\bar u)-(\mathcal{S}^\ell)^\star P_m(\hat y_k^\ell+\mathcal{S}^\ell\bar u^\ell)\big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

From

$$
\mathcal{S}^\star P_m\mathcal{S}\bar u-(\mathcal{S}^\ell)^\star P_m\mathcal{S}^\ell\bar u^\ell=\mathcal{S}^\star P_m\mathcal{S}\bar u-(\mathcal{S}^\ell)^\star P_m\mathcal{S}^\ell\bar u+(\mathcal{S}^\ell)^\star P_m\mathcal{S}^\ell(\bar u-\bar u^\ell)
$$

and

$$
\int_\Omega \big((\mathcal{S}^\ell)^\star P_m\mathcal{S}^\ell(\bar u-\bar u^\ell)\big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}=-\|\mathcal{S}^\ell(\bar u-\bar u^\ell)\|^2_{L^2(\Omega_m)}\le 0.
$$

we infer that

$$
\begin{aligned}
&\int_\Omega \big(\mathcal{S}^\star P_m\bar y-(\mathcal{S}^\ell)^\star P_m\bar y^\ell\big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}\\
&\quad= \int_\Omega \big(\mathcal{S}^\star P_m\hat y_k-(\mathcal{S}^\ell)^\star P_m\hat y_k^\ell+\mathcal{S}^\star P_m\mathcal{S}\bar u-(\mathcal{S}^\ell)^\star P_m\mathcal{S}^\ell\bar u^\ell\big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}\\
&\quad= \int_\Omega \big(\mathcal{S}^\star P_m(\hat y_k+\mathcal{S}\bar u)-(\mathcal{S}^\ell)^\star P_m(\hat y_k^\ell+\mathcal{S}^\ell\bar u)\big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x}\\
&\quad= \int_\Omega \big(\mathcal{S}^\star P_m\bar y-(\mathcal{S}^\ell)^\star P_m\tilde y^\ell\big)(\bar u^\ell-\bar u)\,\mathrm{d}\mathbf{x},
\end{aligned}
$$

where $\tilde{y}^\ell = \hat{y}_\ell^k + \mathcal{S}\bar{u}$. Consequently,

$$\begin{aligned}
&\int_\Omega y^k(\bar{p}-\bar{p}^\ell)(\bar{u}^\ell-\bar{u})\,\mathrm{d}\mathbf{x}\\
&= \int_\Omega \Big(\big(\mathcal{S}^\star-(\mathcal{S}^\ell)^\star\big)\big(P_m(y^k-y_d)\big)+\mathcal{S}^\star(p^k\bar{u})-(\mathcal{S}^\ell)^\star(p^k\bar{u}^\ell)\Big)(\bar{u}^\ell-\bar{u})\,\mathrm{d}\mathbf{x}\\
&\quad+\int_\Omega \big(\mathcal{S}^\star P_m\bar{y}-(\mathcal{S}^\ell)^\star P_m\bar{y}^\ell\big)(\bar{u}^\ell-\bar{u})\,\mathrm{d}\mathbf{x}\\
&= \int_\Omega \Big(\big(\mathcal{S}^\star-(\mathcal{S}^\ell)^\star\big)\big(P_m(y^k-y_d)\big)+\mathcal{S}^\star(p^k\bar{u})-(\mathcal{S}^\ell)^\star(p^k\bar{u}^\ell)\Big)(\bar{u}^\ell-\bar{u})\,\mathrm{d}\mathbf{x}\\
&\quad+\int_\Omega \big(\mathcal{S}^\star P_m\bar{y}-(\mathcal{S}^\ell)^\star P_m\tilde{y}^\ell\big)(\bar{u}^\ell-\bar{u})\,\mathrm{d}\mathbf{x}\\
&= \int_\Omega \mathcal{S}^\star\big(P_m(y^k+\bar{y}-y_d)+\bar{u}p^k\big)(\bar{u}^\ell-\bar{u})\,\mathrm{d}\mathbf{x}\\
&\quad-\int_\Omega (\mathcal{S}^\ell)^\star\big(P_m(y^k+\tilde{y}^\ell-y_d)+\bar{u}^\ell p^k\big)(\bar{u}^\ell-\bar{u})\,\mathrm{d}\mathbf{x}=\int_\Omega y^k(\bar{p}-\tilde{p}^\ell)(\bar{u}^\ell-\bar{u})\,\mathrm{d}\mathbf{x},
\end{aligned}$$

where $\tilde{p}^\ell$ solves (4.6) with $y^\ell = \tilde{y}^\ell$ and $u = \bar{u}$. Inserting this estimate into (A.6) we find

$$\begin{aligned}
&\sigma\|\bar{u}-\bar{u}^\ell\|^2_{L^2(\Omega)}\\
&\le \int_\Omega \big(\beta p^k(\bar{y}-\bar{y}^\ell)+y^k(\bar{p}-\tilde{p}^\ell)\big)(\bar{u}^\ell-\bar{u})\,\mathrm{d}\mathbf{x}\\
&\le C_1\big(\|\bar{y}-\bar{y}^\ell\|_{L^2(\Omega)}+\|\bar{y}-\bar{y}^\ell\|_{L^3(\Omega)}+\|\bar{p}-\tilde{p}^\ell\|_{L^3(\Omega)}\big)\|\bar{u}^\ell-\bar{u}\|_{L^2(\Omega)}
\end{aligned}$$

with

$$C_1 = \max\big(\beta\|p^k\|_{L^6(\Omega)}, \|y^k\|_{L^6(\Omega)}\big)$$

Since Ω is bounded, there exists a constant $C_2 > 0$ satisfying

$$\|\bar{u}-\bar{u}^\ell\|^2_{L^2(\Omega)} \le C_2\Big(\|\bar{y}-\bar{y}^\ell\|_{L^3(\Omega)}+\|\bar{p}-\tilde{p}^\ell\|_{L^3(\Omega)}\Big) \tag{A.7}$$

which was the claim. □

References

[1] W. Alt. The Lagrange-Newton method for infinite-dimensional optimization problems. *Numer. Funct. Anal. Optim.*, 11:201-224, 1990.

[2] K. Afanasiev and M. Hinze. Adaptive control of a wake flow using proper orthogonal decomposition. *Lect. Notes Pure Appl. Math.*, 216:317-332, 2001.

[3] A.C.ntoulas. *Approximation of Large-Scale Dynamical Systems.* Advances in Design and Control, SIAM, Philadelphia, 2005.

[4] N. Arada, E. Casas, and F. Tröltzsch. Error estimates for the numerical approximation of a semilinear elliptic control problem. *Computational Optimization and Applications*, 23:201–229, 2002.

[5] J.A. Atwell, J.T. Borggaard, and B.B. King. Reduced order controllers for Burgers' equation with a nonlinear observer. *Int. J. Appl. Math. Comput. Sci.*, 11:1311-1330, 2001.

[6] E. Arian, M. Fahl, and E.W. Sachs. Trust-region proper orthogonal decomposition for flow control. Technical Report 2000-25, ICASE, 2000.

[7] P. Benner and E.S. Quintana-Ortí. Model reduction based on spectral projection methods. In Reduction of Large-Scale Systems, P. Benner, V. Mehrmann, D. C. Sorensen (eds.), Lecture Notes in Computational Science and Engineering, 45, 5-48, 2005.

[8] E. Casas and F. Tröltzsch. Error estimates for the finite-element approximation of a semilinear elliptic control problem. *Control and Cybernetics* 31:695-712, 2002.

[9] P. Deuflhard. *Newton Methods for Nonlinear Problems: Affine Invariance and Adaptive Algorithms*. Springer Series in Computational Mathematics, vol. 35, Springer-Verlag, 2004.

[10] F. Diwoky and S. Volkwein. Nonlinear boundary control for the heat equation utilizing proper orthogonal decomposition. *International Series of Numerical Mathematics*, 138:73-87, 2001.

[11] A.L. Dontchev. Local analysis of a Newton-type method based on partial linearization. In *proc. of the AMS-SIAM Summer Seminar in Applied Mathematics on Mathematics and Numerical Analysis: Real Number Algorithms*, edited by J. Renegar, M. Shub and S. Smale. AMS, *Lectures in Appl. Math.*, 32:295-306, 1996.

[12] L.C. Evans. *Partial Differential Equations*. Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, Rhode Island, 2002.

[13] T. Gänzler, S. Volkwein, and M. Weiser. SQP methods for parameter identification problems arising in hyperthermia. *Optimization Methods and Software*, 21:869-887, 2006.

[14] M.A. Grepl, Y. Maday, N.C. Nguyen und A.T. Patera. Efficient reduced-basis treatment of nonaffine and nonlinear partial differential equations. *ESAIM: Math. Model. Numer. Anal.*, 41:575-605, 2007.

[15] M. Hintermüller. On a globalized augmented Lagrangian SQP-algorithm for nonlinear optimal control problems with box constraints. In: *Fast solution methods for discretized optimization problems*. K.-H. Hoffmann, R.H.W. Hoppe, V. Schulz (eds.), International Series of Numerical Mathematics 138, Birkhuser publishers, Basel, 2001, pp. 139-153, 2001.

[16] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. *Optimization with PDE Constraints*. Mathematical Modelling: Theory and Applications, vol. 23, Springer Verlag, 2009.

[17] M. Hinze and S. Volkwein. Error estimates for abstract linear-quadratic optimal control problems using proper orthogonal decomposition. *Computational Optimization and Applications*, 39:319-345, 2008.

[18] P. Holmes, J.L. Lumley, and G. Berkooz. *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*. Cambridge Monographs on Mechanics. Cambridge University Press, 1996.

[19] M. Heinkenschloss and L.N. Vicente. Analysis of inexact trust-region SQP algorithms. *SIAM J. Optimization*, 12:283-302, 2001.

[20] K. Ito and S.S. Ravindran. A reduced basis method for control problems governed by PDEs. In W. Desch, F. Kappel, and K. Kunisch, eds., *Conrol and Estmation of Distributed Parameter Systems*. Proceedings of the International Conference in Vorau, 1996, Birkhäuser-Verlag, Basel, 126:153-168, 1998.

[21] M. Kahlbacher and S. Volkwein. Galerkin proper orthogonal decomposition methods for parameter dependent elliptic systems. *Discussiones Mathematicae: Differential Inclusions, Control and Optimization*, 27:95-117, 2007.

[22] A. Kröner and B. Vexler. A priori error estimates for elliptic optimal control problems with bilinear state equation. *Journal of Computational and Applied Mathematics*, 230:781-802, 2009.

[23] K. Kunisch and K. Ito. *Lagrange Multiplier Approach to Variational Problems and Applications*. Advances in Design and Control, SIAM, Philadelphia, 2008.

[24] K. Kunisch and S. Volkwein. Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics. SIAM J. Numerical Analysis, 40:492-515, 2002.

[25] K. Kunisch and S. Volkwein. Proper orthogonal decomposition for optimality systems. *ESAIM: Mathemematical Modelling and Numerical Analysis*, 42:1-23, 2008.

[26] S. Lall, J.E. Marsden, and S. Glavaski. A subspace approach to balanced truncation for model reduction of nonlinear control systems. *Int. J. Robust Nonlinear Control*, 12:519-535, 2002.

[27] H.V. Ly and H.T. Tran. Modeling and control of physical processes using proper orthogonal decomposition. *Mathematical and Computer Modeling*, 33:223-236, 2001.

[28] K. Malanowski, C. Büskens, H. Maurer. Convergence of approximations to nonlinear control problems. In: *Mathematical Programming with Data Perturbation*, ed.: A. V. Fiacco; Marcel Dekker, Inc., New York (1997), 253-284.

[29] H. Maurer and J. Zowe. First and second order necessary and sufficient optimality conditions for infinite-dimensional programming problems. *Mathematical Programming*, 16:98-110, 1979.

[30] A.T. Patera and G. Rozza. *Reduced Basis Approximation and A Posteriori Error Estimation for Parametrized Partial Differential Equations*. MIT Pappalardo Graduate Monographs in Mechanical Engineering, 2006.

[31] S.S. Ravindran. Adaptive reduced order controllers for a thermal flow system using proper orthogonal decomposition. *SIAM J. Sci. Comput.*, 28:1924-1942, 2002.
[32] S.M. Robinson. Strongly regular generalized equations Math. Oper. Res., 5:43-62, 1980.
[33] C.W. Rowley. Model reduction for fluids, using balanced proper orthogonal decomposition. *Int. J. on Bifurcation and Chaos*, 15:997-1013, 2005.
[34] M. Read and B. Simon. *Methods of Modern Mathematical Physics I: Functional Analysis*. Academic Press, Boston, 1980.
[35] L. Sirovich. Turbulence and the dynamics of coherent structures, parts I-III. *Quart. Appl. Math.*, XLV:561-590, 1987.
[36] T. Tonn, K. Urban, and S. Volkwein. Comparison of the reduced-basis and POD a-posteriori error estimators for an elliptic linear-quadratic optimal control problem. Submitted, 2010.
[37] F. Tröltzsch and S. Volkwein. POD a-posteriori error estimates for linear-quadratic optimal control problems. *Computational Optimization and Applications*, 44:83-115, 2009.
[38] F. Tröltzsch and S. Volkwein. The SQP method for control constrained optimal control of the Burgers equation. *ESAIM: Control. Optimisation and Calculus of Variations*, 6:649-674, 2001.
[39] M. Vallejos and A. Borzì. Multigrid optimization methods for linear and bilinear elliptic optimal control problems. *Computing*, 82:31-52, 2008.
[40] S. Volkwein. Mesh-independence of an augmented Lagrangian-SQP method in Hilbert spaces. *SIAM J. Control and Optimization*, 38:767-785, 2000.
[41] S. Volkwein. Augmented Lagrange-SQP methods with Lipschitz-continuous Lagrange multiplier updates. *SIAM J. Numerical Analysis*, 40:233-253, 2002.
[42] K. Willcox and J. Peraire. Balanced model reduction via the proper orthogonal decomposition. *American Institute of Aeronautics and Astronautics (AIAA)*, 2323-2330, 2002.
[43] K. Yoshida. *Functional Analysis*. Springer-Verlag, Berlin, 1995.

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