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Abstract. In this paper sufficient second order optimality conditions for optimal control problems subject to stationary variational inequalities of obstacle type are derived. Since optimality conditions for such problems always involve measures as Lagrange multipliers, which impede the use of efficient Newton type methods, a family of regularized problems is introduced. Second order sufficient optimality conditions are derived for the regularized problems as well. It is further shown that these conditions are also sufficient for superlinear convergence of the semi-smooth Newton algorithm to be well-defined and superlinearly convergent when applied to the first order optimality system associated with the regularized problems.

Keywords. Variational inequalities, optimal control, sufficient optimality conditions, semi-smooth Newton method.

1 Introduction, problem statement, regularization

We consider the optimal control problem:

$$
\text{(P)} \qquad \begin{cases} \min \quad J(y,u) = g(y) + j(u) \\ \text{over } u \in L^2(\Omega) \text{ under the variational inequality constraint} \\ a(y, \phi - y) \geq (u, \phi - y), \quad y \in K, \ \text{for all } \phi \in K, \end{cases}
$$

where $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is a bilinear form satisfying

$$
\nu_1 |v|^2_{H_0^1} \leq a(v,v), \ \text{ and } a(v,w) \leq \nu_2 |v|_{H_0^1} |w|_{H_0^1}, \tag{1.1}
$$

with $0 < \nu_1 \leq \nu_2$, and

$$
K = \{ v \in H_0^1(\Omega) : v \leq \psi \}. \tag{1.2}
$$

Throughout it will be convenient to alternatively use the operator representation of the bilinear form, i.e.

$$
a(v_1, v_2) = \langle A v_1, v_2 \rangle, \ \text{ for } v_1, v_2 \in H_0^1(\Omega),
$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. It is well known that under the conditions to be specified below the variational inequality in (P) can equivalently be expressed as

$$Ay + \lambda = u, \quad y \le \psi,\ \lambda \ge 0,\ \langle \lambda, y - \psi\rangle = 0, \tag{1.3}$$

where $\lambda \in H^{-1}$ and $\lambda \ge 0$ is short for $\langle \lambda, v\rangle \ge 0$ for all $v \in H^1_0(\Omega)$, $v \ge 0$, and $\psi \in H^1(\Omega)$ with $\psi \ge 0$ on the boundary of Ω. In this way (P) is an optimization problem subject to a complementary condition constraint. If $\lambda \in L^2(\Omega)$, then the complementarity condition in (1.3) can equivalently be expressed as

$$\lambda = \max(0, \lambda + c(y - \psi)), \tag{1.4}$$

for any $c > 0$.

The variational inequality constraint is also equivalent to the optimization problem

$$\min \tfrac{1}{2}\, a(y, y) - (y, u)_{L^2} \text{ over } y \in K, \tag{1.5}$$

so that (P) can alternatively be considered as a bilevel optimization problem with the additional constraint $y \in K$. Here $(\cdot, \cdot)_{L^2}$ denotes the scalar product in $L^2(\Omega)$. In the sequel the index L^2 will frequently be omitted.

Let us briefly describe the structure and contributions of this paper. Section 2 is devoted to necessary and sufficient optimality conditions for (P). The literature essentially offers two approaches to obtain necessary optimality conditions. One is based on convex analysis techniques, see for instance Mignot and Puel [13], the other one on approximation methods, we refer to Barbu [1], and Ito and Kunisch [9], and further references given there. The main concern here is to obtain a system of conditions as complete as possible, so that its solutions in turn provide the solution to (P). Difficulties arise due to the weakly active set also called bi-active set given by $B = \{x : \lambda = y - \psi = 0\}$. While the derivation of optimality conditions for multi-level optimization problems has received repeated attention in the finite-dimensional context, see for instance Scheel and Scholtes [14], this is rather recent in infinite dimensional spaces, we refer to the treatment in Hintermüller and Kopacka [6], also studying the optimal control of variational inequalities. In the present work we use the results from Mignot and Puel [13] and Ito and Kunisch [9] on necessary optimality conditions. Subsequently conditions are obtained under which the first order optimality system in fact provides sufficient condtions for a minimum of (P). Here we can use techniques which were recently successful in the context of second order conditions for state constrained optimal control problems, see Casas, Reyes, and Tröltzsch [4], for example.

Let us compare the optimal control problem considered here, to the following optimal control problem subject to state constraints:

$$\begin{cases} \min \quad J(y,u) = g(y) + j(u) \\ \text{over } u \in L^2(\Omega),\ y \in K, \text{ under the elliptic equation} \\ Ay = u. \end{cases}$$

We write both problems as bilevel optimization problems. Problem (P) can be written as:

$$\begin{aligned} &\min J(y,u) \text{ over all } u \in L^2(\Omega) \text{ and } y \in H_0^1(\Omega) \text{ with} \\ &\qquad\qquad y = \operatorname{argmin}_{y \in K} \tfrac{1}{2} a(y,y) - (y,u)_{L^2}, \end{aligned}$$

whereas the state constrained problem can be written as

$$\begin{aligned} &\min J(y,u) \text{ over all } u \in L^2(\Omega) \text{ and } y \in K \text{ with} \\ &\qquad\qquad y = \operatorname{argmin}_{y \in H_0^1(\Omega)} \tfrac{1}{2} a(y,y) - (y,u)_{L^2}. \end{aligned}$$

The essential difference between both problems is now evident: In the variational inequality control problem, the constraint $y \le \psi$ is prescribed in the inner problem, whereas for the state constrained problem this constraint appears in the outer problem. This implies that for (P) *every* control is feasible, whereas for the state constrained problem the inequality $y \le \psi$ is in fact a restriction on the set of feasible controls. Naturally the first-order necessary optimality conditions differ as well. Problem (P) relies on two multipliers associated to the constraint $y \in K$, denoted by λ and μ below, where one, λ, is non-negative. The state-constrained problem involves only one multiplier for $y \in K$, which is a non-negative measure.

In the subsequent Section 3 we investigate properties of a regularization of (P). This does justice to the fact that the Lagrange multiplier λ associated to the inequality constrained $y \le \psi$ as well as the multiplier, which will be called μ below and is associated to the complementary system in (1.3), are measures only, and such constitute quantities that are not amenable to numerical discretization and realisation. The regularized problems that will be utilized are given by

$$
(\mathrm{P}_c) \qquad \begin{cases} \min \quad J(y,u) = g(y) + j(u) \\ \text{over } u \in L^2(\Omega) \text{ subject to} \\ Ay + \max_c(0, \bar\lambda + c(y-\psi)) = u, \end{cases}
$$

where $\bar\lambda \geq 0$, $\bar\lambda \in L^2(\Omega)$ is given, and $\max_c$ is the C^1-approximation of $x \to \max(0,x)$ constructed by

$$
(1.6) \qquad \max_c(0,x) = \begin{cases} x, & \text{for } x \geq \frac{1}{2c} \\ \frac{c}{2}(x + \frac{1}{2c})^2, & \text{for } |x| \leq \frac{1}{2c} \\ 0, & \text{for } x \leq -\frac{1}{2c}. \end{cases}
$$

For properly chosen $\bar\lambda$ the solutions y_c to (P_c) are feasible, i.e. $y_c \leq \psi$, see Section 3.2 below. This was observed in [9] and will be further analysed and used in the present paper. If g and j are C^1-regular, then the first order optimality system for (P_c) is given by

$$
(1.7) \qquad \begin{cases} Ay_c + \max_c(0,\, \bar\lambda + c(y_c - \psi)) = u_c, \\ A^* p_c + c\,\mathrm{sgn}_c(\bar\lambda + c(y_c - \psi))\, p_c + g'(y_c) = 0, \\ j'(u_c) - p_c = 0, \end{cases}
$$

where

$$
(1.8) \qquad \mathrm{sgn}_c(x) = \begin{cases} 1, & \text{for } x \geq \frac{1}{2c} \\ c(x + \frac{1}{2c}), & \text{for } |x| \leq \frac{1}{2c} \\ 0, & \text{for } x \leq -\frac{1}{2c}, \end{cases}
$$

and the expressions $\lambda_c := \max_c(0,\, \bar\lambda + c(y_c - \psi))$ and $\mu_c := \mathrm{sgn}_c(\bar\lambda + c(y_c - \psi))$ in (1.7) tend to measure-valued Lagrange multipliers as $c \to \infty$. Section 3 also contains a discussion of second order sufficient optimality conditions for (P_c).

Solving (1.8) numerically by Newton-type methods is impeded by the lack of C^1 regularity of the sgn_c operator. In Section 4 it will be shown that semi-smooth Newton methods are applicable to (1.7), [11]. This requires the verification of Newton differentiability, which is quite standard by now, as well as well-posedness of the Newton-step and uniform boundedness of the inverse of the generalized derivatives, which is more delicate to verify. Here this will be achieved under a second-order sufficient optimality condition.

In a followup paper we shall address numerical issues related to solving (P_c) and in particular on the choice of the parameter c. This will make use of the properties of the path associated to (P_c), i.e. on the mapping $c \to (y_c, u_c, \lambda_c)$.

Standing assumptions

Throughout the paper we rely on the following regularity assumptions.

(i) The domain $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$ is bounded and either convex and polygonal or of the class $C^{1,1}$.

(ii) The operator A is an elliptic differential operator defined by

$$(Ay)(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}y(x)\right) + \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j}y(x) + a_0(x)y(x)$$

with functions $a_{ij} \in C^{0,1}(\bar{\Omega})$, $a_j, \frac{\partial}{\partial x_j}a_j, a_0 \in L^\infty(\Omega)$ satisfying the conditions $a_{ij}(x) = a_{ji}(x)$ and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta_0|\xi|^2 \qquad \text{a.e. on } \Omega \text{ for all } \xi \in \mathbb{R}^n$$

with some $\delta_0 > 0$. Additionally, we require $a_0(x) \geq \delta_1 \geq 0$ with δ_1 sufficiently large such that the bilinear form $a(\cdot,\cdot)$ induced by A fulfills the coercivity condition (1.1).

(iii) The obstacle $\psi \in H^1(\Omega)$ fulfills $\psi \geq 0$ on Γ and $A\psi \in L^2(\Omega)$.

(iv) The bilinear form a associated to A satisfies

$$a(y, y^+) = 0 \text{ implies that } y^+ = 0, \text{ for all } y \in H_0^1(\Omega).$$

Here $y^+ = \max(0, y)$. Some of the results can be obtained under weaker regularity assumption on the domain Ω and the coefficients of A, specifically in these cases it suffices that the boundary of Ω is Lipschitz continuous and the coefficients of A are sufficiently regular so that the bilinear form a is welldefined on $H_0^1(\Omega) \times H_0^1(\Omega)$ and satisfies (1.1).

The functions g, j satisfy:

(v) $g : L^2(\Omega) \to \mathbb{R}$ is twice continuously Fréchet-differentiable and bounded from below,

(vi) $j : L^2(\Omega) \to \mathbb{R}$ is twice continuously Fréchet-differentiable and radially unbounded.

Let us introduce the adjoint operator A^* to A by

$$(A^*p)(x) = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\sum_{i=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} p(x) + a_j(x) p(x) \right) + a_0(x) p(x).$$

Due to the assumptions on the coefficients, the equations $Ay = f$ and $A^*p = g$ admit solutions in $H^2(\Omega)$ for right-hand sides $f, g \in L^2(\Omega)$.

Further assumptions will be introduced in the context of second order sufficient optimality conditions and well-posedness of the semi-smooth Newton method.

2 Necessary and sufficient optimality conditions

Let us briefly summarize known results about unique solvability of the underlying variational inequality (1.3).

Lemma 2.1. *For each $u \in H^{-1}(\Omega)$ the variational inequality admits a unique solution $y \in H_0^1(\Omega)$, and the mapping $u \mapsto y$ is Lipschitz continuous from $u \in H^{-1}(\Omega)$ to $H_0^1(\Omega)$.*

This lemma does not depend on the strong regularity assumptions (i)-(iii) above but the weaker ones mentioned below (iii) suffice. Employing the technique of [15], one obtains L^∞-regularity of y.

Lemma 2.2. *For $u \in L^2(\Omega)$ the solution y belongs to $L^\infty(\Omega)$, and the mapping $u \mapsto y$ is Lipschitz continuous from $u \in L^2(\Omega)$ to $L^\infty(\Omega)$.*

Proof. Let $u_1, u_2 \in L^2(\Omega)$ be given, and denote the associated solutions of the variational inequality by y_1, y_2. For $k \in \mathbb{R}$, $k \geq 0$ we introduce the truncation operator $[\cdot]_k : H_0^1(\Omega) \to H_0^1(\Omega)$ by

$$([v]_k)(x) = \min(\max(v(x), -k), k).$$

Then the functions $v_1 = y_2 - [y_2 - y_1]_k$ and $v_2 = y_1 + [y_2 - y_1]_k$ are suitable as test functions for the variational inequality. In fact, if $y_2(x) - y_1(x) > k$ then $v_1(x) = y_2(x) - k \leq \psi(x) - k$ and $v_2(x) = y_1(x) + k < y_2(x) - k + k \leq \psi(x)$. If $y_2(x) - y_1(x) < -k$ then $v_1(x) = y_2(x) + k < y_1(x) - k + k = y_1(x) \leq \psi(x)$ and $v_2(x) = y_1(x) - k \leq \psi(x)$, which altogether implies feasibility of v_1, v_2, i.e. $v_1 \leq \psi$ and $v_2 \leq \psi$.

Using v_1 and v_2 as test functions in the variational inequality for y_1 and y_2, respectively, and adding both inequalities, gives

$$a(y_1 - y_2,\, y_1 - y_2 - [y_1 - y_2]_k) \leq (u_1 - u_2, y_1 - y_2 - [y_1 - y_2]_k).$$

With the notation $v := y_1 - y_2$, $v_k := y_1 - y_2 - [y_1 - y_2]_k$, we obtain by the properties of the differential operator

$$a(v_k, v_k) \leq (u_1 - u_2, v_k).$$

Now, we can proceed as in Stampacchia [15, Theorem 4.1], to obtain the existence of a constant $c > 0$ with

$$\|y_1 - y_2\|_{L^\infty} = \|v\|_{L^\infty} \leq c\|u_1 - u_2\|_{L^2},$$

where c is independent of u_1, u_2. ☐

Under the strong regularity assumptions above one has

Lemma 2.3. *For $u \in L^2(\Omega)$ the unique solution (y, λ) of (1.3) belongs to $H_0^1(\Omega) \cap H^2(\Omega) \times L^2(\Omega)$. If in addition $u \in L^p(\Omega)$ and $\max(0, A\psi - u) \in L^p(\Omega)$, for $p \in [2, \infty)$, then $(y, \lambda) \in W^{2,p}(\Omega) \times L^p(\Omega)$.*

Proof. The result can be obtained from Brezis and Stampacchia [3]. Using Grisvard [5], the regularity result transfers to domains Ω as specified above. For a different approach we refer to [8]. ☐

Lemma 2.4. *The mapping* $u \mapsto (y(u), \lambda(u))$ *is directionally differentiable from* $H^{-1}(\Omega)$ *to* $H^1_0(\Omega) \times H^{-1}(\Omega)$. *The directional derivative* $y'(u; h) = z$ *satisfies* $z \in S(y)$ *and it is given as the solution of the variational inequality*

$$a(z, \phi - z) \geq (h, \phi - z) \text{ for all } \phi \in S(y), \tag{2.1}$$

where

$$S(y) = \{\phi \in H^1_0(\Omega) : \langle \lambda, \phi \rangle = 0, \text{ and } \phi \leq 0 \text{ whenever } y - \psi = 0\}.$$

Moreover, the directional derivative $\lambda'(u; h) = \eta \in H^{-1}(\Omega)$ *satisfies*

$$Az + \eta = h, \ \langle \eta, y - \psi \rangle = 0, \ \langle \eta, z \rangle = 0$$

and

$$\langle \eta, \phi \rangle \leq 0, \quad \text{for all } \phi \in S(y).$$

This lemma again holds under the weaker regularity assumptions on Ω and the coefficients. Under the stronger assumptions, which imply L^2- regularity of λ, the set $S(y)$ can equivalently be expressed as

$$\begin{aligned} S(y) = \{\phi \in H^1_0(\Omega) : \ & \phi = 0 \text{ whenever } \lambda > 0 \\ & \text{and } \phi \leq 0 \text{ whenever } \lambda = y - \psi = 0\}. \end{aligned}$$

Proof. The variational inequality (2.1) for $y'(u; h)$ was proven by Mignot [12]. Setting $\eta = h - Az \in H^{-1}(\Omega)$ and testing

$$\langle \eta, \phi - z \rangle \leq 0$$

with $\phi = 0$, $\phi = 2z$ and $\phi = z + \tilde{\phi}$, for $\tilde{\phi} \in S(y)$, we find $\langle \eta, z \rangle = 0$ and $\langle \eta, \phi \rangle \leq 0$ for all $\phi \in S(y)$. Note that $\phi = 0, 2z, z + \tilde{\phi}, \tilde{\phi} \in S(y)$, are suitable as test functions in (2.1).

To verify the $\langle \eta, y - \psi \rangle = 0$, we pass to the limit in $t \to \frac{1}{t}\langle \lambda_t, y_t - \psi \rangle = 0$ as $t \to 0^+$ using the product rule. Here (y_t, λ_t) denote the solution to (1.3) with $u = u + th$. Since $y'(u; h) = z \in S(y)$ we find $\langle \eta, y - \psi \rangle = 0$ as desired. ☐

If $\lambda \in L^2(\Omega)$ the lemma implies that pointwise a.e. $z\lambda = 0$ is satisfied. If additionally the mapping $u \mapsto \lambda$ would be directionally differentiable with values in $L^2(\Omega)$, then using (1.5) for example, the following relations could be obtained on the biactive set B:

$$z \leq 0, \ \eta \geq 0, \ z\eta = 0 \text{ on } B = \{\lambda = 0, \ y = \psi\}.$$

Remark 2.1. If the biactive set B is empty, then the mapping $u \mapsto y(u)$ is Gâteaux differentiable at u, see [12]. Then the system in Lemma 2.4 can be simplified as follows: The set $S(y)$ is now given by

$$S(y) = \{\phi \in H_0^1(\Omega) : \ \phi = 0 \text{ whenever } y - \psi = 0\}.$$

The directional derivative $z = y'(u; h) \in S(y)$ is given as the solution of the variational equation

$$a(z, \phi) = (h, \phi) \text{ for all } \phi \in S(y),$$

which results in the following property of $\eta = \lambda'(u; h)$:

$$\langle \eta, \phi \rangle = 0 \text{ for all } \phi \in S(y).$$

Moreover, as proven in [12] the set of all $u \in L^2(\Omega)$ such that $y(\cdot)$ Gâteaux differentiable at u is dense in $L^2(\Omega)$.

2.1 Necessary optimality condition

Conditions (v) and (vi) of the standing assumptions together with Lemma 2.1 imply the existence of at least one solution (y^*, u^*) with $y^* = y(u^*)$ to (P).

Let us first state a consequence of local optimality, see e.g. [13] for a proof.

Lemma 2.5. *Let* (y^*, u^*) *be a locally optimal pair for the optimal control problem* (P). *Then*

$$g'(y^*)z + j'(u^*)h \geq 0$$

for all $h \in L^2(\Omega)$ *and* $z = y'(u^*; h)$.

The result of the previous Lemma is a necessary optimality condition that is based solely on directional (Bouligand, conical) derivatives. In analogy to the terminology in mathematical programming with complementarity constraints (mpcc), we call this property *B-stationarity.* This stationarity result does not give any information about gradients and their representation by dual quantities. For this purpose we have the following result.

Theorem 2.1. *Let* (y^*, u^*) *be a locally optimal pair for the optimal control problem* (P) *with associated multiplier* $\lambda^* \in L^2(\Omega)$. *Then there exist uniquely* *determined adjoint states* $p^* \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *and* $\mu^* \in H^{-1}(\Omega) \cap (L^\infty(\Omega))^*$ *such that*

$$A^* p^* + \mu^* + g'(y^*) = 0 \quad \textit{and} \quad p^* \geq 0 \textit{ where } y^* = \psi, \tag{2.2}$$

$$\lambda^* p^* = 0 \textit{ a.e. on } \Omega, \quad \textit{and } \langle \mu^*, p^* \rangle \geq 0 \tag{2.3}$$

$$\langle \mu^*, \varphi(y^* - \psi) \rangle = 0 \textit{ for all } \varphi \in C^1(\bar{\Omega}) \textit{ such that } \varphi\psi|_\Gamma = 0, \tag{2.4}$$

(2.5)
$$\langle \mu^*, \phi \rangle \geq 0 \quad \textit{for all } \phi \in H_0^1(\Omega) \textit{ with } \phi \geq 0 \textit{ on } \{y^* = \psi\} \textit{ and } \langle \lambda^*, \phi \rangle = 0,$$

$$j'(u^*) - p^* = 0. \tag{2.6}$$

Moreover, we have the following sign condition for μ^* *on* B*:*

$$\langle \mu^*, \phi \rangle \geq 0 \quad \textit{for all } \phi \in H_0^1(\Omega), \ \phi \geq 0 \textit{ on } B, \ \phi = 0 \textit{ on } \Omega \setminus B. \tag{2.7}$$

Proof. The existence and uniqueness of the adjoint state p^* for given (y^*, u^*) satisfying the following properties was proven for instance in [12, 13]:

$$\begin{aligned} &a(\phi, p^*) + (g'(y^*), \phi) \leq 0 \\ &\text{for all } \phi \in H_0^1(\Omega) \text{ with } \phi \geq 0 \text{ on } \{y^* = \psi\} \text{ and } \langle \lambda^*, \phi \rangle = 0 \end{aligned} \tag{2.8}$$

and

$$p^* \geq 0 \text{ on } \{y^* = \psi\}, \ \langle \lambda^*, p^* \rangle = 0, \text{ and } j'(u^*) = p^*. \tag{2.9}$$

Setting $\mu^* = -A^* p^* - g'(y^*)$ together with the previous properties implies (2.2) and (2.6). From (2.8) it follows that (2.5) is satisfied. Choosing $\phi \geq 0$ on B and $\phi = 0$ on $\Omega \setminus B$, (2.7) follows from (2.5). The property $\mu^* \in (L^\infty(\Omega))^*$ was obtained in [9, Theorem 5.1].

From $\lambda^* \in L^2(\Omega)$ we conclude that $\lambda^* p^* = 0$ a.e. on Ω. In fact, $\lambda^* = 0$ on $\{y^* < \psi\}$, and $\lambda^* \geq 0$, $p^* \geq 0$ on $\{y^* = \psi\}$ together with $\langle \lambda^*, p^* \rangle$ from (2.9) implies the claim.

Testing (2.8) with p^* we have $\langle \mu^*, p^* \rangle \geq 0$, and thus (2.3) holds. Choosing $\phi = \pm\varphi(y^* - \psi)$ in (2.5) we find $\langle \mu^*, \varphi(y^* - \psi) \rangle = 0$ if $\varphi\psi = 0$ on $\partial\Omega$. This

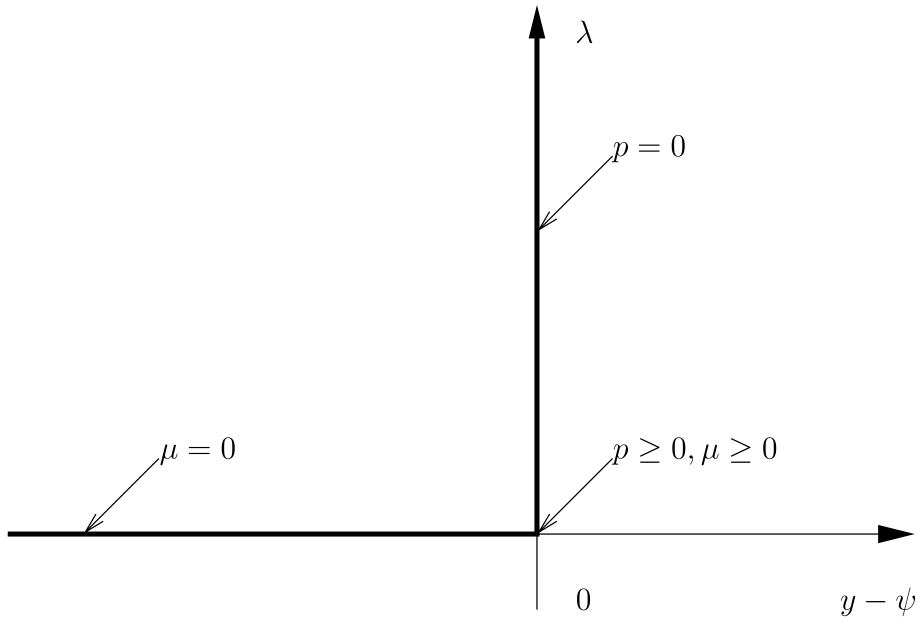

Figure 1: Complementarity relations

gives (2.4). If $\psi = 0$ on $\partial\Gamma$ then φ can be chosen as the constant function with value 1.

The regularity $p^* \in L^\infty(\Omega)$ can be proven with arguments similar to the ones in [15] applied to the variational inequality. Here we use that $g'(y^*) \in L^2(\Omega)$ and $(\lambda^*, (p^* - k)^+) = 0$ and $(\lambda^*, (p^* + k)^-) = 0$ for each $k \geq 0$, which holds since $p^* \lambda^* = 0$. ☐

Figure 1 illustrates the relationsships of primal and dual variables on the active, bi-active and inactive sets.

Under special assumptions on the function j, one can conclude higher regularity of the optimal controls.

Corollary 2.1. *Let* $j(u) = \frac{\alpha}{2}\|u\|^2_{L^2}$*,* $\alpha > 0, \psi \in W^{2,p}(\Omega)$*,* $p > 2$*, and let* (y^*, u^*) *be locally optimal. Then* $u^* \in H^1(\Omega) \cap L^\infty(\Omega)$ *and* $y^* \in W^{2,p}(\Omega)$*.*

Proof. From (2.6) of Theorem 2.1 it follows that $u^* \in L^\infty(\Omega)$. The regularity result for y^* is a consequence of Lemma 2.3. ☐

If the control-to-state mapping $u \mapsto y(u)$ is Gâteaux-differentiable, see Remark 2.1, then the optimality system can be rephrased as follows, see [12, p. 161].

Corollary 2.2. *Let the assumptions of Theorem 2.1 be satisfied. Let* $y(\cdot)$ *be Gâteaux differentiable at* u^*. *Then we have*

$$a(\phi, p^*) + (g'(y^*), \phi) = 0 \quad \textit{for all } \phi \in S(y).$$

In particular, it holds $\langle \eta, p^* \rangle = \langle \mu^*, z \rangle = 0$ *for* $z = y'(u^*; h)$, $\eta = \lambda'(u^*; h)$, $h \in L^2(\Omega)$.

As already mentioned, optimal control of variational inequalities can be interpreted as optimization with complementarity constraints. In analogy to the analysis of mpcc problems in finite dimensions, the system (2.2)–(2.7) describes strong stationarity, see also Hintermüller and Kopacka [6]. Without the information on the sign of p^* and μ^* on the biactive set, the system (2.2)–(2.6) denotes C-stationarity, which is a weaker form of stationarity. For a survey of different stationarity concepts for finite-dimensional optimization problems with complementarity constraints we refer to Scheel and Scholtes [14].

If one would formally derive the optimality system using the Lagrangian

$$L(y, u, \lambda, p, \mu, q) = g(y) + j(u) + \langle Ay + \lambda - u, p \rangle + \langle \mu, y - \psi \rangle + \langle \lambda, q \rangle$$

then the adjoint state p could be identified with the multiplier q of the constraint $\lambda \geq 0$. The measure μ would play the role of a multiplier to the state constraint $y \leq \psi$. The constraint $\langle \lambda, y - \psi \rangle = 0$ is treated implicitly, since the existence of a Lagrange multiplier to $\langle \lambda, y - \psi \rangle = 0$ *does not* follow from first-order optimality conditions, see the discussion and counterexample in Bergounioux and Mignot [2].

Remark 2.2. (i) In the case of strict complementarity, i.e. $\lambda^* > 0$ on $y^* = \psi$, we have $p^* = 0$ on the active set $\{\psi = y^*\}$ by (2.2) and (2.3).

(ii) The support of μ^* is contained on the active set $\{y^* = \psi\}$: In fact, by (2.5) we have $\langle \mu^*, \phi \rangle = 0$ for all $\phi \in H_0^1(\Omega)$ with $\phi = 0$ on the active set. Moreover, (2.7) gives information on the sign of μ on the bi-active set.

Since we have $\mu^* \in (L^\infty(\Omega))^*$, we can write $\mu^* = \mu_r + \mu_p$, where $\mu_r \in L^1(\Omega)$ is the regular part of μ^*, and μ_p is purely finitely additive in the sense of Yosida and Hewitt [16]. The addends μ_r and μ_p are uniquely determined, [16, Theorem 1.24]. Regarding the structure of μ the following result holds.

Proposition 2.1. *Under the assumptions of Theorem 2.1 we have*

$$\operatorname{supp}\mu_p \subset \overline{\{\lambda^* = 0\}} \cap \overline{\{y^* = \psi\}}, \quad \mu_r = -g'(y^*) \text{ on } \{\lambda^* > 0\}.$$

In the case of strict complementarity ($B = \emptyset$) it holds

$$\operatorname{supp}\mu_p \subset \partial\{y^* = \psi\} = \partial\{\lambda^* = 0\}.$$

Proof. Take $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\phi = 0$ on $\{\lambda^* = 0\}$. Then we have $\phi p^* = 0$ a.e. on Ω, which gives $a(\phi, p^*) = 0$. Utilizing ϕ as test function in the adjoint equation (2.2) implies

$$\langle \mu^*, \phi\rangle = \langle \mu_p, \phi\rangle + \int_{\{\lambda^*>0\}} \mu_r\phi = -\int_{\{\lambda^*>0\}} g'(y^*)\phi.$$

Therefore $\langle \mu_p, \phi\rangle = 0$ and $\mu_r = g'(y^*)$ on $\{\lambda^* = 0\}$. Moreover, we obtain from Theorem 2.1, (2.5) that $\langle \mu, \tilde{\phi}\rangle = 0$ for all regular $\tilde{\phi}$ with $\tilde{\phi} = 0$ on $\overline{\{y^* = \psi\}}$, which gives $\langle \mu_p, \tilde{\phi}\rangle = 0$. Hence, the support of μ_p is contained in $\overline{\{\lambda^* = 0\}} \cap \overline{\{y^* = \psi\}}$.

In general, $\partial\{y^* = \psi\}$ and $\partial\{\lambda^* = 0\}$ are subsets of $\overline{\{\lambda^* = 0\}} \cap \overline{\{y^* = \psi\}}$. In the case $B = \emptyset$ the reverse inclusion can be proven, and the claim follows. ☐

If in addition y^*, ψ, λ^* would be continuous, then $\operatorname{supp}\mu_p \subset B$ could be obtained.

2.2 Sufficient optimality condition

Now let us introduce the coercivity condition that will ensure local optimality.

Assumption 1. *(i) There is $\gamma > 0$ such that*

$$j''(u^*)(h,h) \ge \gamma\|h\|_{L^2}^2 \quad \text{for all } h \in L^2(\Omega). \tag{2.10}$$

(ii) For all $h \in L^2(\Omega) \setminus \{0\}$ and $z = y'(u^*; h)$ with $j'(u^*)h + g'(y^*)z = 0$, we have

$$g''(y^*)(z,z) + j''(u^*)(h,h) > 0. \tag{2.11}$$

(iii) There exists a constant $\tau > 0$ such that

$$p^* \ge 0 \text{ on } \{\psi - \tau < y^* < \psi\}. \tag{2.12}$$

(iv) Moreover, μ satisfies

$$\langle \mu^*, \phi \rangle \geq 0, \quad \phi \in H_0^1(\Omega), \phi \geq 0. \tag{2.13}$$

The following theorem asserts that the first order optimality condition given by Theorem 2.1 together with Assumption 1 imply local optimality.

Theorem 2.2. *Let $(y^*, u^*, \lambda^*, p^*, \mu^*)$ satisfy the first-order optimality system (2.2)–(2.6) given by Theorem 2.1. If Assumption 1 is fulfilled, then (y^*, u^*) is locally optimal, and there exist $\alpha > 0, r > 0$ such that*

$$\begin{aligned} &J(y^*, u^*) + \alpha \|v - u^*\|_{L^2}^2 \leq J(y(v), v) \\ &\textit{for all } v \in L^2(\Omega) \textit{ with } \|u^* - v\|_{L^2} < r. \end{aligned} \tag{2.14}$$

Proof. Let us assume that the quadratic growth condition (2.14) does not hold. Then there exist sequences u_k, $u_k \to u^*$ in $L^2(\Omega)$, $y_k = y(u_k), \lambda_k = \lambda(u_k)$, such that

$$J(y^*, u^*) + \frac{1}{k}\|u_k - u^*\|_{L^2}^2 > J(y_k, u_k).$$

Introducing the quantities $\rho_k = \|u_k - u^*\|_{L^2}$ and $h_k = \frac{1}{\rho_k}(u_k - u^*)$, this inequality becomes

$$J(y^*, u^*) + \frac{\rho_k^2}{k} > J(y_k, u_k).$$

By Lemma 2.1 and (1.3), the sequences $h_k := \frac{1}{\rho_k}(u_k - u^*)$, $z_k := \frac{1}{\rho_k}(y_k - y^*)$, $\eta_k := \frac{1}{\rho_k}(\lambda_k - \lambda^*)$ are bounded in $L^2(\Omega)$, $H_0^1(\Omega)$, and $H^{-1}(\Omega)$, respectively. Thus we have by compact embeddings and after extracting subsequences

$$\begin{aligned} &h_k \rightharpoonup h \text{ in } L^2(\Omega), \quad h_k \to h \text{ in } H^{-1}(\Omega), \\ &z_k \to z \text{ in } H_0^1(\Omega), \quad \eta_k \to \eta \text{ in } H^{-1}(\Omega). \end{aligned} \tag{2.15}$$

Let us investigate the difference of the values of the objective functional J.

Using the optimality system, we find

$$\begin{aligned} J(y_k, u_k) - J(y^*, u^*) &= g'(y^*)(y_k - y^*) + \frac{1}{2} g''(\tilde{y}_k)(y^* - y_k)^2 \\ &\quad + j'(u^*)(u_k - u^*) + \frac{1}{2} j''(\tilde{u}_k)(u^* - u_k)^2 \\ &\quad + \langle A(y_k - y^*) + \lambda_k - \lambda^* - (u_k - u^*), p^* \rangle \\ &= \frac{1}{2} g''(\tilde{y}_k)(y^* - y_k)^2 + \frac{1}{2} j''(\tilde{u}_k)(u^* - u_k)^2 \\ &\quad + \langle \lambda_k - \lambda^*, p^* \rangle - \langle \mu^*, y_k - y^* \rangle < \frac{\rho_k^2}{k}, \end{aligned}$$

where $\tilde{y}_k$ and $\tilde{u}_k$ denote elements between y_k and y^*, and u_k and u^*, respectively. We obtain

$$\begin{aligned} &\langle \frac{1}{\rho_k}(\lambda_k - \lambda^*), p^* \rangle - \langle \mu^*, \frac{1}{\rho_k}(y_k - y^*) \rangle \\ &\qquad < \frac{\rho_k}{k} - \frac{\rho_k}{2} \Bigg(g''(\tilde{y}_k) \left(\frac{1}{\rho_k}(y^* - y_k) \right)^2 + j''(\tilde{u}_k) \left(\frac{1}{\rho_k}(u^* - u_k) \right)^2 \Bigg), \end{aligned} \tag{2.16}$$

or with the notation introduced above

$$\langle \eta_k, p^* \rangle - \langle \mu^*, z_k \rangle < \frac{\rho_k}{k} - \frac{\rho_k}{2} \left(g''(\tilde{y}_k)(z_k, z_k) + j''(\tilde{u}_k)(h_k, h_k) \right). \tag{2.17}$$

Due to Lemma 2.4, the mapping $u \mapsto (y, \lambda)$ is directionally differentiable, which yields that $z = y'(u^*; h)$ and $\eta = \lambda'(u^*; h)$. Together with (2.17), the general assumptions (v) and (vi), and the fact that $\lim_{k \to \infty} \rho_k = 0$ this implies

$$\langle \eta, p^* \rangle - \langle \mu^*, z \rangle \leq 0.$$

Since $Az + \eta = h$ equations (2.2) and (2.6) of the first order optimality system imply the identity

$$\langle \eta, p^* \rangle - \langle \mu^*, z \rangle = j'(u^*)h + g'(y^*)z. \tag{2.18}$$

By B-stationarity, cf. Lemma 2.5, it holds that $j'(u^*)h + g'(y^*)z \geq 0$, which in turn gives $\langle \eta, p^* \rangle - \langle \mu^*, z \rangle \geq 0$, and hence $\langle \eta, p^* \rangle - \langle \mu^*, z \rangle = 0$, which, in turn implies that

$$j'(u^*)h + g'(y^*)z = 0. \tag{2.19}$$

Now, let us prove that the left-hand side of (2.16) is non-negative for k large enough. Due to Lemma 2.2 we can choose k_0 sufficiently large, such that $\|y^* - y_k\|_{L^\infty} < \tau$ for all $k > k_0$, where τ is chosen according to Assumption 1(iii). Then the inclusions $\operatorname{supp} \lambda_k \subset \{y_k = \psi\} \subset \{\psi - \tau < y^* \le \psi\}$ hold. We obtain using $\lambda^* p^* = 0$ and $p^* \ge 0$ on $\{\psi - \tau < y^* \le \psi\}$, see (2.2),(2.12),

$$\langle \lambda_k - \lambda^*, p^* \rangle = \int_{\psi - \tau < y^* \le \psi} \lambda_k p^* \ge 0. \tag{2.20}$$

Since $\langle \mu^*, y^* - \psi \rangle = 0$ by (2.3), we have as a consequence of (2.13)

$$-\langle \mu^*, y_k - y^* \rangle = -\langle \mu^*, y_k - \psi \rangle \ge 0 \tag{2.21}$$

for all k. Combining (2.16)–(2.21), we obtain

$$\begin{aligned}\frac{1}{2}\left(g''(y^*)(z_k, z_k) + j''(u^*)(h_k, h_k)\right) < \frac{1}{k} - \frac{1}{2}\big(&g''(\tilde{y}_k)(z_k, z_k) - g''(y^*)(z_k, z_k) \\ &+ j''(\tilde{u}_k)(h_k, h_k) - j''(u^*)(h_k, h_k)\big).\end{aligned}$$

Since j'' and g'' are continuous by assumption, and the sequences h_k and z_k are bounded, the right-hand side vanishes for $k \to \infty$. Turning to the left hand side, we recall that $z_k \to z$ in $H_0^1(\Omega)$ strongly by (2.15). The positivity requirement Assumption 1(i) on j'', implies that $j''(u^*)(h, h) \le \liminf j''(u^*)(h_k, h_k)$. Consequently

$$g''(y^*)(z, z) + j''(u^*)(h, h) \le 0.$$

By Assumption 1(ii), which is applicable due to (2.19), we find $h = 0$. This in turn gives $z = 0$, since the variational inequality (2.1) determining z is uniquely solvable. Due to $\|h_k\|_{L^2} = 1$ and Assumption 1(i) we have

$$0 < \gamma \le j''(u^*)(h_k, h_k) < \frac{1}{k} - \frac{1}{2}\left(g''(\tilde{y}_k)(z_k, z_k) + j''(\tilde{u}_k)(h_k, h_k) - j''(u^*)(h_k, h_k)\right).$$

Since $z_k \to z = 0$ strongly in $H_0^1(\Omega)$, the right-hand side vanishes for $k \to \infty$ yielding finally the contradiction. ☐

For sufficiency, it was necessary to impose a sign condition on μ^* as well as on p^* on the almost bi-active set $\{\psi - \tau < y^* < \psi\} \cap \{\lambda = 0\}$. This is a major difference in comparison to second-order sufficient optimality conditions for state-constrained optimal control problems, where the sign conditions (2.12) and (2.13) are not necessary, since there in particular $\mu^* \ge 0$ follows from the first-order necessary optimality conditions. This is not the case here, where we have information on the sign of μ only on the bi-active set itself.

Remark 2.3. (i) In case $g'(y^*) \leq 0$ a.e. in Ω we have $p^* \geq 0$ a.e. in Ω and hence Assumption 1(iii) holds. In fact, decomposing p^* as $(p^*)^+ - (p^*)^-$ we have

$$-a((p^*)^-, p^*) - \langle \mu^*, (p^*)^- \rangle = (g'(y^*), (p^*)^-) \leq 0.$$

By (2.2)-(2.5) we have $\langle \mu^*, (p^*)^- \rangle = 0$ and hence the general assumption (iv) implies that $(p^*)^- = 0$. For the tracking type functional $g(y) = \frac{1}{2}\|y - z\|_{L^2}^2$, we have $g'(y^*) \leq 0$ if $z \geq y^*$ a.e.in Ω.

(ii) If one would have strong convergence $\lambda_k \to \lambda^*$ in $L^\infty(\Omega)$, then one could weaken the non-negativity assumption on $\mu^* = \mu_p + \mu_r$ to: $\mu_p \geq 0$ and $\mu_r \geq 0$ on $0 < \lambda^* < \tau$, and the corresponding arguments are analogous to (2.20) to obtain the conclusion of Theorem 2.2.

(iii) If the mapping $y(\cdot)$ is Gâteaux differentiable at u^*, the condition $j'(u^*)h + g'(y^*)z = 0$ in Assumption 1(ii) is redundant. In fact, then one has by (2.18) and Corollary 2.2

$$j'(u^*)h + g'(y^*)z = \langle \eta, p^* \rangle - \langle \mu^*, z \rangle = 0.$$

3 Convergence properties and feasibility of regularized problems

For numerical purposes the optimality system obtained in Theorem 2.1 has the disadvantage of involving Lagrange multipliers that are measures. This is one of the motivations to introduce regularisation techniques, which we already announced in Section 1 in form of problem (P$_c$).

3.1 Convergence properties for the regularized problems

This subsection is devoted to a brief summary of the convergence properties of the regularized problems (P$_c$) as $c \to \infty$. Let us commence with considering the regularized equation

$$Ay + \max_c(0, \bar{\lambda} + c(y - \psi)) = u, \tag{3.1}$$

and define

$$\lambda_c = \max{}_c(0, \bar{\lambda} + c(y - \psi)).$$

The following convergence result is taken from [10].

Lemma 3.1. *For* $u \in L^2(\Omega)$ *let* (y_c, λ_c) *denote the solution to* (3.1). *Then* (y_c, λ_c) *converge to the unique solution* (y, λ) *of* (1.3) *in the sense that* $y_c \to y = y(u)$ *strongly in* $H_0^1(\Omega)$ *and* $\lambda_c \to \lambda$ *weakly in* $H^{-1}(\Omega)$ *as* $c \to \infty$.

In addition, we obtain convergence rate results in $L^\infty(\Omega)$.

Lemma 3.2. *Let* $\bar{\lambda} \in L^\infty(\Omega)$ *be given. Then for each* $c > 0$

$$y \leq y_c + \frac{\|\bar{\lambda}\|_{L^\infty}}{c} + \frac{1}{2c^2} \quad \text{a.e. on } \Omega.$$

Proof. Let us define for $k \geq 0$ the function $\phi_k = \max(y - y_c - k, 0) \in H_0^1(\Omega)$. Subtracting (3.1) from the first equation in (1.3) and testing with ϕ_k gives

$$a(y - y_c - k, \phi_k) + \langle \lambda - \max{}_c\left(\bar{\lambda} + c((y_c) - \psi)\right), \phi_k \rangle + a(k, \phi_k) = 0.$$

Let us set $k := \frac{\|\bar{\lambda}\|_{L^\infty}}{c} + \frac{1}{2c^2}$. Since $\lambda \geq 0$, we have $\langle \lambda, \phi_k \rangle \geq 0$. Moreover, since $y \leq \psi$ we obtain on the set $\{\phi_k > 0\}$

$$\begin{aligned} \max{}_c(\bar{\lambda} + c(y_c - \psi)) &\leq \max{}_c(\bar{\lambda} + c(y_c - y)) \\ &\leq \max{}_c(\bar{\lambda} - ck)) \leq \max{}_c(-\tfrac{1}{2c}) = 0, \end{aligned}$$

which, together with $a(1, \phi) \geq 0$ implies that $a(y - y_c - k, \phi_k) \leq 0$, and hence $\phi_k = 0$, which implies the desired result. ☐

Remark 3.1. In [10] a bilateral L^∞ bound on the difference of $y - y_c$ at the expense of additional regularity assumptions was obtained. Let us set

$$\mathcal{A} = \{x \in \Omega \colon y(x) = \psi(x)\}, \quad \mathcal{A}_c = \{x \in \Omega \colon y_c(x) = \psi(x)\}.$$

Assume that $u \in L^\infty(\Omega)$. If, moreover, the boundary $\partial\mathcal{A}$ of the active set is a $C^{1,1}$ manifold in $\mathbb{R}^{n-1}$ and for every $c > 0$ the boundary $\partial\mathcal{A}_c$ of $\mathcal{A}_c$ is a Lipschitzian manifold in $\mathbb{R}^{n-1}$, then

$$\|y_c - y^*\|_{L^\infty(\Omega)} \leq \tfrac{1}{c}\|f - A\psi\|_{L^\infty(\Omega)}.$$

In the following subsection we shall see that setting $\bar{\lambda}$ large enough yields feasibility of y_c. This enables to give L^∞-convergence results of the following type.

Proposition 3.1. *Let* $\bar{\lambda} \in L^\infty(\Omega)$ *be given and suppose that* $y_c \leq \psi$. *Then*

$$\|y - y_c\|_{L^\infty} \leq \frac{\|\bar{\lambda}\|_{L^\infty}}{c} + \frac{1}{2c^2}.$$

Proof. Let us define the test function

$$\phi_k = \begin{cases} y - y_c - k & \text{if } y - y_c > k \\ y - y_c + k & \text{if } y - y_c < -k \\ 0 & \text{otherwise,} \end{cases}$$

where $k \in \mathbb{R}$. Proceeding as in the proof for Lemma 3.2 we get

$$a(y - y_c - k, \phi_k) + \langle \lambda - \max_c \left(\bar{\lambda} + c((y_c) - \psi)\right), \phi_k \rangle + a(k, \phi_k) = 0.$$

We can split $\phi_k = \max(\phi_k, 0) + \min(\phi_k, 0)$, since both addends belong to $H_0^1(\Omega)$. As in the previous proof we obtain $\langle \lambda, \max(\phi_k, 0)\rangle \geq 0$. By feasibility of y_c we find

$$0 \geq \min(\phi_k, 0) = \min(y - y_c + k, 0) \geq \min(y - \psi + k, 0) \geq \min(y - \psi, 0) = y - \psi.$$

This implies

$$0 \geq \langle \lambda, \min(\phi_k, 0)\rangle \geq \langle \lambda, y - \psi \rangle = 0.$$

On the set $\{\phi_k > 0\}$, we find

$$\max_c \left(\bar{\lambda} + c((y_c) - \psi)\right) = 0 \text{ for } k \geq \frac{\|\bar{\lambda}\|_{L^\infty}}{c} + \frac{1}{2c^2},$$

and hence $(\max_c \left(\bar{\lambda} + c((y_c) - \psi)\right), \max(\phi_k, 0)) = 0$. Since from the definition $\max_c \left(\bar{\lambda} + c((y_c) - \psi)\right) \geq 0$, we have

$$-(\max_c \left(\bar{\lambda} + c((y_c) - \psi)\right), \min(\phi_k, 0)) \geq 0.$$

Thus, we obtained $a(y - y_c - k, \phi_k) \leq 0$ for $k \geq \frac{\|\bar{\lambda}\|_{L^\infty}}{c} + \frac{1}{2c^2}$, which implies $\phi_k = 0$. $\square$

Remark 3.2. Analogously to Remark 3.1 a slightly tighter estimate for the feasible case was obtained in [10] under additional regularity requirements. In fact, if $\mathcal{A}_c$ is a domain with a $C^{1,1}$ boundary, then

$$\|y_c - y\|_{L^\infty(\Omega)} \leq \tfrac{1}{c}\, \|\bar{\lambda}\|_{L^\infty(\Omega)}.$$

Up to now, we studied convergence of solutions for fixed right-hand side u in (3.1). Let us now turn to the case, where the right-hand side is a (possibly weakly) convergent sequence. Due to the monotonicity of the $\max_c$-function we obtain the following Lipschitz continuity result for the solutions of the regularized equation.

Lemma 3.3. *Let $u_1, u_2 \in H^{-1}(\Omega)$ be given. Then there exists a constant $L > 0$ independent of c such that*

$$\|y_c(u_1) - y_c(u_2)\|_{H_0^1} \leq L\|u_1 - u_2\|_{H^{-1}}.$$

Lemma 3.4. *Let $u_c \rightharpoonup u$ in $L^2(\Omega)$. Then $y_c \to y$ in $H_0^1(\Omega)$ strongly.*

Proof. It holds

$$\|y_c(u_c) - y(u)\|_{H^1} \leq \|y_c(u_c) - y_c(u)\|_{H^1} + \|y_c(u) - y(u)\|_{H^1}.$$

The first addend can be majorized by $L\|u_c - u\|_{H^{-1}}$ due to Lemma 3.3. By compact embeddings this term tends to zero for $c \longrightarrow \infty$. The second addend tends to zero according to Lemma 3.1. ☐

The final result of this section addresses convergence of the solutions of the regularized optimal control problem (P_c) to those of original problem (P).

Proposition 3.2. *Let $j : L^2(\Omega) \to \mathbb{R}$ be weakly lower semi-continuous.*

For every $c > 0$ problem (P_c) admits a global solution $(y_c, u_c) = (y(u_c), u_c) \in H_0^1(\Omega) \times L^2(\Omega)$. For every subsequence of controls $\{u_{c_n}\}$ converging weakly in $L^2(\Omega)$ (of which there exists at least one) to some u^*, the corresponding states $y_{c_n} = y(u_{c_n})$ converge strongly in $H_0^1(\Omega)$ to $y^* = y(u^*)$, and (y^*, u^*) is a global solution to (P). Moreover $\lambda_{c_n} = \max_c(0, \bar{\lambda} + c_n(y_{c_n} - \psi)) \rightharpoonup \lambda(y^*)$ weakly in $H^{-1}(\Omega)$.

In addition, in the feasible case with $y_{c_n} \leq \psi$ for all n, $\{(p_{c_n}, \mu_{c_n})\}$ converge weakly in $H_0^1(\Omega)$ and weakly star in $L^\infty(\Omega)^*$ to $(p^*, \mu^*) \in H_0^1(\Omega) \times L^\infty(\Omega)^*$ satisfying (2.2)–(2.6).

Proof. Since j is radially unbounded and g is bounded from below, every minimizing sequence $\{(y_c(u_n), u_n)\}$ to (P_c) has a weakly convergent subsequence, denoted by the same symbol, with weak limit $u_c \in L^2(\Omega)$. By Lemma 3.3 we find $y_c(u_n) \to y_c(u_c)$ strongly in $H_0^1(\Omega)$. Weak lower semi-continuity of j and continuity of $g : H_0^1(\Omega) \to \mathbb{R}$ implies that $(y_c(u_c), u_c)$ is a solution to (P).

Next consider a family of solutions $\{(y_c, u_c)\}$ to (P_c). Let $y_c(0)$ denote the solution to the equality constraint in (P_c) with $u = 0$, and note that $\{y_c(0)\}_{c\geq 1}$ is bounded in $H_0^1(\Omega)$. Hence $\{g(y_c(0))\}_{c\geq 1}$ is bounded as well. Then $(y_c(0), 0)$ is a feasible pair for (P_c) for every $c > 0$, and $J(y_c, u_c) \leq J(y_c(0), 0)$. Thus $\{j(u_c)\}_{c\geq 1}$ is bounded and radial unboundedness of j implies that $\{u_c\}_{c\geq 1}$ is bounded in $L^2(\Omega)$. Consequently there exists a weakly convergent subsequence u_{c_n} in $L^2(\Omega)$ with weak limit $u^* \in L^2(\Omega)$. By Lemma 3.4 the sequence $y_{c_n} = y(u_{c_n}) \to y(u^*)$ strongly in $H_0^1(\Omega)$. Moreoever $\lambda_{c_n} = \max_{c_n}(0, \bar\lambda + c_n(y_{c_n} - \psi)) \to \lambda(y^*)$ weakly in $H^{-1}(\Omega)$, where $Ay^* + \lambda(y^*) = u^*$. As above we can now pass to the limit in (P_{c_n}) as $n \to \infty$ and obtain that (y^*, u^*) is a solution to (P), with associated Lagrange multiplier $\lambda(y^*)$. By Theorem 2.1 there exists a uniquely associated adjoint state $p^* \in H_0^1(\Omega)$ and $\mu^* \in H^{-1}(\Omega) \cap L^\infty(\Omega)^*$ satisfying (2.2)–(2.6). Convergence of $p_{c_n} \to p^*$ and $\mu_{c_n} \to \mu^*$ weakly in $H_0^1(\Omega)$ and weakly star in $L^\infty(\Omega)^*$ was proved in [9] for the feasible case. ☐

3.2 Feasibility of solutions for large $\bar\lambda$

In this short subsection we give a sufficient condition so that for large $\bar\lambda$ sufficiently large the optimal states y_c of (P_c) are feasible, i.e. $y_c \leq \psi$. The principle idea can be found in Theorem 5.1 of [9], but uniformity with respect to c is not clarified there.

Proposition 3.3. *Let $\rho \geq 0$ and let $y_c \in H_0^1(\Omega)$ denote the solution to $A_c y_c + \max_c(0, \bar\lambda + c(y_c - \psi)) = u$ with $u \in B_\rho = \{u : \|u\|_{L^\infty} \leq \rho\}$. If $\bar\lambda \geq \max(0, -A\psi + \rho)$ then y_c is feasible, i.e. $y_c \leq \psi$ for each $c > 0$.*

Proof. Let $u \in B_\rho$ and consider the regularized equation

$$Ay_c + \max_c(0,\, \bar\lambda + c\,(y_c - \psi)) = u,$$

. Testing this equation with $\phi = \max(0,\, y_c - \psi)$ and using the fact that $\max_c(0,\, x) \geq \max(0,\, x) \geq x$ we obtain

$$\langle Ay_c - A\psi,\, \phi\rangle + (\max(0,\, \bar\lambda + c(y_c - \psi)),\, \phi) + (A\psi - u,\, \phi) = 0,$$

and hence

$$\langle Ay_c - A\psi,\, \phi\rangle + (\max(0,\, -A\psi), \phi) + (\rho - u + A\psi, \phi) + c\,\|\phi\|_{L^2}^2 \leq 0.$$

This implies that $\langle (A(y_c - \psi), y_c - \psi) \leq 0$ and hence $y_c \leq \psi$.

□

Corollary 3.1. *Assume that* $\|j'(u)\|_{L^\infty(\Omega)} \geq K(\|u\|_{L^\infty(\Omega)} - 1)$ *for a constant* K *independent of* $u \in L^\infty(\Omega)$*. Then there exists* $\rho > 0$ *such that the optimal controls* u_c *to* (P_c) *satisfy* $\|u_c\|_{L^\infty(\Omega)} \leq \rho$ *for all* $c \geq 1$ *and hence Proposition 3.3 applies.*

Proof. The family of solutions $\{u_c\}_{c\geq 1}$ is bounded in $L^2(\Omega)$. Hence $\{y_c\}_{c\geq 1}$ is bounded in $H_0^1(\Omega)$ and $\{g'(y_c)\}_{c\geq 1}$ is bounded in $L^2(\Omega)$. Next consider the adjoint equations

$$A^*p + \mathrm{sgn}_c(\bar\lambda + c(y_c - \psi))\, p = -g'(y_c).$$

By the Stampacchia method the family of solutions $\{p_c\}_{c\geq 1}$ is bounded in $L^\infty(\Omega)$ independently of $\bar\lambda$. The claim now follows from the assumption that $\|p_c\|_{L^\infty(\Omega)} = \|j'(u_c)\|_{L^\infty(\Omega)} \geq K(\|u_c\|_{L^\infty(\Omega)} - 1)$. □

3.3 Existence of approximating sequences

Let (y^*, u^*) be a strictly locally optimal pair for (P), that is, there exists $\rho > 0$ such that

$$J(y^*, u^*) < J(y, u) \quad \text{for all } (y, u) \text{ satisfying (1.3) and } \|u - u^*\|_{L^2} < \rho. \tag{3.2}$$

We will show that for each strictly locally optimal pair (y^*, u^*) there is a sequence of local solutions (y_c, u_c) of (P_c) that converges strongly to (y^*, u^*) in $H_0^1(\Omega) \times L^2(\Omega)$. This means in particular, we show existence of a path of solutions $\{y_c, u_c\}_{c>c_0}$.

Theorem 3.1. *Let* j *be weakly lower semi-continuous. Moreover, we require for* j *that*

$$u_n \rightharpoonup u \text{ in } L^2(\Omega), \text{ and } j(u_n) \to j(u) \text{ imply } u_n \to u \text{ in } L^2(\Omega). \tag{3.3}$$

Let (y^*, u^*) *be a strictly locally optimal pair for* (P)*. Then there exists a sequence of local solutions* (y_c, u_c) *of* (P_c) *that converges strongly to* (y^*, u^*) *in* $H_0^1(\Omega) \times L^2(\Omega)$*.*

Proof. Let ρ be given by (3.2) and take ρ' with $0 < \rho' < \rho$. Consider the auxiliary problem

$$(\mathrm{P}_c^{\rho'}) \qquad \begin{cases} \text{Minimize} \quad J(y,u) = g(y) + j(u) \\ \text{over } u \in L^2(\Omega) \text{ with } \|u - u^*\|_{L^2} \le \rho' \text{ and subject to} \\ Ay + \max_c(0, \bar\lambda + c(y-\psi)) = u. \end{cases}$$

Clearly, this optimal control problem is solvable for every $c > 0$. Let u_c denote a global solution of $(\mathrm{P}_c^{\rho'})$. By construction, the set $\{u_c\}_{c>0}$ is bounded, which yields weak convergence of a subsequence $u_{c_n} \rightharpoonup \tilde u$ in $L^2(\Omega)$ with $\|\tilde u - u^*\|_{L^2} \le \rho'$. Due to Lemma 3.4, the corresponding states converge $y_{c_n} \to \tilde y$ in $H_0^1(\Omega)$, where $\tilde y$ is a solution of the variational inequality (1.3) for right-hand side $\tilde u$. This implies $J(y^*, u^*) \le J(\tilde y, \tilde u)$ by optimality of (y^*, u^*).

Denoting by $y_c(u^*)$ the solution of the regularized equation to the control u^*, we have $J(y_c, u_c) \le J(y_c(u^*), u^*)$. By Lemma 3.1, we have $y_c(u^*) \to y^*$ in $H_0^1(\Omega)$.

Then we obtain by (3.2) and the optimality and convergence properties above

$$\begin{aligned} g(y^*) + j(u^*) &\le g(\tilde y) + j(\tilde u) \le \lim g(y_{c_n}) + \liminf j(u_{c_n}) \\ &\le \lim g(y_{c_n}) + \limsup j(u_{c_n}) \le \lim g(y_{c_n}(u^*)) + j(u^*) \\ &= g(y^*) + j(u^*). \end{aligned} \tag{3.4}$$

This implies $J(y^*, u^*) = J(\tilde y, \tilde u)$ and hence $\tilde u = u^*$ by strict local optimality of (y^*, u^*). Moreover, it follows that $\lim j(u_{c_n}) = j(\tilde u)$ which yields $u_{c_n} \to u^*$ in $L^2(\Omega)$ by (3.3). Since u^* is the unique local solution in the L^2- neighborhood of u^* of radius ρ', the whole sequence u_c converges to u^*.

Convergence of $u_c \to u^*$ also implies the existence of c_0 such that $\|u_c - u^*\|_{L^2} < \rho'$ for all $c > c_0$. Consequently, if $c > c_0$, then (y_c, u_c) is locally optimal for (P_c). ☐

The pre-requisite (3.3) is fulfilled for instance for the standard choice $j(u) = \frac{\alpha}{2}\|u\|_{L^2}^2$, $\alpha > 0$.

Regarding convergence of adjoint states and multipliers, we obtain

Corollary 3.2. *Let (y_c, u_c) be a sequence of local solutions of (P_c) converging strongly in $H_0^1(\Omega) \times L^2(\Omega)$ to (y^*, u^*). Let (y^*, u^*) solve the variational inequality and satisfy together with (λ^*, p^*, μ^*) the first-order optimality system*

(2.2)–(2.7) *given by Theorem 2.1. Then we have*

$$\lambda_c \to \lambda^* \text{ and } \mu_c \rightharpoonup \mu^* \text{ in } H^{-1}(\Omega),\ p_c \rightharpoonup p^* \text{ in } H_0^1(\Omega).$$

where (λ_c, p_c, μ_c) *are the corresponding multipliers and adjoint state for* (P_c)*, see* (1.7)*.*

Proof. Due to the strong convergence of y_c, u_c, the strong convergence of λ_c follow immediately

$$\lambda_c = u_c - Ay_c \to u^* - Ay^* = \lambda^* \text{ in } H^{-1}(\Omega).$$

Multiplying the second equation in (1.7) by p_c gives

$$\nu_1 \|p_c\|_{H^1}^2 \leq \|g'(y_c)\|_{L^2} \|p_c\|_{L^2},$$

which gives boundedness of $\{p_c\}$ in $H_0^1(\Omega)$. Hence, we find a subsequence p_{c_n} converging weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ to $\tilde{p}$. The third equation in (1.7) implies

$$j'(u_{c_n}) = p_{c_n} \to j'(u^*) = \tilde{p},$$

which gives $\tilde{p} = p^*$ by optimality condition (2.6). Since the adjoint state p^* is uniquely determined by (y^*, u^*), the whole sequence p_c converges weakly in $H_0^1(\Omega)$ to p^*. Arguing as above, we find for μ_c

$$\mu_c = -A^* p_c - g'(y_c) \rightharpoonup -A^* p^* - g'(y^*) = \mu^* \text{ in } H^{-1}(\Omega),$$

which finishes the proof. ☐

3.4 Sufficient optimality condition for the regularized problems

Here, we will frequently use the second derivative of the $\max_c$-function, which is given by

$$\mathrm{sgn}_c'(x) = \begin{cases} 0, & \text{for } |x| > \frac{1}{2c} \\ c, & \text{for } |x| \leq \frac{1}{2c}. \end{cases} \tag{3.5}$$

Let us denote by (y_c, u_c) a sequence of local minimizers for the regularized problems (P_c) converging to a local minimum (y^*, u^*) of (P) as given by Theorem 3.1. Then Corollary 3.2 provides multipliers (λ_c, p_c, μ_c).

The following coercivity condition is sufficient for local optimality of solutions of the regularized problem (P_c) for sufficiently large c.

Assumption 2. *There exists* $\alpha > 0$ *and* $C_0 > 0$ *such that for all* $c > C_0$

$$g''(y_c)(z,z) + j''(u_c)(h,h) + \left(c^2 sgn_c'\left(\bar\lambda + c(y_c - \psi)\right) p_c z, z\right) \geq \alpha \|h\|_{L^2}^2 \tag{3.6}$$

for all $(z,h) \in H_0^1(\Omega) \times L^2(\Omega)$ *satisfying*

$$Az + c\, sgn_c\left(\bar\lambda + c(y_c - \psi)\right) z = h. \tag{3.7}$$

Theorem 3.2. *Let a sequence* $(y_c, u_c, \lambda_c, p_c, \mu_c)$ *of solutions of the first-order system* (1.7) *be given. Let Assumption 2 be satisfied. Then for all* $c > C_0$ *the pair* (y_c, u_c) *is a strict local minimum of* (P_c) *and it holds*

$$J(y_c(u), u) \geq J(y_c, u_c) + \alpha_c \|u - u_c\|_{L^2}^2$$

for all $u \in L^2(\Omega)$ *with* $\|u - u_c\|_{L^2} < \rho_c$ *with positive constants* α_c, ρ_c, *which depend on* c.

Since we consider a sequence (y_c, u_c) converging strongly to a solution (y^*, u^*) of the original problem (P), the following question arises: If (y^*, u^*) satisfies the second-order condition Assumption 1, does (y_c, u_c) satisfies the sufficient condition above ? Unfortunately, Assumption 1 is not enough to prove Assumption 2. We will need additional assumptions. The first one is due to the discontinuity of the function sgn_c' that appears in (3.6).

Assumption 3. *The following sign condition on the adjoint state holds:* $p_c \geq 0$ *on* $\{|\bar\lambda + c(y_c - \psi)| < 1/2c\}$ *for all* c *sufficiently large.*

Moreover, the coercivity condition in Assumption 1(ii) is not enough, the test space in (2.11) is too small. In particular, we were not able to proof that the solutions z_c of the linearized equation (3.7) tend for $c \to \infty$ to a directional derivative $z = y'(u^*; h)$. Hence we require coercivity on a larger subspace.

Assumption 4. *(i) There is* $\gamma > 0$ *such that*

$$j''(u^*)(h,h) \geq \gamma \|h\|_{L^2}^2 \quad \text{for all } h \in L^2(\Omega).$$

(ii) We assume that

$$g''(y^*)(z,z) + j''(u^*)(h,h) > 0. \tag{3.8}$$

holds for all $(z,h,\eta) \in H_0^1(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$ *satisfying*

$$j'(u^*)h + g'(y^*)z = 0,$$

and

$$\begin{gathered} Az + \eta = h \\ \langle \eta, y^* - \psi \rangle = 0,\ \langle \eta, z \rangle \geq 0,\ \lambda^* z = 0 \ \textit{a.e. on}\ \Omega. \end{gathered} \tag{3.9}$$

Observe, that the test space in Assumption 4(ii) is larger than in Assumption 1(ii): the system (3.9) allows more test functions than the system in Lemma 2.4 that characterizes the directional derivative $z = y'(u^*; h)$, which is used in Assumption 1. Differently to Assumption 1, we do not impose sign conditions on p^* and μ^*, instead the uniform sign condition on p_c, see Assumption 3, is used.

Theorem 3.3. *Let $\bar\lambda$ be chosen as $\bar\lambda \geq \Lambda$ according to Proposition 3.3. Let Assumptions 3 and 4 be satisfied. Then Assumption 2 is fulfilled.*

Proof. Let us suppose that the claim does not hold. Then there exists a sequence (z_c, h_c) such that

$$g''(y_c)(z_c,z_c) + j''(u_c)(h_c,h_c) + \left(c^2 \operatorname{sgn}_c'\left(\bar\lambda + c(y_c - \psi)\right) p_c z_c, z_c\right) \leq \frac{1}{c}\|h_c\|_{L^2}^2 \tag{3.10}$$

with

$$Az_c + c\operatorname{sgn}_c\left(\bar\lambda + c(y_c - \psi)\right) z_c = h_c. \tag{3.11}$$

W.l.o.g. we can choose h_c with $\|h_c\|_{L^2} = 1$. Then the sequence z_c is bounded in $H^1(\Omega)$. Hence after extracting a subsequence if necessary, we find

$$h_c \rightharpoonup h \text{ in } L^2(\Omega) \text{ and } z_c \rightharpoonup z \text{ in } H^1(\Omega).$$

Due to the compact embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ we have $h_c \to h$ in $H^{-1}(\Omega)$. Passing to the limit in equation (3.10) gives $Az + \eta = h$ together with the convergence $\eta_c := c\operatorname{sgn}_c\left(\bar\lambda + c(y_c - \psi)\right) z_c \rightharpoonup \eta$ in $H^{-1}(\Omega)$.

Arguing as in [9, pp. 356–357], which uses feasibility $y_c \leq \psi$, we can prove $\langle \eta, y^* - \psi \rangle = 0$ and $z\lambda^* = 0$ a.e. on Ω. Additionally, we obtain $\langle \eta, z \rangle \geq 0$ from $\left(c\operatorname{sgn}_c\left(\bar\lambda + c(y_c - \psi)\right) z_c, z_c\right) \geq 0$ by taking the limit. These facts establish (3.9).

We have using $h_c \to h$ in $H^{-1}(\Omega)$, $z_c \rightharpoonup z$, $y_c \to y^*$, and $p_c \rightharpoonup p^*$ in $H_0^1(\Omega)$ and equations (1.7), (3.7)

$$\begin{aligned}0 = (\eta_c, p_c) - (c \operatorname{sgn}_c \left(\bar\lambda + c(y_c - \psi)\right) p_c, z_c)&\\ &= \langle h_c, p_c\rangle - a(z_c, p_c) + a(z_c, p_c) + (g'(y_c), z_c)\\ &= \langle h_c, p_c\rangle + (g'(y_c), z_c)\\ &\to \langle h, p^*\rangle + (g'(y^*), z)\\ &= \langle h, p^*\rangle - a(z, p^*) + a(z, p^*) + (g'(y^*), z)\\ &= \langle \eta, p^*\rangle - \langle \mu^*, z\rangle,\end{aligned}$$

which yields

$$\langle \eta, p^*\rangle - \langle \mu^*, z\rangle = 0.$$

By (2.18), we obtain $j'(u^*)h + g'(y^*)z = 0$, which shows that (h, z) are feasible as test functions in Assumption 4.

By Assumption 3, it holds $(c^2 \operatorname{sgn}_c' \left(\bar\lambda + c(y_c - \psi)\right) p_c z_c, z_c) \geq 0$ for sufficiently large $c > C_1$. Hence by (3.10)

$$g''(y_c)(z_c, z_c) + j''(u_c)(h_c, h_c) \leq \frac{1}{c} \quad \text{for all } c > C_1. \tag{3.12}$$

We can write

$$j''(u_c)(h_c, h_c) = (j''(u_c)(h_c, h_c) - j''(u^*)(h_c, h_c)) + j''(u^*)(h_c, h_c),$$

which gives due to $j''(u_c) \to j''(u^*)$ and Assumption 4(i)

$$\liminf j''(u_c)(h_c, h_c) = \liminf j''(u^*)(h_c, h_c) \geq j''(u^*)(h, h).$$

Passing to the limit in (3.12) gives then

$$g''(y^*)(z, z) + j''(u^*)(h, h) \leq 0.$$

Hence, it follows by Assumption 4 $h = 0$. The system (3.9) implies the estimate $\|z\|_{H^1} \leq C\|h\|_{H^{-1}}$, which yields $z = 0$.

Due to Assumption 4(i) there is c_1 such that

$$j''(u_c)(h_c, h_c) \geq \gamma/2 \|h_c\|_{L^2}^2$$

for all $c > c_1$. Inequality (3.12) can then be written as

$$g''(y_c)(z_c, z_c) \leq \frac{1}{c} - \frac{\gamma}{2} \quad \text{for all } c > c_1,$$

which finally yields the contradiction, since the left-hand side tends to zero for $c \to \infty$. $\square$

Remark 3.3. If one omits the assumption on $\bar{\lambda}$, one cannot expect feasibility of the regularized solutions y_c. Then it seems to be impossible to prove the relations $\langle \eta, y^* - \psi \rangle = 0$ and $z\lambda^* = 0$. Hence, one has to strengthen Assumption 4(ii) again, i.e. skip the restrictions $\langle \eta, y^* - \psi \rangle = 0$ and $z\lambda^* = 0$, to obtain a result similar as in the previous theorem.

Let us conclude this section with a result about the sets that appear in Assumption 3.

Lemma 3.5. *For every sequence* $c_n \to \infty$ *it holds*

$$\bigcap_{c_n} \{|\bar{\lambda} + c_n(y_{c_n} - \psi)| < 1/2c_n\} \subset B,$$

where B *is the bi-active set.*

Proof. Let us denote $N_n := \{|\bar{\lambda} + c_n(y_{c_n} - \psi| < 1/2c_n\}$ and $N := \bigcap_{c_n} N_n$. Then we have on N_n

$$-\frac{1}{2c_n} \leq \bar{\lambda} + c_n(y_{c_n} - \psi) \leq \frac{1}{2c_n},$$

which implies

$$-\frac{1}{2c_n} - \bar{\lambda}\frac{1}{c_n} \leq y_{c_n} - \psi \leq \frac{1}{2c_n} - \bar{\lambda}\frac{1}{c_n}.$$

Since $y_{c_n} \to y$ in $L^p(\Omega)$, we obtain $y = \psi$ on N and $y_{c_n} \to y$ in $L^\infty(N)$. In addition, we have

$$0 \leq \lambda_{c_n} \leq \frac{c_n}{2}\left(\bar{\lambda} + c_n(y_{c_n} - \psi) + \frac{1}{2c_n}\right)^2 \leq \frac{1}{8c_n}$$

on N_n, which proves $\lambda_{c_n} \to 0 = \lambda$ in $L^\infty(N)$. Hence, it holds $N \subset B$. ☐

4 Semi-smooth Newton method: wellposedness and convergence

The direct use of Newton-type methods is impeded by the non-smoothness of the constraints appearing in the original problem (P). Even the optimality system (1.7) of the regularized problem (P_c) contains the nondifferentiable

sgn_c term, which is not C^1, so that Newton's method is not directly applicable. We therefore investigate the use of a semi-smooth Newton method. In view of recent results on semi-smooth methods for non-differentiable problems in function spaces the proof of the semi-smooth part is quite straightforward, see e.g. [6, 10]. The stability part, however, is more delicate.

The main assumption, under which we obtain well-posed and superlinear convergence of the semi-smooth Newton method, is the second order sufficient optimality condition for the regularized problem that was developed in Section 3.4, specifically (3.6). This condition will be required in a neighborhood of (y_c, u_c, p_c). Since (3.6) contains the discontinuous term sgn_c' which is unstable with respect to pertubations of the reference point (y_c, u_c, p_c) we require the following additional assumption.

Assumption 5. *The mapping $y \to sgn_c'(\bar\lambda + c(y - \psi))$ is continuous from a $H^2(\Omega) \cap H_0^1(\Omega)$-neighborhood of y_c to $L^1(\Omega)$.*

Assumption 5 essentially requires that y_c is nowhere tangential to the level sets $\pm\frac{1}{2c}$. In Remark 4.1 below, we shall give extra conditions on g, j and p_c, which allow to bypass Assumption 5.

We henceforth abbreviate $W := H^2(\Omega) \cap H_0^1(\Omega)$ and define the function $F : W \times L^2(\Omega) \times W \to L^2(\Omega)^3$ by

$$F(y,u,p) = \begin{pmatrix} Ay + \max_c\left(\bar\lambda + c(y-\psi)\right) - u \\ A^*p + c\operatorname{sgn}_c\left(\bar\lambda + c(y-\psi)\right)p + g'(y) \\ j'(u) - p \end{pmatrix}. \tag{4.1}$$

A generalized derivative [10] G_F of F at $(y, u, p) \in W \times L^2(\Omega) \times W$ in the direction $(z, h, q) \in W \times L^2(\Omega) \times W$ is given by

$$G_F(y,u,p)(z,h,q) = \begin{pmatrix} Az + c\operatorname{sgn}_c\left(\bar\lambda + c(y-\psi)\right)z - h \\ A^*q + c\operatorname{sgn}_c\left(\bar\lambda + c(y-\psi)\right)q + c^2\operatorname{sgn}_c'\left(\bar\lambda + c(y-\psi)\right)pz + g''(y)z \\ j''(u)h - q \end{pmatrix}.$$

Theorem 4.1. *Let Assumptions 2 and 5 be satisfied at $c \geq C_0$, where C_0 is given by Assumption 2. Then the semi-smooth Newton method applied to $F(y, u, p) = 0$ converges locally superlinearly to (y_c, u_c, p_c) in $W \times L^2(\Omega) \times W$.*

The proof uses two preparatory lemmas.

Lemma 4.1. *Under the assumptions of Theorem 4.1 the generalized derivatives $G_F(y,u,p)$ are uniformly bounded invertible in a $W \times L^2(\Omega) \times W-$ neigborhood of (y_c,u_c,p_c). In particular, the semi-smooth Newton algorithm is well defined in this neighborhood.*

Proof. From Assumption 2, (3.6), and Assumption 5 it follows that there exists $\rho > 0$ such that

$$g''(y)(z,z) + j''(u)(h,h) + (c^2 \mathrm{sgn}_c' \left(\bar{\lambda} + c(y-\psi)\right) p\, z, z) \geq \frac{\alpha}{2}\|h\|_{L^2}^2 \tag{4.2}$$

holds for all (y,u,p) with

$$\|y-y_c\|_{H^2} + \|u-u_c\|_{L^2} + \|p-p_c\|_{H^2} \leq \rho \tag{4.3}$$

and all pairs (h,z) satisfying

$$Az + c\,\mathrm{sgn}_c(\bar{\lambda} + c(y-\psi))z = h. \tag{4.4}$$

Here we use the facts that $x \longrightarrow \mathrm{sgn}_c(x)$ is globally Lipschitz continuous and that

$$\|z_y - z_{y_c}\|_{H^2} \leq K\|y-y_c\|_{L^\infty}\|h\|_{L^2}, \text{ for all } h \in L^2(\Omega),$$

where z_y, z_{y_c} denote the solutions to (4.4) and (3.7) respectively, and the constant K is uniform with respect to y in bounded subsets of $H^2(\Omega)$.

To argue the asserted wellposedness of $G_F(y,u,p)$ with (y,u,p) satisfying (4.3) let $r = (r_1,r_2,r_3) \in L^2(\Omega)^3$ be arbitrary and consider the equation

$$G_F(y,u,p)(z,h,q) = r, \tag{4.5}$$

for $(z,h,q) \in W \times L^2(\Omega) \times W$.

For verifying existence of a solution to (4.5) we set

$$B = A + c\,\mathrm{sgn}_c(\bar{\lambda} + c(y-\psi)) \text{ and } b = c^2\,\mathrm{sgn}_c'(\bar{\lambda} + c(y-\psi))p$$

and consider the quadratic minimization problem in $L^2(\Omega)$

$$\begin{aligned}\min\nolimits_h \tfrac{1}{2}\big(g''(y)B^{-1}(h+r_1), B^{-1}(h+r_1)\big) + \tfrac{1}{2}\big(b\,B^{-1}(h+r_1), B^{-1}(h+r_1)\big)\\ +\tfrac{1}{2}(j''(u)h,h) - (r_2, B^{-1}h) - (r_3,h).\end{aligned}$$

The necessary optimality condition for this problem is

$$B^{-*}\big(g''(y)\,B^{-1}(h+r_1) + b\,B^{-1}(h+r_1) - r_2\big) + j''(u)h - r_3 = 0, \tag{4.6}$$

and the second order sufficient condition is (4.2) with the substitution $B^{-1}h = z$ as in (4.4). Since the second order condition holds for the quadratic problem, (4.6) admits a unique solution. Setting

$$z = B^{-1}(h + r_1) \text{ and } q = -B^{-*}(g''(y)z + bz - r_2)$$

we have

$$\begin{aligned} &Az + c\,\mathrm{sgn}_c(\bar\lambda + c(y-\psi))z - h = r_1, \\ &A^*q + c\,\mathrm{sgn}_c(\bar\lambda + c(y-\psi))q - g''(y)z + bz = r_2, \qquad (4.7) \\ &j''(u)h - q = r_3, \end{aligned}$$

which is (4.5).

Continuous invertibility of the linear operator

$$h \to B^{-*}\big(g''(y)\,B^{-1}h + b\,B^{-1}h\big) + j''(u)h,$$

implies the continuous dependence of h on r. Continuous dependence of (z, q) on r follows from (4.7). ☐

Lemma 4.2. *Let $f : \mathcal{Y} \to \mathcal{Z}$ and $g : \mathcal{X} \to \mathcal{Y}$ be Newton differentiable in open sets V and U, respectively, with $U \subset \mathcal{X}$, $g(U) \subset V \subset \mathcal{Y}$. Assume that g is locally Lipschitz continuous and that there exists a Newton map $G_f(\cdot)$ of f which is bounded on $g(U)$. Then the superposition $f \circ g : \mathcal{X} \to \mathcal{Z}$ is Newton differentiable in U with a Newton map $G_f G_g$.*

For the proof of this lemma we refer to the Appendix of [7].

Proof of Theorem 4.1 According to the superlinear convergence theorem for Newton differentiable mappings [11, 10] it suffices to verify that the inverses of the generalized gradients $G_F(y, u, p)$ are uniformly bounded in a neighborhood of (y_c, u_c, p_c), which was achieved in Lemma 4.1, and that $(y, u, p) \to F(y, u, p)$ is Newton-differentiable. This is clear for all terms appearing in F except for $y \to \mathrm{sgn}_c(\bar\lambda + c(y-\psi))$ as mapping from $W \to L^2(\Omega)$. Note that

$$\mathrm{sgn}_c(x) = \max(1, \max(0, c(x + \frac{1}{2c}))).$$

Recall that the max operation is Newton differentiable from $L^p(\Omega)$ to $L^q(\Omega)$ if $1 \le q < p$ and

$$(G_{\max} y)(x) = \begin{cases} 0 & \text{if } y(x) > 0 \\ \delta & \text{if } y(x) = 0 \\ 1 & \text{if } y(x) < 0, \end{cases}$$

for any $\delta \in \mathbb{R}$ and $y \in L^p(\Omega)$. Together with Lemma 4.2 this implies Newton differentiability of $y \to \operatorname{sgn}_c(\bar\lambda + c(y-\psi))$ from $W \to L^2(\Omega)$. $\square$

Remark 4.1. Assumption 5 was used to guarantee (4.2) in a neighborhood of (y_c, u_c, p_c) specified in (4.3) and all pairs (h, z) satisfying

$$Az + c\operatorname{sgn}_c(\bar\lambda + c(y-\psi))z = h. \tag{4.8}$$

Alternatively to Assumption 5, let us assume that

$$g''(y_c)(z,z) + j''(u_c)(h,h) \geq \alpha \|h\|^2_{L^2} \tag{4.9}$$

for all pairs (h, z) satisfying (4.8) and that

$$\left\{ \begin{array}{l} \text{there exist } \eta > 0 \text{ and } \rho_1 > 0 \text{ such that} \\ |\frac{\bar\lambda(x)}{c} + y_c(x) - \psi(x)| \leq \rho_1 \text{ implies } p_c(x) \geq \eta, \quad \text{for a.e. } x \in \Omega. \end{array} \right. \tag{4.10}$$

Now ρ can be chosen sufficiently small such that

$$g''(y)(z,z) + j''(u)(h,h) \geq \frac{\alpha}{2}\|h\|^2_{L^2},$$

for all (h, z) satisfying (4.8), and (y, u) satisfying (4.3). Let $\bar k$ denote the embedding constant of W into $C(\Omega)$ and set $S = \{x : |(\bar\lambda + c(y-\psi))(x)| \leq \frac{1}{2c}\}$. Then ρ and C_0 can be chosen such that for $x \in S$

$$\left|(\frac{\bar\lambda}{c} + y_c - \psi)(x)\right| \leq |y_c - y|_{L^\infty} + \left|(\frac{\bar\lambda}{c} + y - \psi)(x)\right| \leq \bar k\rho + \frac{1}{c^2} \leq \rho_1$$

so that by (4.10) we have $p_c(x) \geq \eta$ on S. After possibly further decreasing ρ this implies that for p satisfying (4.3) we have $p \geq 0$ on S.

Summarizing this discussion there exists $\rho >$ such that (4.9) and (4.10) imply (4.2)–(4.4).

Note also that (4.9) and (4.10) imply Assumption 2 with $\rho = \frac{1}{2C_0}$.

Combining the results of Section 3.1 and this section we showed that the solutions of the regularized solutions converge as $c \to \infty$ and that each regularized problem with a fixed value of c can be solved with superlinear rate. In a follow up paper we shall address the issue of combining these two asymptotic processes. This will rely on a study of the path $c \to (y_c, u_c, \lambda_c)$ associated to (P_c).

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