SpezialForschungsBereich F 32

Karl–Franzens Universität Graz
Technische Universität Graz
Medizinische Universität Graz

A note on the stable coupling of finite and boundary elements

O. Steinbach

SFB-Report No. 2009–074 November 2009

A–8010 GRAZ, HEINRICHSTRASSE 36, AUSTRIA

Supported by the
Austrian Science Fund (FWF)

SFB sponsors:

- **Austrian Science Fund** (FWF)
- **University of Graz**
- **Graz University of Technology**
- **Medical University of Graz**
- **Government of Styria**
- **City of Graz**

A note on the stable coupling of finite and boundary elements

O. Steinbach

Institut für Numerische Mathematik, TU Graz,
Steyrergasse 30, A 8010 Graz, Austria
o.steinbach@tugraz.at

Abstract

In this note we prove in the case of a Lipschitz interface the stability of the coupling of finite and boundary element methods when the direct boundary integral equation with single and double layer potentials is used only. In particular we prove an ellipticity estimate of the coupled bilinear form. Hence we can use standard arguments to derive stability and error estimates for the Galerkin discretization for all pairs of finite and boundary element trial spaces.

1 Introduction

The coupling of finite and boundary element methods is of increasing interest in many applications in engineering and science, e.g. in acoustic and electromagnetic scattering, in electromagnetism, and in elasticity to name a few of them. In particular, boundary integral equation methods can be used to handle partial differential equations with constant coefficients in unbounded domains, while finite element methods are more favourable when dealing with partial differential equations in bounded domains with varying coefficients, or even nonlinear equations.
First approaches to couple finite and boundary element methods are based on the use of either indirect single or double layer potentials, or the direct approach with both single and double layer potentials [1, 5], see also [6]. When the stability analysis of the coupled scheme is based on the use of a Gårding inequality of the related bilinear form, the compactness of the double layer potential has to be assumed which allows the consideration of smooth interface boundaries only. An alternative approach is to consider a sufficient accurate discretization, i.e. by using a much finer boundary element mesh to approximate the Neumann data, of the boundary integral equation to ensure the ellipticity of the boundary element approximation of the related Dirichlet to Neumann map [17], see also [7]. While a rigorous mathematical analysis was not yet available at that time, several numerical

examples indicated the stability of this coupling scheme for more general situations, see, e.g., [4].
In [2], a symmetric coupling of finite and boundary element methods was introduced which allows a rigorous stability and error analysis of the coupled scheme. Moreover, preconditioned parallel iterative solution strategies are available for the symmetric coupling, see, e.g., [9]. The symmetric formulation of boundary integral equations is based on the use of a second, the so–called hypersingular boundary integral equation. Despite of the fact, that the bilinear form of the hypersingular boundary integral operator can be rewritten as a weakly singular integral due to integration by parts [11], the use of the symmetric formulation is still not popular in engineering and for more advanced applications. Hence there is still a great interest in the coupling of finite elements with the first boundary integral equation only, which is also simpler to implement.
In a recent paper [13] the stability of the standard finite and boundary element coupling scheme was proven for the first time. The proof in [13] is based on a variational argument in the context of convolution quadrature methods.
In this paper we will present an alternative proof for the stability of the finite and boundary element coupling in the case of a Lipschitz interface. We restrict our considerations to the case of a space free Poisson equation, but the stability results can be stated also for more general situations. In particular we prove an ellipticity estimate of the combined bilinear form which allows us to use Cea's lemma to derive stability and related error estimates. An essential ingredient is the use of different variational and boundary integral formulations of the Steklov–Poincaré operator [14] which is involved in the Dirichlet to Neumann map associated to the interior Dirichlet boundary value problem. The second important tool is the use of some natural Sobolev norms in $H^{\pm 1/2}(\Gamma)$ which are induced by the single layer potential and its inverse as introduced in [16].
For a review on boundary integral equation methods and results on the mapping properties of boundary integral operators we refer in particular to [3, 8, 10, 15].

2 Non–symmetric BEM/FEM coupling

For a bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ with a Lipschitz boundary $\Gamma = \partial\Omega$ we consider the free space transmission boundary value problem

$$-\text{div}[A(x)\nabla u_i(x)] \,=\, f(x) \quad \text{for } x \in \Omega, \quad -\Delta u_e(x) \,=\, 0 \quad \text{for } x \in \Omega^c := \mathbb{R}^n \backslash \overline{\Omega} \tag{2.1}$$

with the interface boundary conditions

$$u_i(x) \,=\, u_e(x), \quad n_x \cdot [A(x)\nabla u_i(x)] \,=\, \frac{\partial}{\partial n_x} u_e(x) \quad \text{for } x \in \Gamma, \tag{2.2}$$

and with the radiation boundary condition

$$u(x) \,=\, \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty. \tag{2.3}$$

Note that $A = A^\top > 0$ is a uniformly positive definite and symmetric matrix, $f \in L_2(\Omega)$ is a given function, and n_x is the exterior normal vector which is defined for almost all $x \in \Gamma$. In the two–dimensional case $n = 2$ we assume the scaling condition $\operatorname{diam}\Omega < 1$.
The variational formulation of the interior Poisson equation in (2.1) is to find $u_i \in H^1(\Omega)$ such that

$$\int_\Omega [A(x)\nabla u_i(x)] \cdot \nabla v(x)dx = \int_\Omega f(x)v(x)dx + \int_\Gamma n_x \cdot [A(x)\nabla u_i(x)]v(x)ds_x \tag{2.4}$$

is satisfied for all $v \in H^1(\Omega)$. The solution of the exterior Laplace equation in (2.1) satisfying the radiation condition (2.3) is given by the representation formula

$$u_e(x) = -\int_\Gamma U^*(x,y)\frac{\partial}{\partial n_y}u_e(y)ds_y + \int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)u_e(y)ds_y \quad \text{for } x \in \Omega^c, \tag{2.5}$$

where

$$U^*(x,y) = \begin{cases} -\dfrac{1}{2\pi}\log|x-y| & \text{for } n = 2, \\ \dfrac{1}{4\pi}\dfrac{1}{|x-y|} & \text{for } n = 3 \end{cases}$$

is the fundamental solution of the Laplace operator. To ensure that the solution u_e as given by the representation formula (2.5) fulfils the radiation condition (2.3) also in the two–dimensional case $n = 2$ we need to assume that the normal derivative

$$t(x) := \frac{\partial}{\partial n_x}u_e(x)$$

satisfies the scaling condition, see, e.g., [15, Lemma 6.21],

$$\int_\Gamma t(x)\, ds_x = 0\,. \tag{2.6}$$

From (2.5) we obtain the boundary integral equation for $x \in \Gamma$

$$(Vt)(x) = -\frac{1}{2}u_e(x) + (Ku_e)(x), \tag{2.7}$$

where

$$(Vt)(x) = \int_\Gamma U^*(x,y)t(y)ds_y, \quad (Ku_e)(x) = \int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)u_e(y)ds_y$$

denote the single and double layer potential, respectively. When inserting the interface boundary conditions (2.2) into the variational formulation (2.4) and into the boundary integral equation (2.7) we end up with a variational problem to find $u_i \in H^1(\Omega)$ and $t \in H^{-1/2}(\Gamma)$ such that

$$\int_\Omega [A(x)\nabla u_i(x)] \cdot \nabla v(x)dx - \int_\Gamma t(x)v(x)ds_x = \int_\Omega f(x)v(x)dx \tag{2.8}$$

is satisfied for all $v \in H^1(\Omega)$ and

$$\langle Vt, \tau\rangle_\Gamma + \langle(\frac{1}{2}I - K)u_i, \tau\rangle_\Gamma \; = \; 0 \tag{2.9}$$

is satisfied for all $\tau \in H^{-1/2}(\Gamma)$.
When choosing in (2.8) as test function $v = 1$ this gives

$$-\int_\Gamma t(x)\, ds_x \; = \; \int_\Omega f(x)\, dx\,. \tag{2.10}$$

Hence, in the two–dimensional case $n = 2$ we have to assume the solvability condition

$$\int_\Omega f(x)\, dx \; = \; 0 \tag{2.11}$$

to ensure the required scaling condition (2.6).
Since the single layer potential $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bounded and $H^{-1/2}(\Gamma)$–elliptic, we may choose the natural density $w_{\text{eq}} = V^{-1}1$ as a test function in (2.9),

$$\langle t, 1\rangle_\Gamma + \langle(\frac{1}{2}I - K)u_i, V^{-1}1\rangle_\Gamma \; = \; 0\,.$$

By using the equilibrium condition (2.10) and the symmetry relation $KV = VK'$, see, e.g., [8, 15, 16], with the adjoint double layer potential operator K' we obtain

$$\begin{aligned}\int_\Omega f(x)dx \;&=\; \langle(\frac{1}{2}I - K)u_i, V^{-1}1\rangle_\Gamma \\ &=\; \langle u_i, w_{\text{eq}}\rangle_\Gamma - \langle u_i, (\frac{1}{2}I + K')V^{-1}1\rangle_\Gamma \\ &=\; \langle u_i, w_{\text{eq}}\rangle_\Gamma - \langle u_i, V^{-1}(\frac{1}{2}I + K)1\rangle_\Gamma \;=\; \langle u_i, w_{\text{eq}}\rangle_\Gamma\end{aligned}$$

due to $(\frac{1}{2}I + K)1 = 0$. Hence we may introduce the splitting

$$u_i(x) \; = \; u_0 + \widetilde{u}_i(x) \quad \text{for } x \in \Omega \tag{2.12}$$

where $\widetilde{u}_i \in H^1(\Omega)$ satisfies the scaling condition

$$\langle \widetilde{u}_i, w_{\text{eq}}\rangle_\Gamma \; = \; 0, \tag{2.13}$$

and where the constant u_0 is given by

$$u_0 \; = \; \frac{1}{\langle 1, w_{\text{eq}}\rangle_\Gamma} \int_\Omega f(x)\, dx \; = \; \frac{1}{\langle Vw_{\text{eq}}, w_{\text{eq}}\rangle_\Gamma} \int_\Omega f(x)\, dx\,. \tag{2.14}$$

Due to the solvability condition (2.11) in the two–dimensional case $n = 2$ we obtain $u_0 = 0$ in this case.
Instead of the coupled variational formulation (2.8) and (2.9) we now consider a modified variational problem to find $(\widetilde{u}_i, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\int_\Omega [A(x)\nabla\widetilde{u}_i(x)] \cdot \nabla v(x)dx + \langle\widetilde{u}_i, w_{\rm eq}\rangle_\Gamma\langle v, w_{\rm eq}\rangle_\Gamma - \langle t, v\rangle_\Gamma = \int_\Omega f(x)v(x)dx \quad (2.15)$$

$$\langle Vt, \tau\rangle_\Gamma + \langle(\frac{1}{2}I - K)\widetilde{u}_i, \tau\rangle_\Gamma = -\langle u_0, \tau\rangle_\Gamma \quad (2.16)$$

is satisfied for all $(v, \tau) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$.
By construction we have seen that any solution of the coupled variational problem (2.8) and (2.9) is also a solution of the modified variational problem (2.15)–(2.16). But also the reverse is true.

Lemma 2.1 *Any solution* $(\widetilde{u}_i, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *of the modified variational problem* (2.15)–(2.16) *implies a solution* $(\widetilde{u}_i + u_0, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *of the coupled variational problem* (2.8) *and* (2.9)*. Moreover,* $\widetilde{u}_i$ *satisfies the scaling condition* (2.13)*.*

Proof. Let $(\widetilde{u}_i, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ be any solution of the modified variational problem (2.15)–(2.16). By choosing $v = 1$ and $\tau = 0$ this gives

$$\langle\widetilde{u}_i, w_{\rm eq}\rangle_\Gamma\langle 1, w_{\rm eq}\rangle_\Gamma - \langle t, 1\rangle_\Gamma = \int_\Omega f(x)dx$$

while by choosing $v = 0$ and $\tau = w_{\rm eq} = V^{-1}1$ we obtain

$$\langle t, 1\rangle_\Gamma = -\langle u_0, w_{\rm eq}\rangle_\Gamma = -\int_\Omega f(x)\, dx\,.$$

Hence we conclude

$$\langle\widetilde{u}_i, w_{\rm eq}\rangle_\Gamma = 0\,.$$

Then it is a direct consequence that $(\widetilde{u}_i + u_0, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ is also a solution of the coupled variational problem (2.8) and (2.9). ■

To ensure unique solvability of the modified variational formulation (2.15)–(2.16) as well as stability of a related Galerkin scheme we need to establish some ellipticity or coercivity estimate of an associated bilinear form. First we note that

$$\|v\|^2_{H^1(\Omega),\Gamma} = \int_\Omega |\nabla v(x)|^2 dx + [\langle v, w_{\rm eq}\rangle_\Gamma]^2$$

defines an equivalent norm in $H^1(\Omega)$, see, e.g., [15, Theorem 2.6], while

$$\|\tau\|^2_V = \langle V\tau, \tau\rangle_\Gamma$$

defines an equivalent norm in $H^{-1/2}(\Gamma)$. Associated to the modified variational formulation (2.15)–(2.16) we introduce the bilinear form

$$\begin{aligned}\widetilde{a}(u,t;v,\tau) \;&:=\; \int_\Omega [A(x)\nabla u(x)]\cdot\nabla v(x)dx + \langle u, w_{\text{eq}}\rangle_\Gamma \langle v, w_{\text{eq}}\rangle_\Gamma - \langle t, v\rangle_\Gamma \\ &\quad +2\left[\langle Vt,\tau\rangle_\Gamma + \langle(\frac{1}{2}I - K)u,\tau\rangle_\Gamma\right]\end{aligned} \tag{2.17}$$

which is bounded for all $(u,t),(v,\tau) \in H^1(\Omega)\times H^{-1/2}(\Gamma)$. For $(v,\tau)\in H^1(\Omega)\times H^{-1/2}(\Gamma)$ we then obtain

$$\widetilde{a}(v,\tau;v,\tau) \;\geq\; \lambda_{\min}(A)\,\|v\|^2_{H^1(\Omega),\Gamma} + \|\tau\|^2_V - 2\langle Kv,\tau\rangle_\Gamma. \tag{2.18}$$

The form (2.18) is coercive satisfying a Gårding inequality when we assume that the double layer potential $K : H^{1/2}(\Gamma)\to H^{1/2}(\Gamma)$ is compact. In this case, stability of the related Galerkin scheme follows asymptotically when using standard results for the Galerkin approximation of compact perturbations of elliptic equations. But the compactness of the double layer potential K allows the consideration of smooth interface boundaries Γ only. Note that the bilinear form (2.17) corresponds to the variational problem (2.15)–(2.16), where we multiply (2.16) with the factor 2.
Instead of a Gårding inequality as in (2.18) we will prove an ellipticity estimate for the bounded bilinear form

$$\begin{aligned}a(u,t;v,\tau) \;&:=\; \int_\Omega [A(x)\nabla u(x)]\cdot\nabla v(x)dx + \langle u, w_{\text{eq}}\rangle_\Gamma \langle v, w_{\text{eq}}\rangle_\Gamma - \langle t, v\rangle_\Gamma \\ &\quad +\langle Vt,\tau\rangle_\Gamma + \langle(\frac{1}{2}I - K)u,\tau\rangle_\Gamma\end{aligned} \tag{2.19}$$

from which we can derive stability and error estimates in a standard way. In what follows we state the main result of this paper.

Theorem 2.2 *Let $\lambda_{\min}(A) > \frac{1}{4}$ be satisfied. The bilinear form as defined in* (2.19) *is then $H^1(\Omega)\times H^{-1/2}(\Gamma)$–elliptic, i.e.*

$$a(v,\tau;v,\tau) \;\geq\; \frac{1}{2}\Big[(1+\lambda_{\min}(A)) - \sqrt{(1-\lambda_{\min}(A))^2+1}\Big]\left[\|v\|^2_{H^1(\Omega),\Gamma} + \|\tau\|^2_V\right] \tag{2.20}$$

for all $(v,\tau)\in H^1(\Omega)\times H^{-1/2}(\Gamma)$.

The proof of Theorem 2.2 will be given in Sect. 4.
Now we are in a position to describe the Galerkin discretization of the modified variational problem (2.15)–(2.16). Let

$$V_H = \text{span}\{\varphi_i\}_{i=1}^{M_i} \subset H^1(\Omega) \tag{2.21}$$

be some finite element subspace which is defined with respect to some admissible domain mesh with the finite element mesh size H, and let

$$Z_h = \mathrm{span}\{\psi_k\}_{k=1}^{N_e} \subset H^{-1/2}(\Gamma) \tag{2.22}$$

be some boundary element space which is defined with respect to some admissible boundary element mesh with mesh size h. For example, we may consider piecewise linear and continuous basis functions φ_i, and piecewise constant basis functions ψ_k, but any other choice of trial spaces satisfying an approximation property can be used as well.
The Galerkin discretization of the modified variational problem (2.15)–(2.16) leads to a linear system of algebraic equations

$$\begin{pmatrix} V_h & \frac{1}{2}M_h - K_h \\ -M_h^\top & A_H \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{\widetilde{u}} \end{pmatrix} = \begin{pmatrix} \underline{g} \\ \underline{f} \end{pmatrix} \tag{2.23}$$

where the boundary element block matrices are defined as

$$V_h[\ell,k] = \langle V\psi_k, \psi_\ell\rangle_\Gamma, \quad M_h[\ell,i] = \langle \varphi_i, \psi_\ell\rangle_\Gamma, \quad K_h[\ell,i] = \langle K\varphi_i, \psi_\ell\rangle_\Gamma$$

for $k,\ell = 1,\ldots,N_e$, $i = 1,\ldots,M_i$, and the modified finite element stiffness matrix is given by

$$A_H[j,i] = \int_\Omega [A(x)\nabla\varphi_i(x)] \cdot \nabla\varphi_j(x)\,dx + \langle \varphi_i, w_{\mathrm{eq}}\rangle_\Gamma \langle \varphi_j, w_{eq}\rangle_\Gamma$$

for $i,j = 1,\ldots,M_i$. The block vectors of the right hand side in (2.23) are given by

$$f_j = \int_\Omega f(x)\varphi_j(x)dx \quad \text{for } j = 1,\ldots,M_i, \quad g_\ell = -\langle u_0, \psi_\ell\rangle_\Gamma \quad \text{for } \ell = 1,\ldots,N_e.$$

Since the bilinear form (2.19) is $H^1(\Omega) \times H^{-1/2}(\Gamma)$–elliptic, stability of the Galerkin approximation of the modified variational problem (2.15)–(2.16) follows, and by Cea's lemma we have the energy error estimate

$$\|\widetilde{u}_i - \widetilde{u}_{i,H}\|_{H^1(\Omega)} + \|t - t_h\|_{H^{-1/2}(\Gamma)} \le c \left\{ \inf_{v_H \in V_H} \|\widetilde{u}_i - v_H\|_{H^1(\Omega)} + \inf_{\tau_h \in Z_h} \|t - \tau_h\|_{H^{-1/2}(\Gamma)} \right\}. \tag{2.24}$$

Therefore, convergence follows from the regularity of the solution $(\widetilde{u}_i, t)$ and from the approximation properties of both the finite and boundary element trial spaces, V_H and Z_h.

3 Dirichlet to Neumann maps

The proof of Theorem 2.2 is essentially based on different representations of the Dirichlet to Neumann map which is related to the solution of the interior Dirichlet boundary value problem

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \quad u(x) = g(x) \quad \text{for } x \in \Gamma \tag{3.1}$$

whose weak solution $u \in H^1(\Omega)$, $u(x) = g(x)$ for $x \in \Gamma$, is the unique solution of the variational problem

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx \;=\; 0 \quad \text{for all } v \in H^1_0(\Gamma). \tag{3.2}$$

Since $u \in H^1(\Omega)$ is the weak solution of the Dirichlet boundary value problem (3.1) we can define the related normal derivative $\lambda(x) := n_x \cdot \nabla u(x)$ for almost all $x \in \Gamma$ by means of Green's first formula, i.e. $\lambda \in H^{-1/2}(\Gamma)$ satisfies

$$\int_\Gamma \lambda(x) v(x) ds_x \;=\; \int_\Omega \nabla u(x) \cdot \nabla v(x) dx \quad \text{for all } v \in H^1(\Omega). \tag{3.3}$$

Hence, by solving the Dirichlet boundary value problem (3.1) and by defining the normal derivative λ via (3.3) we obtain the Dirichlet to Neumann map

$$\lambda(x) \;=\; (S_i g)(x) \quad \text{for almost all } x \in \Gamma, \tag{3.4}$$

where the Steklov–Poincaré operator $S_i : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is defined implicitly by solving the Dirichlet boundary value problem (3.1).
Instead of the variational definition (3.3) of the Steklov–Poincaré operator S_i we now consider equivalent definitions of the Steklov–Poincaré operator which are based on the use of boundary integral equations.
The solution of the Dirichlet boundary value problem (3.1) is given by the representation formula

$$u(x) \;=\; \int_\Gamma U^*(x,y)\lambda(y) ds_y - \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) g(y) ds_y \quad \text{for } x \in \Omega \tag{3.5}$$

where $U^*(x,y)$ is again the Laplace fundamental solution. From (3.5) we obtain a system of two boundary integral equations on Γ,

$$\begin{pmatrix} g \\ \lambda \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} g \\ \lambda \end{pmatrix} \tag{3.6}$$

where in addition to the single and double layer potentials V and K we used the adjoint double layer potential K' and the hypersingular boundary integral operator D for $x \in \Gamma$,

$$(K'\lambda)(x) = \int_\Gamma \frac{\partial}{\partial n_x} U^*(x,y)\lambda(y) ds_y, \quad (Dg)(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) g(y) ds_y.$$

Since the single layer potential V is invertible, we solve the first equation in (3.6) to obtain the Dirichlet to Neumann map

$$\lambda(x) \;=\; V^{-1}(\frac{1}{2}I + K) g(x) \quad \text{for almost all } x \in \Gamma. \tag{3.7}$$

Inserting (3.7) into the second equation of (3.6) this results in a second boundary integral representation of the Dirichlet to Neumann map,

$$\lambda(x) \;=\; (Dg)(x) + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2} + K)g(x) \quad \text{for almost all } x \in \Gamma. \tag{3.8}$$

Since the solution of the Dirichlet boundary value problem (3.1) is unique, the Dirichlet to Neumann map $\lambda = S_i g$ is unique for all $g \in H^{1/2}(\Gamma)$, and therefore, all representations as introduced in (3.3), (3.7) and (3.8) coincide.
In the proof of Theorem 2.2 we will replace the Dirichlet to Neumann map $\lambda = S_i g$ which is originally defined via the variational formulation (3.3) by the equivalent symmetric boundary integral representation (3.8), in particular we will use the Steklov–Poincaré operator

$$S_i \;=\; D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K) \;:\; H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)\,. \tag{3.9}$$

4 Proof of Theorem 2.2

To prove the ellipticity estimate of Theorem 2.2 we first consider the form (2.19), i.e.

$$\begin{aligned} a(v,\tau;v,\tau) &= \int\limits_{\Omega} [A(x)\nabla v(x)] \cdot \nabla v(x) dx + [\langle v, w_{\text{eq}}\rangle_\Gamma]^2 + \langle V\tau,\tau\rangle_\Gamma - \langle(\frac{1}{2}I + K)v,\tau\rangle_\Gamma \\ &\geq \lambda_{\min}(A) \int\limits_{\Omega} |\nabla v(x)|^2 dx + [\langle v, w_{\text{eq}}\rangle_\Gamma]^2 + \langle V\tau,\tau\rangle_\Gamma - \langle(\frac{1}{2}I + K)v,\tau\rangle_\Gamma. \end{aligned} \tag{4.1}$$

For $v \in H^1(\Omega)$ we consider the splitting $v = v_\Gamma + \widetilde{v}$ where v_Γ is the harmonic extension of $v_{|\Gamma}$, i.e. $v_\Gamma \in H^1(\Omega)$ is the weak solution of the Dirichlet boundary value problem

$$-\Delta v_\Gamma(x) = 0 \quad \text{for } x \in \Omega, \quad v_\Gamma(x) = v(x) \quad \text{for } x \in \Gamma.$$

In particular, $v_\Gamma \in H^1(\Omega)$, $v_\Gamma(x) = v(x)$ for $x \in \Gamma$, solves

$$\int\limits_{\Omega} \nabla v_\Gamma(x) \cdot \nabla z(x) dx \;=\; 0 \quad \text{for all } z \in H^1_0(\Omega).$$

By construction we also have $\widetilde{v} \in H^1_0(\Omega)$. Hence we obtain

$$\int\limits_{\Omega} |\nabla v(x)|^2 dx \;=\; \int\limits_{\Omega} |\nabla(v_\Gamma(x) + \widetilde{v}(x))|^2 dx \;=\; \int\limits_{\Omega} |\nabla v_\Gamma(x)|^2 dx + \int\limits_{\Omega} |\nabla \widetilde{v}(x)|^2 dx\,. \tag{4.2}$$

By using Greens first formula we further conclude

$$\int\limits_{\Omega} |\nabla v_\Gamma(x)|^2 dx \;=\; \int\limits_{\Gamma} \frac{\partial}{\partial n_x} v_\Gamma(x) v_\Gamma(x) ds_x \;=\; \int\limits_{\Gamma} (S_i v_\Gamma)(x) v_\Gamma(x) ds_x$$

where $S_i : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the Steklov–Poincaré operator which realizes the Dirichlet to Neumann map when considering an interior Dirichlet boundary value problem for the Laplace equation. Hence we can rewrite (4.1) as

$$\begin{aligned} a(v,\tau;v,\tau) \ \geq \ & \lambda_{\min}(A) \int\limits_\Omega |\nabla \widetilde{v}(x)|^2 dx + [\langle v, w_{\rm eq}\rangle_\Gamma]^2 \\ & + \lambda_{\min}(A) \, \langle S_i v_\Gamma, v_\Gamma\rangle_\Gamma + \langle V\tau,\tau\rangle_\Gamma - \langle (\frac{1}{2}I+K)v,\tau\rangle_\Gamma \, . \end{aligned} \tag{4.3}$$

By definition we have $v_\Gamma(x) = v(x)$ for $x \in \Gamma$. Therefore, by using the symmetric boundary integral representation (3.9) of the Steklov–Poincaré operator S_i we further obtain

$$\begin{aligned} & \lambda_{\min}(A) \, \langle S_i v, v\rangle_\Gamma + \langle V\tau,\tau\rangle_\Gamma - \langle (\frac{1}{2}I+K)v,\tau\rangle_\Gamma \\ & = \lambda_{\min}(A) \Big[\langle Dv, v\rangle_\Gamma + \langle V^{-1}(\frac{1}{2}I+K)v, (\frac{1}{2}I+K)v\rangle_\Gamma \Big] + \langle V\tau,\tau\rangle_\Gamma - \langle (\frac{1}{2}I+K)v,\tau\rangle_\Gamma \\ & = \lambda_{\min}(A) \langle Dv, v\rangle_\Gamma + \lambda_{\min}(A) \|(\frac{1}{2}I+K)v\|^2_{V^{-1}} + \|\tau\|^2_V - \langle (\frac{1}{2}I+K)v,\tau\rangle_\Gamma \end{aligned}$$

where

$$\|\cdot\|_{V^{-1}} \ := \ \sqrt{\langle V^{-1}\cdot,\cdot\rangle_\Gamma}$$

defines an equivalent norm in $H^{1/2}(\Gamma)$. With the bound, see, e.g. [16],

$$\Big| \langle (\frac{1}{2}I+K)v,\tau\rangle_\Gamma \Big| \ \leq \ \|(\frac{1}{2}I+K)v\|_{V^{-1}} \|\tau\|_V ,$$

we conclude

$$\begin{aligned} & \lambda_{\min}(A) \langle S_i v, v\rangle_\Gamma + \langle V\tau,\tau\rangle_\Gamma - \langle (\frac{1}{2}I+K)v,\tau\rangle_\Gamma \\ & \geq \ \lambda_{\min}(A) \langle Dv, v\rangle_\Gamma + \lambda_{\min}(A) \|(\frac{1}{2}I+K)v\|^2_{V^{-1}} + \|\tau\|^2_V - \|\langle (\frac{1}{2}I+K)v\|_{V^{-1}} \|\tau\|_V \\ & = \ \lambda_{\min}(A) \langle Dv, v\rangle_\Gamma + \Big(\lambda_{\min}(A) - \frac{1}{2}\frac{1}{\gamma^2}\Big) \|(\frac{1}{2}I+K)v\|^2_{V^{-1}} + \Big(1 - \frac{1}{2}\gamma^2\Big) \|\tau\|^2_V \\ & \qquad + \frac{1}{2}\Big(\gamma \, \|\tau\|_V - \frac{1}{\gamma} \, \|(\frac{1}{2}I+K)v\|_{V^{-1}}\Big)^2 \\ & = \ \lambda_{\min}(A) \langle Dv, v\rangle_\Gamma + \Big(\lambda_{\min}(A) - \frac{1}{2}\frac{1}{\gamma^2}\Big) \|(\frac{1}{2}I+K)v\|^2_{V^{-1}} + \Big(1 - \frac{1}{2}\gamma^2\Big) \|\tau\|^2_V \\ & \geq \ \min \Big\{ \lambda_{\min}(A), 1 - \frac{1}{2}\gamma_*^2 \Big\} \ \Big[\langle Dv, v\rangle_\Gamma + \|(\frac{1}{2}I+K)v\|^2_{V^{-1}} + \|\tau\|^2_V \Big] \end{aligned}$$

if

$$\lambda_{\min}(A) - \frac{1}{2}\frac{1}{\gamma_*^2} \ = \ 1 - \frac{1}{2}\gamma_*^2 \tag{4.4}$$

is satisfied. From (4.4) we find

$$\gamma_*^2 \ = \ 1 - \lambda_{\min}(A) + \sqrt{(1-\lambda_{\min}(A))^2 + 1}$$

and therefore

$$1-\frac{1}{2}\gamma_*^2 \;=\; \frac{1}{2}\left[(1+\lambda_{\min}(A))-\sqrt{(1-\lambda_{\min}(A))^2+1}\right] \;>\; 0$$

if we assume $\lambda_{\min}(A) > \frac{1}{4}$. Due to

$$\frac{1}{2}\left[(1+\lambda_{\min}(A))-\sqrt{(1-\lambda_{\min}(A))^2+1}\right] \;\leq\; \lambda_{\min}(A)$$

we finally obtain

$$\begin{aligned}
a(v,\tau;v,\tau) \;&\geq\; \lambda_{\min}(A)\int\limits_\Omega |\nabla\widetilde{v}(x)|^2dx + [\langle v, w_{\rm eq}\rangle_\Gamma]^2 \\
&\quad +\frac{1}{2}\left[(1+\lambda_{\min}(A))-\sqrt{(1-\lambda_{\min}(A))^2+1}\right]\,\left[\langle S_i v_\Gamma, v_\Gamma\rangle_\Gamma + \|\tau\|_V^2\right] \\
&\geq\; \frac{1}{2}\left[(1+\lambda_{\min}(A))-\sqrt{(1-\lambda_{\min}(A))^2+1}\right]\cdot \\
&\qquad\cdot\left[\int\limits_\Omega |\nabla\widetilde{v}(x)|^2dx + \int\limits_\Omega |\nabla v_\Gamma(x)|^2dx + [\langle v, w_{\rm eq}\rangle_\Gamma]^2 + \|\tau\|_V^2\right] \\
&=\; \frac{1}{2}\left[(1+\lambda_{\min}(A))-\sqrt{(1-\lambda_{\min}(A))^2+1}\right]\left[\|v\|^2_{H^1(\Omega),\Gamma} + \|\tau\|_V^2\right]
\end{aligned}$$

as stated in Theorem 2.2.

5 Extensions

The solution of the free space partial differential equation (2.1)–(2.3) requires the consideration of the modified variational problem (2.15)–(2.16), which involves the use of the natural density $w_{\rm eq} = V^{-1}1$. Although it is possible to compute a boundary element approximation $w_{\rm eq,h} \in Z_h$ in advance, see, e.g., [12], such an approach is not needed when considering boundary conditions in addition, e.g., Dirichlet boundary conditions on some (interior) Dirichlet part of the multiple connected domain Ω, as considered, for example, in [6]. This reflects only the unique solvability of the partial differential equation, but not the stable coupling of finite and boundary elements across the interface.

References

[1] F. Brezzi, C. Johnson: On the coupling of boundary integral and finite element methods. Calcolo 16 (1979) 189–201.

[2] M. Costabel: Symmetric methods for the coupling of finite elements and boundary elements. In: Boundary Elements IX (C. A. Brebbia, G. Kuhn, W. L. Wendland eds.), Springer, Berlin, pp. 411–420, 1987.

[3] M. Costabel: Boundary integral operators on Lipschitz domains: Elementary results. SIAM J. Math. Anal. 19 (1988) 613–626.

[4] M. Costabel, V. J. Ervin, E. P. Stephan: Experimental convergence rates for various couplings of boundary and finite elements. Math. Comput. Modelling 15 (1991) 93–102.

[5] C. Johnson, J.–C. Nedelec: On the coupling of boundary integral and finite element methods. Math. Comp. 35 (1980) 1063–1079.

[6] G. C. Hsiao: The coupling of boundary element and finite element methods. Z. Angew. Math. Mech. 70 (1990) T493–T503.

[7] G. C. Hsiao, E. Schnack, W. L. Wendland: A hybrid coupled finite–boundary element method in elasticity. Comp. Methods Appl. Mech. Eng. 173 (1999) 287–316.

[8] G. C. Hsiao, W. L. Wendland: Integral Equation Methods for Boundary Value Problems. Springer, Heidelberg, 2008.

[9] U. Langer: Parallel iterative solution of symmetric coupled fe/be equations via domain decomposition. Contemp. Math. 157 (1994) 335–344.

[10] W. McLean: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, 2000.

[11] J.–C. Nedelec: Integral equations with non integrable kernels. Int. Eq. Operator Th. 5 (1982) 562–572.

[12] G. Of, O. Steinbach: A fast multipole boundary element method for a modified hypersingular boundary integral equation. In: Proceedings of the International Conference on Multifield Problems (M. Efendiev, W. L. Wendland eds.), Springer Lecture Notes in Applied Mechanics, vol. 12, Springer, Berlin, pp. 163–169, 2003.

[13] F.–J. Sayas: The validity of Johnson–Nedelec's BEM–FEM coupling on polygonal interfaces. Submitted, 2009.

[14] O. Steinbach: Stability estimates for hybrid coupled domain decomposition methods. Springer Lecture Notes in Mathematics, vol. 1809, 2003.

[15] O. Steinbach: Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements. Springer, New York, 2008.

[16] O. Steinbach, W. L. Wendland: On C. Neumann's method for second order elliptic systems in domains with non–smooth boundaries. J. Math. Anal. Appl. 262 (2001) 733–748.

[17] W. L. Wendland: On asymptotic error estimates for the combined BEM an FEM. In: In Finite Element and Boundary Element Techniques from Mathematical and Engineering Point of View (E. Stein, W. L. Wendland eds.), CISM Lecture Notes, vol. 301, Springer, Wien, pp. 273–333, 1988.