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Abstract

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1 Introduction

In this paper we investigate optimal control problems for parabolic variational inequalities in the Hilbert space $H = L^2(\Omega)$ which are of the type

$$(1.1)\quad \begin{cases} \langle \frac{d}{dt}y^*(t) + Ay^*(t) - f(t), y - y^*(t)\rangle_{X^*,X} \geq 0, \text{ for all } y - \psi \in C, \\ y^* - \psi \in C, \\ y^*(0) = y_0, \end{cases}$$

where the closed convex subset C in H is given by

$$C = \{y \in H : y \geq 0\}.$$

Here X be a Hilbert space that is continuously embedded into H, and V be a separable closed linear subspace of X, which is endowed with the induced norm of X and is dense in H. Further A is a closed linear operator in H, Ω denotes an open domain in $\mathbb{R}^n$, and $y \geq 0$ in H is interpreted in the pointwise almost everywhere sense. Throughout it is assumed that

$$\psi \in X, \quad f \in L^2(0,T;V^*),$$

and that

$$\phi^+ = \sup(0,\phi) \in V, \qquad \text{for all } \phi \in V,$$

which requires that V has a Hilbert lattice structure, so that the *sup* operation is well-defined.

Note that (1.1) is considered without assuming that V is embedded compactly into H. In the latter case one can use Aubin's lemma which states that $W(0,T) = L^2(0,T;V) \cap H^1(0,T;V^*)$ is compactly embedded into $L^2(0,T;H)$. This ensures that the weak limit of certain approximating sequences defines the solution, see e.g. [GLT, IK1]. Instead, our analysis uses the monotone trick for variational inequalities. From e.g.[Tan], pg.151, we recall that under the present assumptions $W(0,T)$ embeds continuously into $C([0,T];H)$.

Different combinations of the following assumptions will be used for the operator A.

(1) $A \in \mathcal{L}(X, V^*)$, i.e., there exists $\bar{M}$ such that

$$|\langle Ay, \phi\rangle_{V^*\times V}| \le \bar{M}|y|\,|\phi|, \quad \text{for all } y \in X \text{ and } \phi \in V,$$

and A is closed in H with

$$\mathrm{dom}(A) = \{y \in X : Ay \in H\} \subset X,$$

where $\mathrm{dom}(A)$ is a Hilbert space equipped with the graph norm.
(2) There exist $\omega > 0$ and $\rho \in \mathbb{R}$ such that for all $\phi \in V$

$$\langle A\phi, \phi\rangle_{V^*,V} \ge \omega|\phi|_V^2 - \rho|\phi|_H^2$$

(3) For all $\phi \in V$,

$$\langle A\phi, \phi^+\rangle_{V^*,V} \ge \langle A\phi^+, \phi^+\rangle.$$

(4) There exists $\bar{\lambda} \in H$ satisfying $\bar{\lambda} \le 0$ a.e. such that

$$\langle \bar{\lambda} + A\psi - f(t), \phi\rangle_{V^*,V} \le 0$$

for a.e. t and all $\phi \in V$ satisfying $\phi \ge 0$ a.e.
(5) There exits a $\bar{\psi} \in \mathrm{dom}(A)$ such that $\bar{\psi} - \psi \in V \cap C =: \mathcal{C}$.
(6) Let a_s be the symmetric form on $V \times V$ defined by

$$a_s(y, \phi) = \frac{1}{2}(\langle Ay, \phi\rangle_{V^*,V} + \langle A\phi, y\rangle_{V^*,V})$$

for $y, \phi \in V$ and assume that the skew-symmetric form satisfies

$$\frac{1}{2}|\langle Ay, \phi\rangle_{V^*,V} - \langle A\phi, y\rangle_{V^*,V}| \le M|y|_V|\phi|_H,$$

for a constant M independent of $y, \phi \in V$.

(7) $\quad (\phi - \gamma)^+ \in V$ for any $\gamma \in \mathbb{R}^+$ and $\phi \in V$, and $\langle A1, (\phi - \gamma)^+\rangle \ge 0$.

Assumptions (1)-(5) apply to second order elliptic differential operators. Assumption (6) applies to the bi-harmonic operator Δ^2 and to self-adjoint operators. For the bi-harmonic operator and systems of equations as the elasticity system, for instance, the monotone property (3) does not hold.

We give the definitions of strong and weak solutions to (1.1).

Definition 1.1. (Strong Solution) *Given $y_0 - \psi \in \mathcal{C}$ and $f \in L^2(0,T;H)$, an X-valued function $y^*(t)$, with $y^* - \psi \in L^2(0,T;V) \cap H^1(0,T;V^*)$ is called strong solution of (1.1), if $y^*(0) = y_0$, $y^* \in H^1(\delta,T;H)$ for every $\delta > 0$, $y^*(t,x) \geq \psi(x)$ a.e. in $(0,T) \times \Omega$, and for a.e. $t \in (0,T)$,*

$$\langle \frac{d}{dt} y^*(t) + Ay^*(t) - f(t), y - y^*(t) \rangle \geq 0,$$

for a.e. $t \in (0,T)$, and for all $y - \psi \in \mathcal{C}$.

Defining $\lambda^* = -\frac{d}{dt}y^* - Ay^* + f(t) \in L^2(0,T;V^*)$, we have in the distributional sense that y^* satisfies

$$\left\{\begin{array}{l} \dfrac{d}{dt}y^*(t) + Ay^*(t) + \lambda^*(t) = f(t), \quad y^*(0) = y_0, \\ \langle \lambda^*(t), y - \psi \rangle \leq 0 \text{ for all } y - \psi \in \mathcal{C} \quad \text{and} \quad \langle \lambda^*, y^* - \psi \rangle = 0. \end{array}\right. \tag{1.2}$$

If y^* is a strong solution satisfying $y^* \in L^2(\delta,T;\mathrm{dom}(A))$ for every $\delta > 0$, then $\lambda^* \in L^2(\delta,T;H)$ and (1.1) can equivalently be written as a variational inequality in the form

$$\left\{\begin{array}{l} \dfrac{d}{dt}y^*(t) + Ay^*(t) + \lambda^*(t) = f(t), \quad y^*(0) = y_0, \\ \lambda^*(t) \leq 0, \quad y^*(t) \geq \psi, \quad (y^*(t) - \psi, \lambda^*(t))_H = 0, \\ \text{for a.e. } t > 0. \end{array}\right. \tag{1.3}$$

Definition 1.2. (Weak Solution) *Given $y_0 \in H$ and $f \in L^2(0,T,V^*)$, a function $y^* - \psi \in L^2(0,T;V)$ satisfying $y^*(t,x) \geq \psi(x)$ a.e. in $(0,T) \times \Omega$ is called weak solution to (1.1) if*

$$\begin{aligned} \int_0^T \big[\langle \tfrac{d}{dt}y(t), y(t) - y^*(t) \rangle &+ \langle Ay^*(t), y(t) - y^*(t) \rangle \\ &- \langle f(t), y(t) - y^*(t) \rangle \big]\, dt + \tfrac{1}{2}|y(0) - y_0|_H^2 \geq 0 \end{aligned} \tag{1.4}$$

is satisfied for all $y-\psi \in \mathcal{K}$, where

$$\mathcal{K} = \{y \in W(0,T) : y(t,x) \geq 0 \text{ a.e. in } (0,T)\times\Omega\}. \tag{1.5}$$

Since for y^* and y in $W(0,T)$

$$\int_0^T [\langle \frac{d}{dt} y(t) - y^*(t), y(t) - y^*(t)\rangle = \frac{1}{2}(|y(T)-y^*(T)|_H^2 - |y(0)-y_0|_H^2),$$

it follows that a strong solution to (1.1) is also a weak solution.

Let us briefly outline this paper. Section 1 is devoted to proving existence and uniqueness of strong solutions to (1.1). Extra regularity of solutions is obtained in section 2. Continuous dependence of the solution with respect to parameters in A is investigated in section 3. Section 4 focuses on weak solutions obtained as the limit of approximating difference schemes.

Optimal control problems related to (1.1) are investigated in section 5, and section 6 is devoted to the application of the presented theory to the Black-Scholes model with American options.

2 Strong solutions

In this section we establish the existence of the strong solution to (1.1) under assumptions (1)-(5) and (1)-(2),(5)-(6) respectively.

For $\bar\lambda \in H$ satisfying $\bar\lambda \leq 0$ we consider the regularized equations of the form

$$\begin{cases} \frac{d}{dt} y_c + Ay_c + \min(0, \bar\lambda + c\,(y_c - \psi)) = f, \quad c > 0 \\ y_c(0) = y_0, \end{cases} \tag{2.1}$$

where $c > 0$.

Proposition 2.1. *If Assumptions (1)–(2) hold and $y_0 \in H$, $f \in L^2(0,T;V^*)$, and $\bar\lambda \in H$, then* (2.1) *has a unique solution y_c satisfying $y_c - \psi \in L^2(0,T;V) \cap H^1(0,T;V^*)$, for each $c > 0$.*

Proof. Existence and uniqueness of the solution to (2.1) follows with monotone techniques, see [ItKa, Lio] for instance. Define $\mathcal{A} : V \to V^*$ by

$$\mathcal{A}\phi = A\phi + \min(-\bar\lambda, c\phi).$$

Then (2.1) can equivalently be expressed as

$$(2.2) \qquad \frac{d}{dt}v + \mathcal{A}v = f - \bar{\lambda} - A\psi \in L^2(0,T;V^*), \quad v(0) = y_0 - \psi \in H,$$

with $v = y_c - \psi$. We note that $\mathcal{A}$ is hemicontinuous, i.e. $s \to \langle \mathcal{A}(\phi_1 + s\phi_2), \phi_3 \rangle$ is continuous from $\mathbb{R} \to \mathbb{R}$, for all $\phi_i \in V, i = 1,..,3$ and

$$|\mathcal{A}\phi|_{V^*} \le |A\phi|_{V^*} + c|\phi|_H \text{ for all } \phi \in V$$

$$\langle \mathcal{A}\phi_1 - \mathcal{A}\phi_2, \phi_1 - \phi_2 \rangle \ge \omega|\phi_1 - \phi_2|_V^2 - \rho|\phi_1 - \phi_2|_H^2 \text{ for all } \phi_1, \phi_2 \in V$$

$$\langle \mathcal{A}\phi, \phi \rangle \ge \omega|\phi|_V^2 - \rho|\phi|_H^2 \text{ for all } \phi \in V.$$

Therefore it follows that (2.2) admits a unique solution $v \in L^2(0,T;V) \cap H^1(0,T;V^*)$ and this gives the desired solution $y_c = v + \psi$ of (2.1), compare [ItKa], Theorem 8.7, [Lio], Theorem II.1.2.

□

Theorem 2.1. *(1) If in addition to the assumptions in Proposition 2.1 Assumptions (3)–(4) hold, and* $y_0 - \psi \in C$*, then* $y_c(t) - \psi \in C$ *and* $y_c(t) \ge y_{\hat{c}}(t)$ *for* $\hat{c} \ge c$*. Moreover,* $y_c - \psi \to y^* - \psi$ *strongly in* $L^2(0,T;V)$ *and weakly in* $H^1(0,T;V^*)$ *as* $c \to \infty$*, where* y^* *is the unique solution of* (1.1) *in the sense that* $y^* - \psi \in \mathcal{K}$*,* (1.2) *is satisfied with* $\lambda^* \in L^2(0,T;H)$ *and the estimate*

$$\frac{1}{2}e^{-2\rho t}|y_c(t) - y^*(t)|_H^2 + \int_0^t e^{-2\rho s}\omega\,|y_c(s) - y^*(s)|_V^2\,ds \le \frac{1}{c}\int_0^t e^{-2\rho s}|\bar{\lambda}|^2\,ds \to 0,$$

for $t \in [0,T]$ *holds. If in addition Assumption (7) is satisfied and* $\bar{\lambda} \in L^\infty(\Omega)$*, then*

$$|y_c(t) - y^*(t)|_{L^\infty} \le \frac{1}{c}|\bar{\lambda}|_{L^\infty}.$$

(2) If Assumptions (1)–(4) hold, $y_0 - \psi \in C$*, and* $f \in L^2(0,T;H)$*, then* y^* *is the unique strong solution to* (1.1) *and* $y^* \in L^2(\delta,T;dom(A))$ *for every* $\delta > 0$*. If moreover* $y_0 \in dom(A)$*, then* δ *can be chosen equal to* 0*.*

Proof. (1) From (2.1) it follows that y_c satisfies

$$\left\{\begin{array}{l} \frac{d}{dt}y_c + A(y_c - \psi) + \lambda_c = f - A\psi \\ y_c(0) = y_0, \end{array}\right. \tag{2.3}$$

where

$$\lambda_c = \min(0, \bar{\lambda} + c\,(y_c - \psi)).$$

If $y_0 - \psi \in C$, then $y_c(t) - \psi \in C$. In fact, let $\phi = \min(0, y_c - \psi) = -(y_c - \psi)^- \in -C \cap V$, where $\phi = \phi(t)$, for $t \in (0, T)$. Since $\bar{\lambda} \le 0$ it follows that

$$\langle \frac{d}{dt}y_c + A(y_c - \psi), \phi\rangle + \langle A\psi - f + \bar{\lambda} + c\,\phi, \phi\rangle = 0,$$

where by Assumptions (3) and (2)

$$\langle A(y_c - \psi), \phi\rangle \ge \langle A\phi, \phi\rangle \ge \omega|\phi|^2 - \rho|\phi|_H^2$$

and by (4)

$$\langle A\psi - f(t) + \bar{\lambda}, \phi\rangle \ge 0.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}|\phi|_H^2 \le \rho\,|\phi|_H^2$$

and consequently

$$e^{-2\rho t}|\phi|_H^2 \le |\phi(0)|_H^2 = 0. \tag{2.4}$$

Since

$$0 \ge \lambda_c = \min(0, \bar{\lambda} + c\,(y_c - \psi)) \ge \bar{\lambda},$$

we have

$$|\lambda_c(t)|_H \le |\bar{\lambda}|_H$$

for all $t \in [0, T]$. From (2.3) we deduce that $\{y_c - \psi\}$ is bounded in $L^2(0, T; V)$. By Assumption (1) and again (2.3) it follows that $\{Ay_c\}$ and

$\{\frac{d}{dt}y_c\}$ are bounded in $L^2(0,T;V^*)$. Thus, there exist $\lambda^* \in L^2(0,T;H)$ satisfying $\lambda^* \le 0$ a.e., and y^* satisfying $y^* - \psi \in \mathcal{K}$, such that for a subsequence denoted again by c,

$$\begin{aligned} &\lambda_c \to \lambda^*, \text{ weakly in } L^2(0,T;H), \\ &Ay_c \to Ay^* \text{ and } \tfrac{d}{dt}y_c \to \tfrac{d}{dt}y^* \text{ weakly in } L^2(0,T;V^*), \end{aligned} \tag{2.5}$$

as $c \to \infty$. Taking the limit in (2.3) implies that

$$\frac{d}{dt}y^* + Ay^* - f = -\lambda^*, \qquad y^*(0) = y_0, \tag{2.6}$$

with equality in the differential equation holding in the sense of $L^2(0,T;V^*)$.

For $\phi = -(y_c - y_{\hat{c}})^-$ with $c \le \hat{c}$ we deduce from (2.3)

$$\langle \frac{d}{dt}(y_c - y_{\hat{c}}) + A(y_c - y_{\hat{c}}), \phi\rangle + (\lambda_c - \lambda_{\hat{c}}, \phi) = 0$$

where

$$(\lambda_c - \lambda_{\hat{c}}, \phi) = \big(\min(0, \bar{\lambda} + c(y_{\hat{c}} - \psi)) - \min(0, \bar{\lambda} + \hat{c}(y_{\hat{c}} - \psi))$$

$$+ \min(0, \bar{\lambda} + c(y_c - \psi)) - \min(0, \bar{\lambda} + c(y_{\hat{c}} - \psi)), \phi\big) \ge 0,$$

since $y_{\hat{c}} \ge \psi$. Hence, using the same arguments as those leading to (2.4), we have $|\phi(t)|_H = 0$ and thus

$$y_c \ge y_{\hat{c}} \quad \text{for} \quad c \le \hat{c}.$$

By Lebesgue dominated convergence theorem and the theorem of Beppo Levi $y_c \to y^*$ strongly to $L^2(0,T;H)$ and pointwise a.e. in $(0,T) \times \Omega$. Since

$$0 \ge \int_0^T (\lambda_c, y_c - \psi)_H \, dt \ge -\frac{1}{c} \int_0^T |\bar{\lambda}|_H^2 \, dt \to 0$$

as $c \to \infty$, we have

$$\int_0^T (\lambda^*, y^* - \psi) \, dt = 0.$$

That is, (y^*, λ^*) satisfies (1.2), with $\lambda^* \in L^2(0,T;H)$.

Suppose that $(y, \lambda) \in \mathcal{K} \in L^2(0,T;H)$ is another pair satisfying (1.2). Then it follows that

$$\frac{1}{2}\frac{d}{dt}|y^*(t) - y(t)|_H^2 + \langle A(y^* - y(t)), y^*(t) - y(t)\rangle \le 0$$

and thus $e^{-2\rho t}|y^*(t)-y(t)|_H^2 \le |y_0-y(0)|_H^2$. This implies that y^* is the unique solution to (1.1) in $\mathcal{K}$ and that the whole family $\{(y_c,\lambda_c)\}$ converges in the sense specified in (2.5). From (1.1) and (2.1)

$$\langle \frac{d}{dt}y^*(t)+Ay^*(t)-f(t), y_c(t)-y^*(t)\rangle \ge 0$$

$$\langle \frac{d}{dt}y_c(t)+Ay_c(t)-f(t), y^*(t)-y_c(t)\rangle \ge (\lambda_c, y_c-\psi)_H.$$

Since $y_c \ge \psi$, we have $(\lambda_c, y_c-\psi)_H \ge -\frac{1}{c}|\bar\lambda|_H^2$. Summing the above inequalities and multiplying by $e^{-2\rho t}$ this gives

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}(e^{-2\rho t}|y_c(t)-y^*(t)|_H^2)+e^{-2\rho t}\langle A(y_c(t)-y^*(t)), y_c(t)-y^*(t)\rangle \\ +\rho\,|y_c(t)-y^*(t)|_H^2 \le \tfrac{1}{c}\,e^{-2\rho t}|\bar\lambda|^2,\end{aligned}$$

which implies the first estimate in the statement of Theorem 2.1 and in particular that $y_c \to y^*$ in $L^2(0,T;V)$ strongly.

Suppose next that in addition $\bar\lambda \in L^\infty(\Omega)$. Let $k \in \mathbb{R}^+$ and $\phi = (y_c - y^* - k)^+$. By assumption (7) we have $\phi \in V$. From (1.2) and (2.1)

$$\langle \frac{d}{dt}y^*+Ay^*-f, \phi\rangle \ge 0,$$

and

$$\langle \frac{d}{dt}y_c+Ay_c+\lambda_c-f, \phi\rangle = 0.$$

If $k \ge \dfrac{1}{c}|\bar\lambda|_{L^\infty}$, then

$$(\lambda_c, \phi) = (\min(0, \bar\lambda + c\,(y_c-\psi)), (y_c-y^*-k)^+) = 0,$$

where we use that $y_c \ge y^* \ge \psi$. Hence, we obtain

$$\langle \frac{d}{dt}(y_c-y^*-k)+A(y_c-y^*-k)+A\,k, \phi\rangle \le 0.$$

By assumptions (3) and (7)

$$\frac{1}{2}\frac{d}{dt}|\phi|^2+\langle A\phi,\phi\rangle \le 0,$$

which implies the second estimate.

(2) Now suppose that $f \in L^2(0,T;H)$ and that Assumptions (1)–(4) hold. Consider (2.3) in the form

$$\begin{cases} \frac{d}{dt}y_c + Ay_c = f - \lambda_c \\ y(0) = y_0. \end{cases} \tag{2.7}$$

We decompose $y_c = y_{c,i} + y_h$, where $y_{c,i}$ and y_h are the solutions to (2.7) with initial condition and forcing functions set zero, respectively. Note that $\{\lambda_c\}$ is bounded in $L^2(0,T;H)$ uniformly with respect to c. Hence by the following lemma $\{Ay_{c,i}\}$ and $\{\frac{d}{dt}y_{c,i}\}$ are bounded in $L^2(0,T;H)$ uniformly for $c > 0$. Moreover $Ay_h \in L^2(\delta,T;H)$ and $\frac{d}{dt}y_h \in L^2(\delta,T;H)$ for every $\delta > 0$, and $\delta = 0$ is admitted if $y_0 \in dom(A)$. Thus y_c is bounded in $H^1(\delta,T;H) \cap L^2(\delta,T;dom(A))$, and converges weakly in $H^1(\delta,T;H) \cap L^2(\delta,T;dom(A))$ to y^* for $c \to \infty$.

□

Lemma 2.1. *Under the assumptions of the previous theorem $-A$ generates an analytic semigroup on H. If $\dfrac{d}{dt}x + Ax = g \in L^2(0,T;H)$, with $x_0 = 0$, then $\dfrac{d}{dt}x(t)$ and $Ax(t) \in L^2(0,T;H)$, and*

$$|Ax|_{L^2(0,T;H)} \leq \bar{k}\,|g|_{L^2(0,T;H)},$$

with $\bar{k}$ independent $g \in L^2(0,T;H)$.

Proof. For the purpose of this proof the Hilbert space H and the operator A are considered as complexified quantities. Let $B = A + \rho I$. For $u \in dom(A)$ and $\lambda \in \mathbb{C}$ with $Re\,\lambda \geq 0$ set

$$\bar{g} = \lambda u + Bu.$$

Then, since

$$Re\,\lambda\,(u,u)_H + Re\,\langle Bu, u\rangle \leq |\bar{g}|_H |u|_H,$$

Assumption (2) implies that

$$\omega |u|_V^2 \leq |\bar{g}|_H |u|_H. \tag{2.8}$$

From Assumption (1) and (2.8)

$$|\lambda|\,|u|_H^2 \le |\bar g|_H |u|_H + \bar M |u|_V^2 \le (1+\frac{\bar M}{\omega})|\bar g|_H |u|_H,$$

and thus

$$|u|_H = |(\lambda I + B)^{-1}\bar g|_H \le (1+\frac{\bar M}{\omega})\,\frac{1}{|\lambda|}\,|\bar g|_H. \tag{2.9}$$

It thus follows from [ItKa, Tan], [Pa], page 61, that $-B$ and hence $-A$ generate analytic semigroups on H related by $e^{-B\,t} = e^{-(\rho+A)\,t}$.

For $g \in L^2(0,\infty;H)$ with $e^{-\rho\,\cdot}g \in L^2(0,\infty;H)$ consider

$$\frac{d}{dt}x + Ax = g, \text{ with } x(0)=0.$$

This is related to

$$\frac{d}{dt}z + Bz = g_\rho := ge^{-\rho\,\cdot}, \text{ with } z(0)=0 \tag{2.10}$$

by $z(t) = e^{-\rho\,t}x$. Taking the Laplace transform of (2.10), we obtain

$$\lambda\hat z + B\hat z = \hat g_\rho, \qquad \text{where } \hat z = \int_0^\infty e^{-\lambda s} z(s)\,ds$$

and thus by (2.9)

$$|B\hat z|_H \le |B(\lambda I + B)^{-1}\hat g_\rho|_H = |(B+\lambda I-\lambda I)(\lambda I+B)^{-1}\hat g_\rho|_H \le (2+\frac{\bar M}{\omega})|\hat g_\rho|_H.$$

From the Fourier-Plancherel theorem we have

$$\int_0^\infty |Bz(t)|_H^2\,dt \le (2+\frac{\bar M}{\omega})\int_0^\infty |g_\rho|^2\,dt.$$

This implies that $|Bx|_{L^2(0,T;H)} \le e^{\rho\,T}(2+\frac{\bar M}{\omega})|g|_{L^2(0,T;H)}$ if $g(t)=0$ for $t \ge T$. Since $-A$ generates a semigroup on H, there exists a constant $\hat c$ such that $|x|_{L^2(0,T;H)} \le \hat c\,|g|_{L^2(0,T;H)}$ for any $g \in L^2(0,T;H)$ and the claim follows. □

To allow $\delta = 0$ for the strong solutions in the previous theorem we let $\bar y$ denote the solution to

$$\begin{cases} \frac{d}{dt}\bar y + A\bar y = 0, \\ \bar y(0) = y_0, \end{cases}$$

and consider

$$\begin{cases} \frac{d}{dt}(y_c - \bar{y}) + A(y_c - \bar{y}) = f - \lambda_c \\ y(0) - \bar{y}(0) = 0. \end{cases}$$

Arguing as in (2) of Theorem 2.1 we obtain the following corollary.

Corollary 2.1. *Under the assumptions of Theorem 2.1 (2) we have* $y^* - \bar{y} \in H^1(0,T;H) \cap L^2(0,T;dom(A))$.

Next we turn to verify existence under a different set of assumptions which, in particular, does not involve the monotone Assumption (3). For $\bar{\lambda} = 0$ in (2.1), let $\hat{y}_c$ denote the corresponding solution, i.e.

$$\frac{d}{dt}\hat{y}_c + A\hat{y}_c + c\,\min(0, c\,(y_c - \psi)) = f, \quad c > 0, \tag{2.11}$$

which exists by Proposition 2.1.

Theorem 2.2. *If Assumptions (1)–(2) and (5)–(6) hold, and* $y_0 - \psi \in C \cap V$ *and* $f \in L^2(0,T;H)$, *then* (1.1) *has a unique, strong solution* $y^*(t)$ *in* $H^1(0,T;H) \cap L^2(0,T;dom(A))$, *and* $\hat{y}_c \to y^*$ *strongly in* $L^2(0,T;V) \cap C(0,T;H)$ *as* $c \to \infty$. *Moreover* $t \to y^*(t) \in V$ *is right continuous. If in addition Assumptions (3)–(4) hold, then* $\hat{y}_c \leq \hat{y}_{\hat{c}}$ *for* $c \leq \hat{c}$ *and* $\hat{y}_c(t) \to y^*(t)$ *strongly in* H *for each* $t \in [0,T]$ *and pointwise almost everywhere in* Ω.

Proof. For $\bar{\lambda} = 0$ we have

$$\frac{1}{2}\frac{d}{dt}|\hat{y}_c - \psi|_H^2 + \langle A(\hat{y}_c - \psi), \hat{y}_c - \psi\rangle + \langle A\psi - f, \hat{y}_c - \psi\rangle + c\,|(\hat{y}_c - \psi)^-|^2 = 0.$$

From Assumptions (1)–(2) we have

$$\begin{aligned} |\hat{y}_c(t) - \psi|_H^2 &+ \int_0^t (\omega\,|\hat{y}_c - \psi|_V^2 + c\,|(\hat{y}_c - \psi)^-|_H^2)\,ds \\ &\leq |y_0 - \psi|_H^2 + \int_0^t (2\rho\,|\hat{y}_c - \psi|_H^2 + \frac{1}{\omega}|A\psi - f|_{V^*}^2)\,ds \end{aligned}$$

and thus

$$\begin{aligned} |\hat{y}_c(t) - \psi|_H^2 &+ \int_0^t (\omega\,|\hat{y}_c - \psi|_V^2 + c\,|(\hat{y}_c - \psi)^-|_H^2)\,ds \\ &\leq e^{2\rho t}(|y_0 - \psi|_H^2 + \frac{1}{\omega}\int_0^t |A\psi - f|_{V^*}^2\,ds). \end{aligned} \tag{2.12}$$

Let us assume, for the moment that $\hat{y}_c \in H^1(0,T;H)$. Then

$$(\frac{d}{dt}\hat{y}_c + A\hat{y}_c + c\min(0,\hat{y}_c - \psi) - f(t), \frac{d}{dt}\hat{y}_c)_H = 0,$$

for a.e. $t \in (0,T)$. Recall that $\hat{y}_c(t) - \psi \in V$. By (5) moreover $\psi - \bar{\psi} \in V$ and hence $\hat{y}_c(t) - \bar{\psi} \in V$ for almost every $t \in (0,T)$. Using (6) we find

$$\begin{aligned}|\frac{d}{dt}\hat{y}_c|_H^2 + \frac{1}{2}\frac{d}{dt}(a_s(\hat{y}_c - \bar{\psi}, \hat{y}_c - \bar{\psi}) + c|(\hat{y}_c - \psi)^-|^2)\\ \le (M\,|\hat{y}_c - \bar{\psi}|_V + |A\bar{\psi} - f|_H)\,|\frac{d}{dt}\hat{y}_c|_H\end{aligned}$$

and consequently

$$|\frac{d}{dt}\hat{y}_c|_H^2 + \frac{d}{dt}(a_s(\hat{y}_c - \bar{\psi}, \hat{y}_c - \bar{\psi}) + c|(\hat{y}_c - \psi)^-|^2) \le 2(M^2|\hat{y}_c - \bar{\psi}|_V^2 + |A\bar{\psi} - f|_H^2).$$

Using the assumption $y_0 - \psi \in C$ this implies that

(2.13)
$$\begin{aligned}a_s(\hat{y}_c(t) - \bar{\psi}, \hat{y}_c(t) - \bar{\psi}) + c\,|(\hat{y}_c(t) - \psi)^-|_H^2 + \int_0^t |\frac{d}{dt}\hat{y}_c|_H^2\,ds\\ \le a_s(y_0 - \bar{\psi}, y_0 - \bar{\psi}) + \int_0^t 2(M^2|\hat{y}_c(s) - \bar{\psi}|_V^2 + |A\bar{\psi} - f(s)|_H^2)\,ds.\end{aligned}$$

It thus follows from (2.12)–(2.13) that

(2.14)
$$\begin{aligned}|\hat{y}_c(t) - \bar{\psi}|_V^2 + c\,|(\hat{y}_c(t) - \psi)^-|_H^2 + \int_0^t |\frac{d}{dt}\hat{y}_c|_H^2\,ds\\ \le K(|y_0 - \bar{\psi}|_V + |\psi - \bar{\psi}|_V + \int_0^t |A\bar{\psi} - f(s)|_H^2\,ds),\end{aligned}$$

for a constant K independent of $c > 0$ and $t \in [0,T]$. Here we use $\psi - \bar{\psi} \in V$.

Inequality (2.14) provides an a-priori estimate for $\frac{d}{dt}\hat{y}_c$ in $L^2(0,T;H)$, which can now be verified with a Galerkin argument, under assumptions (5) and (6).

Further from this estimate we conclude that there exists y^* such that $y^* - \bar{\psi} \in H^1(0,T;H) \cap B(0,T;V)$ and on a subsequence

$$\frac{d}{dt}\hat{y}_c \to \frac{d}{dt}y^*, \quad A\hat{y}_c \to Ay^*$$

weakly in $L^2(0,T;H)$ and $L^2(0,T;V^*)$, respectively. In particular this implies that $y_c \to y^*$ weakly in $L^2(0,T;H)$. Above $B(0,T;V)$ denotes the space of all everywhere bounded measureable functions from $[0,T]$ to V. By assumption (5) we have that $y^* - \psi \in B(0,T;V)$ as well.

Since

$$\int_0^T (\hat{y}_c - \psi, \phi)_H \, dt \geq -\int_0^T ((\hat{y}_c - \psi)^-, \phi)_H \, dt$$

for all $\phi \in L^2(0,T;H)$ with $\phi(t) \in C$ for a.e. t, and since $\lim_{c\to 0} \int_0^T |(\hat{y}_c(t) - \psi)^-|_H^2 \, dt = 0$, by (2.14) we have

$$\text{(2.15)} \qquad \int_0^T (y^*(t) - \psi, \phi)_H \, dt \geq 0 \text{ for all } \phi \in L^2(0,T;H) \text{ with } \phi(t) \in C.$$

This implies that $y^*(t) - \psi \in C$ for a.e. $t \in [0,T]$. For $y - \psi \in \mathcal{C}$

$$-(c(\hat{y}_c(t) - \psi)^-, y - \hat{y}_c(t)) = -\big(c(\hat{y}_c(t) - \psi)^-, y - \psi - (\hat{y}_c(t) - \psi)\big) \leq 0,$$

for a.e. $t \in (0,T)$. It therefore follows from (2.11) that for $y - \psi \in \mathcal{K}$

$$\text{(2.16)} \quad \langle \frac{d}{dt}(\hat{y}_c(t) - y(t) + y(t)) + A(\hat{y}_c(t) - y(t)) + Ay(t) - f(t), y(t) - \hat{y}_c(t)\rangle \geq 0,$$

and hence

$$\begin{aligned} &\textstyle\int_0^t e^{-2\rho s} \langle \frac{d}{ds}(\hat{y}_c(s) - y(s)), \hat{y}_c(s) - y(s)\rangle \, ds + \\ &\qquad \textstyle\int_0^t e^{-2\rho s} \langle \frac{d}{ds} y(s) + A(\hat{y}_c(s) - y(s)), \hat{y}_c(s) - y(s)\rangle \\ &\qquad \leq \textstyle\int_0^t e^{-2\rho s} \langle Ay(s) - f(s), y(s) - \hat{y}_c(s)\rangle \, ds. \end{aligned}$$

For $z \in W(0,T)$ we have

(2.17)

$$\int_0^t e^{-2\rho s} \langle \frac{d}{ds} z(s), z(s)\rangle \, ds = \frac{1}{2}\,(e^{-2\rho t}|z(t)|_H^2 - |z(0)|_H^2) + \int_0^t \rho\, e^{-2\rho s} |z(s)|_H^2 \, ds.$$

This, together with (2.16) implies for $y - \psi \in \mathcal{K}$

$$\frac{1}{2}\,(e^{-2\rho t}|\hat{y}_c(t) - y(t)|_H^2 - |y_0 - y(0)|_H^2)$$

$$+ \int_0^t e^{-2\rho s}\big(\,\langle \frac{d}{ds}y(s) + A(\hat{y}_c(s) - y(s)), \hat{y}_c(s) - y(s)\rangle + \rho\,|\hat{y}_c(s) - y(s)|_H^2\,\big)\,ds$$

$$\leq \int_0^t e^{-2\rho s}\langle Ay(s) - f(s), y(s) - \hat{y}_c(s)\rangle\,ds \to \int_0^t e^{-2\rho s}\langle Ay - f(s), y(s) - y^*(s)\rangle\,ds,$$

as $c \to \infty$. Since norms are weakly lower semi-continuous, we obtain

$$\frac{1}{2}\,(e^{-2\rho t}|y^*(t) - y(t)|_H^2 - |y_0 - y(0)|_H^2)$$

$$+ \int_0^t e^{-2\rho s}(\langle \frac{d}{ds}y(s) + A(y^*(s) - y(s)), y^*(s) - y(s)\rangle + \rho\,|y^*(s) - y(s)|_H^2)\,ds$$

$$\leq \int_0^t e^{-2\rho s}\langle Ay(s) - f(s), y(s) - y^*(s)\rangle\,ds,$$

or equivalently, using (2.17)

$$\int_0^t e^{-2\rho s}\langle \frac{d}{ds}y^*(s) + Ay^*(s) - f(s), y(s) - y^*(s)\rangle\,ds \geq 0 \tag{2.18}$$

for all $y - \psi \in \mathcal{K}$ and $t \in [0, T]$. If $y(\cdot) - \psi \in H^1(0, T; H) \cap B(0, T; V)$ also satisfies (2.18), it follows that

$$\int_0^t e^{-2\rho s}\langle \frac{d}{ds}(y(s) - y^*(s)) + A(y(s) - y^*(s)), y(s) - y^*(s)\rangle\,ds \leq 0.$$

Using (2.17) this implies

$$\frac{1}{2}e^{-2\rho t}|y(t) - y^*(t)|_H^2 + \int_0^t e^{-2\rho s}(\langle A(y(s) - y^*(s)), y(s) - y^*(s)\rangle + \rho\,|y^*(s) - y(s)|_H^2)\,ds \leq 0$$

and thus $y(t) = y^*(t)$. Hence the solution to (2.19) is unique. Integrating (2.16) on (τ, t) with $0 \leq \tau < t \leq T$ we obtain with the arguments that led to (2.18)

$$\int_\tau^t e^{-2\rho s}\,\langle \frac{d}{ds}y^*(s) + Ay^*(s) - f(s), y(s) - y^*(s)\rangle\,ds \geq 0, \tag{2.19}$$

and consequently y^* satisfies (1.1).

To argue that $\hat{y}_c - \psi \to y^* - \psi$ strongly in $L^2(0,T;V) \cap C(0,T;H)$, note that $\hat{\lambda}_c = c\min(0, \hat{y}_c - \psi)$ converges weakly in $L^2(0,T;V^*)$ to λ^*. From (2.11) and (1.2) we have

$$\tfrac{1}{2}\tfrac{d}{dt}|\hat{y}_c - y^*|^2 + \langle A(\hat{y}_c - y^*), \hat{y}_c - y^*\rangle = \langle \lambda^* - \hat{\lambda}_c, \hat{y}_c - y^*\rangle$$
$$\leq \langle \lambda^* - \hat{\lambda}_c, \hat{y}_c - \psi + \psi - y^*\rangle \leq \langle \lambda^*, \hat{y}_c - \psi\rangle + \langle \hat{\lambda}_c, y^* - \psi\rangle =: \eta_c,$$

where $|\eta_c|_{L^1(0,T;\mathbb{R})} \to 0$ for $c \to \infty$. By assumption (2)

$$\frac{1}{2}\frac{d}{dt}|\hat{y}_c - y^*|_H^2 + \omega|\hat{y}_c - y^*|_V^2 - \rho|\hat{y}_c - y^*|_H^2 \leq \eta_c,$$

and hence

$$\tfrac{d}{dt}[e^{-2\rho t}|\hat{y}_c - y^*|_H^2] + \omega\, e^{-2\rho t}\, |\hat{y}_c - y^*|_V^2 \leq 2e^{-2\rho t}\eta_c,$$

which implies that

$$e^{-2\rho t}|\hat{y}_c(t) - y^*(t)|_H^2 + \omega \int_0^t e^{-2\rho s}|\hat{y}_c(s) - y^*(s)|_V^2\, ds \leq \int_0^t e^{-2\rho s}\eta_c(s)\, ds,$$

and the desired convergence of y_c to y^* in $L^2(0,T;V) \cap C(0,T;H)$ follows.

To argue right-continuity of $t \to y^*(t) \in V$, note that from (2.19) it follows that

$$\int_\tau^t \big\langle e^{-2\rho s}(\frac{d}{ds}y^*(s) + Ay^*(s) - f(s))\,,\, \frac{y^*(s-h) - y^*(s)}{-h}\big\rangle\, ds \leq 0, \tag{2.20}$$

where $h > 0$. Using $a(b-a) = \frac{b^2-a^2}{2} - \frac{1}{2}(a-b)^2$ we find

$$\liminf_{h\to 0} \tfrac{1}{-h}\textstyle\int_\tau^t e^{-2\rho s}a_s(y^*(s) - \bar{\psi}, y^*(s-h) - y^*(s))$$
$$= \liminf_{h\to 0}\big(-\tfrac{1}{2h}\textstyle\int_{\tau-h}^{t-h} e^{-2\rho s}(e^{-2\rho h} - 1)\, a_s(y^*(s) - \bar{\psi}, y^*(s) - \bar{\psi})$$
$$-\tfrac{1}{2h}\textstyle\int_{\tau-h}^{t-h} e^{-2\rho s}\, a_s(y^*(s) - \bar{\psi}, y^*(s) - \bar{\psi})\big) + \tfrac{1}{2h}\textstyle\int_\tau^t e^{-2\rho s}\, a_s(y^*(s) - \bar{\psi}, y^*(s) - \bar{\psi})$$
$$+\tfrac{1}{2h}\textstyle\int_\tau^t e^{-2\rho s}\, a_s(y^*(s-h) - y^*(s), y^*(s-h) - y^*(s))$$
$$\geq \rho\textstyle\int_\tau^t e^{-2\rho s}a_s(y^*(s) - \bar{\psi}, y^*(s) - \bar{\psi})\, ds$$
$$+\tfrac{1}{2}e^{-2\rho t}a_s(y^*(t) - \bar{\psi}, y^*(t) - \bar{\psi}) - \tfrac{1}{2}e^{-2\rho\tau}a_s(y^*(\tau) - \bar{\psi}, y^*(\tau) - \bar{\psi}).$$

This estimate and Assumption (6) allow us to pass to the limit in (2.20) to obtain

$$e^{-2\rho t}a_s(y^*(t)-\bar{\psi},y^*(t)-\bar{\psi})-e^{-2\rho\tau}a_s(y^*(\tau)-\bar{\psi},y^*(\tau)-\bar{\psi})+\int_\tau^t e^{-2\rho s}|\frac{d}{ds}y^*(s)|_H^2\,ds$$

$$\le 2M\int_\tau^t e^{-2\rho s}|y^*(s)-\bar{\psi}|_V\,|\frac{d}{dt}y^*(s)|_H\,ds+2\int_\tau^t e^{-2\rho s}|A\bar{\psi}-f(s)|_H\,|\frac{d}{dt}y^*(s)|_H\,ds.$$

Consequently we have

$$e^{-2\rho t}a_s(y^*(t)-\bar{\psi},y^*(t)-\bar{\psi})$$

$$\le e^{-2\rho\tau}a_s(y^*(\tau)-\bar{\psi},y^*(\tau)-\bar{\psi})+2\int_\tau^t e^{-2\rho s}(M^2|y^*(s)-\bar{\psi}|_V^2+|A\bar{\psi}-f(s)|_H^2)\,ds,$$

for $0\le\tau\le t\le T$. This implies that

$$a_s(y^*(\tau)-\bar{\psi},y^*(\tau)-\bar{\psi})+|y^*(\tau)-\bar{\psi}|_H^2\ge\limsup_{t\to\tau}\big(a_s(y^*(t)-\bar{\psi},y^*(t)-\bar{\psi})+|y^*(t)-\bar{\psi}|_H^2\big).$$

Since $a_s(\phi,\phi)+|\phi|_H^2$ defines an equivalent norm on the Hilbert space V, $|y^*(t)-y^*(\tau)|_V\to 0$ as $t\downarrow\tau$. Hence y^* is right continuous.

Now, in addition Assumptions (3)-(4) are supposed to hold. Let

$$\hat{\lambda}_c(t)=c\min(0,\hat{y}_c(t)-\psi).$$

Then for $c\le\hat{c}$ and $\phi=(\hat{y}_c-\hat{y}_{\hat{c}})^+$

$$(\hat{\lambda}_c-\hat{\lambda}_{\hat{c}},\phi)=\big((c-\hat{c})\min(0,\hat{y}_c-\psi)+\hat{c}(\min(0,\hat{y}_c-\psi)-\min(0,\hat{y}_{\hat{c}}-\psi),\phi\big)\ge 0.$$

Hence, using the arguments leading to (2.4), we have $\hat{y}_c\le\hat{y}_{\hat{c}}$ for $c\le\hat{c}$. Then $\hat{y}_c(t)\to y^*(t)$ strongly in H and pointwise almost everywhere in Ω. □

Theorems 2.1 and 2.2 imply in particular the monotone convergence of $\hat{y}_c$ and y_c to y^*. This is expressed in the following corollary.

Corollary 2.2. *If Assumptions (1)-(6) hold,* $y_0-\psi\in\mathcal{C}\cap V$ *and* $f\in L^2(0,T;H)$*, then for every* $t\in[0,T]$

$$\hat{y}_c(t)\le y^*(t)\le y_c(t)$$

and

$$\hat{y}_c(t) \uparrow y^*(t) \text{ and } y_c(t) \downarrow y^*(t)$$

pointwise almost everywhere, monotonically in Ω *as* $c \to \infty$. *Moreover* $y^* - \bar{\psi} \in H^1(0,T;H) \cap L^2(0,T;dom(A)) \cap L^2(0,T;V)$.

3 Regularity

In this section we discuss additional regularity of the solution y^* to (1.1) under the assumptions of Theorem 2.1 (2), i.e. (1)-(4) and $f \in L^2(0,T;H)$, $y_0 - \psi \in C \cap V$. Then, in particular, $\lambda \in L^2(0,T;H)$. In addition we assume that $y_0 \in dom(A)$. Then from the proof of part (2) of Theorem 2.1 it follows that $y^* \in H^1(0,T;H)$.

For $h > 0$ we have by (1.2), suppressing the superscripts *,

$$\frac{d}{dt}(\frac{(y(t+h)-y(t)}{h}) + A(\frac{y(t+h)-y(t)}{h}) + \frac{\lambda(t+h)-\lambda(t)}{h} = \frac{f(t+h)-f(t)}{h}$$

From (1.2) further

$$(\lambda(t+h)-\lambda(t), y(t+h)-y(t)) = -(\lambda(t+h), y(t)-\psi) - (\lambda(t), y(t+h)-\psi) \geq 0$$

and thus

$$\langle \frac{d}{dt}(\frac{(y(t+h)-y(t)}{h}), \frac{(y(t+h)-y(t)}{h}\rangle + \langle A(\frac{y(t+h)-y(t)}{h}), \frac{y(t+h)-y(t)}{h}\rangle$$

$$\leq \langle \frac{f(t+h)-f(t)}{h}, \frac{y(t+h)-y(t)}{h}\rangle.$$

Multiplying this by $t > 0$ we obtain

$$\frac{d}{dt}(\frac{t}{2}|\frac{y(t+h)-y(t)}{h}|_H^2) + \frac{t\omega}{2}|\frac{y(t+h)-y(t)}{h}|_V^2 \leq \frac{1}{2}|\frac{(y(t+h)-y(t)}{h}|^2$$

$$+\rho t\,|\frac{y(t+h)-y(t)}{h}|_H^2 + \frac{t}{2\omega}|\frac{(f(t+h)-f(t)}{h}|_{V^*}^2.$$

Integrating in time,

$$t\,|\frac{y(t+h)-y(t)}{h}|_H^2 + \omega\int_0^t s|\frac{y(s+h)-y(s)}{h}|_V^2\,ds$$

$$\leq e^{2\rho t}\int_0^t (|\frac{y(s+h)-y(s)}{h}|_H^2 + \frac{s}{\omega}|\frac{f(s+h)-f(s)}{h}|_{V^*}^2)\,ds,$$

and letting $h \to 0^+$, we obtain

$$t\,|\frac{d}{dt}y|_H^2 + \omega \int_0^t s|\frac{d}{dt}y|_V^2\,ds \le e^{2\rho t}\int_0^t (|\frac{d}{dt}y|_H^2 + \frac{s}{\omega}|\frac{d}{dt}f|_{V^*}^2)\,ds, \tag{3.1}$$

provided that $f \in H^1(0,T;V^*)$.

Hence we obtain

Theorem 3.1. *Suppose that Assumptions (1)-(4) hold and that* $y_0 - \psi \in C \cap dom(A)$, $f \in L^2(0,T;H) \cap H^1(0,T,V^*)$. *Then, the strong solution satisfies* (3.1) *and thus* $y(t) \in dom\,(A)$, *for all* $t > 0$.

The conclusion of Theorem 3.1 remains correct under the assumptions of the first part of Theorem 2.2, i.e. under Assumptions (1)–(2) and (5)–(6), $y_0 - \psi \in C \cap V$, and $f \in L^2(0,T;H) \cap H^1(0,T,V^*)$.

4 Continuity of $q \to y(q) \in L^\infty(\Omega)$

In this section we analyse the continuous dependence of the strong solution to (1.1) with respect to parameters in the operator A. Let U denote the normed linear space of parameters and let $\tilde{U} \subset U$ be a bounded subset, such that for each $q \in \tilde{U}$ the operator $A(q)$ satisfies the assumptions (1)-(4) of section 1 with $\rho = 0$, $f \in L^2(0,T;H)$ and $y_0 - \psi \in C$. We assume further that $dom(A(q)) = D$ is independent of $q \in \tilde{U}$ and that

$$q \in \tilde{U} \to A(q) \in \mathcal{L}(X, V^*)$$

is Lipschitz continuous with Lipschitz constant κ. Let $y(q)$ denote the strong solution to (1.1) corresponding to $A = A(q)$, $q \in \tilde{U}$. Then for q_1, $q_2 \in \tilde{U}$, we have

$$\langle \frac{d}{dt}(y(q_1)-y(q_2)) + A(q_1)(y(q_1)-y(q_2)) + (A(q_1)-A(q_2))y(q_2)\,,\, y(q_1)-y(q_2)\rangle \le 0,$$

and therefore

$$\begin{aligned} &|y(q_1)(T) - y(q_2)(T)|_H^2 + \omega \int_0^T |y(q_1) - y(q_2)|_V^2\,dt \\ &\le \frac{1}{\omega}\int_0^T |(A(q_2) - A(q_1))y(q_2)|_{V^*}^2\,dt = e(q_1,q_2)^2, \end{aligned}$$

where

$$e(q_1, q_2)^2 \leq \frac{\kappa}{\omega} |q_1 - q_2|_U^2 |y(q_2)|^2_{L^2(0,T;V)}.$$

Since under the assumptions of Theorem 3 we have $y(q)(T) \in D$ for $q \in \tilde{U}$, it follows by interpolation that

$$|y(q_1)(T) - y(q_2)(T)|_{W^\alpha} \leq \bar{C} e(q_1, q_2)^{1-\alpha}, \tag{4.1}$$

where $W^\alpha = [H, D]_\alpha$ is the interpolation space between D and H, see e.g. [Fat], Chapter 8, and $\bar{C}$ is an embedding constant. If $L^\infty(\Omega) \subset W^\alpha$, with $\alpha \in (\alpha_0, 1)$ for some α_0, then Hölder continuity of $q \in \tilde{U} \to y(q)(T) \in L^\infty(\Omega)$ follows.

Next we prove Lipschitz continuity of $q \in \tilde{U} \to y(q) \in L^\infty(0, T; L^\infty(\Omega))$. Some prerequisits are established first. We assume that $A(q)$ generates an analytic semigroup $S(t) = S(t; q)$ on H for every $q \in \tilde{U}$ [Pa]. Then for each $q \in \tilde{U}$ there exists M such that

$$\|A^\alpha S(t)\| \leq \frac{M}{t^\alpha}, \text{ for all } t > 0, \tag{4.2}$$

where A^α denote the fractional powers of A, with $\alpha \in (0, 1)$. We assume that M is independent of $q \in \tilde{U}$. We shall further assume that

$$dom(A^{\frac{1}{2}}) = dom((A^*)^{\frac{1}{2}}) = V, \tag{4.3}$$

for all $q \in \tilde{U}$, which is the case for a large class of second order elliptic differential operators, see e.g. [Fat], Chapter 8. We assume that

$$\|A(q)\|_{L(V,V^*)} \leq \bar{\omega} \text{ for all } q \in \tilde{U}. \tag{4.4}$$

Let $r > 2$ be such that

$$V \subset L^r(\Omega),$$

and let $\bar{M}$ denote the embedding constant so that

$$|\zeta|_{L^r(\Omega)} \leq \bar{M} |\zeta|_V, \text{ for all } \zeta \in V. \tag{4.5}$$

For

$$p = \frac{2r}{r-2} \in (2, \infty),$$

we shall utilize the assumption

$$|A^{-\frac{1}{2}}(q_1)(A(q_1)-A(q_2))y|_{L^p(\Omega)} \le \bar\kappa |q_1-q_2|_U |A(q_2)^\alpha y|_H \quad (4.6)$$
$$\text{for some } \alpha\in(0,1), \text{ and all } q_1,q_2\in\tilde U, y\in D.$$

This assumption is applicable, for example, if the parameter enters as a constant into the leading differential term of $A(q)$ or if it enters into the lower order terms.

Theorem 4.1. *Let $A(q)$ generate an analytic semigroup for every $q\in\tilde U$, and let assumptions (1)-(4), and (7) hold. If further* (4.2)-(4.6), *with M independent of $q\in\tilde U$, are satisfied and $f\in L^\infty(0,T;H), y_0\in\mathcal{C}\cap D$, then $q\to y(q)$ is Lipschitz continuous from $\tilde U\to L^\infty(0,T;L^\infty(\Omega))$.*

Proof. (1) Let $q\in\tilde U$ and $A=A(q)$. Let $f^1,f^2\in L^2(0,T;H)$ with

$$(A^{-\frac{1}{2}})^*(f^1-f^2)\in L^\infty(0,T;L^p(\Omega))),$$

and let y^1,y^2 denote the corresponding strong solutions to (1.1) with associates Lagrange multipliers λ^1 and λ^2. According to Theorem 2.1 they are elements of $L^2(0,T;H)$. For $k>0$ let

$$\phi_k(t)=\max(0,y^1(t)-y^2(t)-k)$$

and

$$\Omega_k(t)=\{x\in\Omega:\phi_k(t)>0\}.$$

By assumption (7), $\phi_k\in V$. Note that

$$(\lambda^1(t)-\lambda^2(t),\,\phi_k(t)\,)_{L^2(\Omega)}\ge 0,$$

for a.e. $t\in(0,T)$. In fact, decomposing Ω into $\{x:\lambda^1(t,x)\ge\lambda^2(t,x)\}$ and its complement, we only need to consider the latter. If $\lambda^1(t,x)<\lambda^2(t,x)$, then $\lambda^1(t,x)<0$ and hence $y^1(t,x)=\psi(x)\le y^2(t,x)$ and consequently $\phi_k(t,x)=0$. This discussion is in the a.e. sense.

Thus, it follows from (1.2) and Theorem 2.1 that

$$\langle\frac{d}{dt}(y^1-y^2),\phi_k\rangle+\langle A(y^1-y^2),\phi_k\rangle\le(f^1-f^2,\phi_k), \quad (4.7)$$

for almost every $t \in (0,T)$. By assumptions (1) and (7) we have

$$\langle A\zeta, (\zeta-k)^+\rangle \geq \omega |(\zeta-k)^+|_V^2$$

for $\zeta \in V$. Note that (4.3) and (4.4) imply that

$$\|A^{\frac{1}{2}}(q)\|_{L(V,H)} \leq \bar{\omega} \text{ for all } q \in \tilde{U}.$$

Hence it follows from (4.7) that

$$\omega \int_0^T |\phi_k|_V^2\, dt \leq \int_0^T |(f,\phi_k)|\, dt \leq \int_0^T |(A^{\frac{-*}{2}}(q) f, A^{\frac{1}{2}}(q)\phi_k)|$$
$$\leq \int_0^T (\int_{\Omega_k(t)} |A^{-\frac{*}{2}}(q) f|^2)^{\frac{1}{2}} (\int_{\Omega_k(t)} |A^{\frac{1}{2}}(q)\phi_k|^2)^{\frac{1}{2}}\, dt$$
$$\leq \bar{\omega} \int_0^T |A^{\frac{-*}{2}}(q) f|_{L^2(\Omega_k(t))} |\phi_k|_V\, dt,$$

where $A^{-\frac{*}{2}} = (A^{\frac{-1}{2}})^*, \phi_k(0) = 0, f = f^1 - f^2$. Thus for any $\beta > 1$

(4.8)

$$\omega(\int_0^T |\phi_k|_V^2\, dt)^{\frac{1}{2}} \leq \bar{\omega}(\int_0^T |A^{-\frac{*}{2}} f|^2_{L^2(\Omega_k)}\, dt)^{\frac{1}{2}} \leq \bar{\omega}\, C^{1-\beta} (\int_0^T |A^{-\frac{*}{2}} f|^2_{L^2(\Omega_k)}\, dt)^{\frac{\beta}{2}}.$$

with

$$C = (\int_0^T |A^{-\frac{*}{2}} f|^2_{L^2}\, dt)^{\frac{1}{2}},$$

where we drop the dependence of A on q. For $p > q = 2$ we have

$$\int_{\Omega_k} |A^{-\frac{*}{2}} f|^q\, dx \leq \left(\int_{\Omega_k} |A^{-\frac{*}{2}} f|^p\right)^{q/p} |\Omega_k|^{(p-q)/p}. \tag{4.9}$$

We denote by h and k arbitrary real numbers satisfying $0 < k < h < \infty$ and we find for $r > 2$ and $\phi = y^1 - y^2$

$$|\phi_k|^r_{L^r} \geq \int_{\Omega_k} |\phi - k|^r\, dx \geq \int_{\Omega_h} |\phi - k|^r\, ds \geq |\Omega_h| |h-k|^r. \tag{4.10}$$

It thus follows from (4.5) and (4.8)–(4.10) that for $\beta > 1$

$$(\textstyle\int_0^T |\Omega_h|^{\frac{2}{r}}\, dt)^{\frac{1}{2}} \leq \frac{\bar{M}}{|h-k|} (\int_0^T |\phi_k|_V^2)^{\frac{1}{2}} \leq \frac{\bar{\omega}\bar{M}C^{1-\beta}}{\omega|h-k|} (\int_0^T |A^{-\frac{1}{2}} f|^2_{L^2(\Omega_k)}\, dt)^{\frac{\beta}{2}}$$

$$\leq \frac{\bar{\omega}\bar{M}C^{1-\beta}}{\omega|h-k|} (\textstyle\int_0^T |A^{-\frac{1}{2}} f|^2_{L^p} |\Omega_k|^{\frac{p-2}{p}}\, dt)^{\frac{\beta}{2}}.$$

For $\frac{1}{P}+\frac{1}{Q}=1$ this implies that

$$(\int_0^T |\Omega_h|^{\frac{2}{r}}\,dt)^{\frac{1}{2}} \le \frac{\bar\omega \bar M C^{1-\beta}}{\omega|h-k|}(\int_0^T |A^{-\frac{1}{2}}f|^{2P}_{L^p}\,dt)^{\frac{\beta}{2P}}\,(\int_0^T |\Omega_k|^{\frac{Q(p-2)}{p}}\,dt)^{\frac{\beta}{2Q}}.$$

For $P=\infty$ and $Q=1$ this implies, using that $p=\frac{2r}{r-2}$,

$$(\int_0^T |\Omega_h|^{\frac{2}{r}}\,dt)^{\frac{1}{2}} \le \frac{K}{|h-k|}(\int_0^T |\Omega_k|^{\frac{2}{r}}\,dt)^{\frac{\beta}{2}}, \tag{4.11}$$

where $K=\frac{\bar\omega\bar M C^{1-\beta}}{\omega}|A^{-\frac{1}{2}}f|^{\beta}_{L^\infty(0,T;L^p(\Omega))}$.

Now, we use the following fact, [Tr], pg. 105: let $\varphi:(k_1,h_1)\to\mathbb{R}$ be a nonnegative, non-increasing function and suppose that there are positive constants K, s, and $\beta>1$ such that

$$\varphi(h)\le K(h-k)^{-s}\varphi(k)^{\beta}\quad\text{for } k_1<k<h<h_1.$$

Then, if $\hat k = K^{\frac{1}{s}}2^{\frac{\beta}{\beta-1}}\varphi(k_1)^{\frac{\beta-1}{s}}$ satisfies $k_1+\hat k<h_1$, it follows that $\varphi(k_1+\hat k)=0$. Here we set

$$\varphi(k)=(\int_0^T |\Omega_k|^{\frac{2}{r}}\,dt)^{\frac{1}{2}}$$

on $(0,\infty)$, $s=1,\beta>1$ and

$$k_1=\sup_{t\in(0,T)}|A^{-\frac{*}{2}}(f^1(t)-f^2(t))|_{L^p(\Omega)}.$$

As in the computation below (4.10) we show that

$$\varphi(k_1)\le\frac{\bar M\bar\omega C}{\omega k_1}.$$

From the definition of $\hat k$ we have

$$\hat k\le 2^{\frac{\beta}{\beta-1}}(\frac{\bar\omega\bar M}{\omega})^{\beta}C^{1-\beta}k_1^{\beta}C^{\beta-1}k_1^{1-\beta}=2^{\frac{\beta}{\beta-1}}(\frac{\bar\omega\bar M}{\omega})^{\beta}k_1,$$

and consequently $k_1+\hat k\le \ell\,k_1$, where $\ell=1+2^{\frac{\beta}{\beta-1}}(\frac{\bar\omega\bar M}{\omega})^{\beta}$. Hence we obtain $y^1-y^2\le \ell k_1$ a.e. in $(0,T)\times\Omega$. Analogously a uniform lower bound for y^1-y^2 is obtained by using $\phi_k=\min(0,y^1-y^2-k)\le 0$ and thus

$$|y^1-y^2|_{L^\infty(0,T;L^\infty(\Omega))}\le \ell\sup_{t\in(0,T)}|A^{-\frac{*}{2}}(f^1(t)-f^2(t))|_{L^p(\Omega)}. \tag{4.12}$$

(2) We use the estimate of step (1) to obtain Lipschitz continuous dependence of the solution on the parameter q. Let $q_1, q_2 \in \tilde{U}$ with corresponding solutions $y(q_1)$ and $y(q_2)$. Since

$$\frac{d}{dt}y(q_2) + A(q_1)y(q_2) + (A(q_2) - A(q_1))y(q_2)) + \lambda(q_2) = f(t),$$

$y(q_2)$ is the solution to (1.1) with $A = A(q_1)$ and $\tilde{f}(t) = f - (A(q_2) - A(q_1))y(q_2)$. Since by assumption $y_0 \in D$ we find from the proof to the second part of Theorem 2.1 that $\tilde{f} \in L^2(0,T;H)$. We can apply the estimate of (1) with $A = A(q_1)$ and $f^1 - f^2 = (A(q_2) - A(q_1))y(q_2)$ provided that $f^1 - f^2 \in L^\infty(0,T;L^p(\Omega))$, which will be argued below. We obtain

$$|y^1 - y^2|_{L^\infty(0,T;L^\infty(\Omega))} \le \ell \sup_{t\in(0,T)} |A(q_1)^{-\frac{*}{2}}(A(q_1) - A(q_2))y(t;q_2)|_{L^p(\Omega)}.$$

Utilizing (4.3) and (4.6) this implies that

$$|y^1 - y^2|_{L^\infty(0,T;L^\infty(\Omega))} \le \ell\,\bar{\kappa}\,|q_1 - q_2|_U \sup_{t\in(0,T)} |A^\alpha(q_2)y(t;q_2)|_H. \tag{4.13}$$

To estimate $A^\alpha(q_2)y(t;q_2)$ recall from the proof of Theorem 2.1 that $\bar{\lambda} \le \lambda(t;q) \le 0$ and thus $\{f - \lambda(q) : q \in \tilde{U}\}$ is uniformly bounded in $L^\infty(0,T;H)$. From (1.1) we have that

$$A(q_2)^\alpha y(t;q_2) = A(q_2)^\alpha S(t;q_2)y_0 + \int_0^t A(q_2)^\alpha S(t-s;q_2)(f(s) - \lambda(s;q_2))\,ds$$
$$\in L^\infty(0,T;H).$$

From (4.2) and since $y_0 \in D$ it follows that $\{A(q_2)^\alpha y(q_2) : q_2 \in \tilde{U}\}$ is bounded in $L^\infty(0,T;H)$ as desired.

□

5 Difference schemes and weak solutions

In this section we establish existence and uniqueness of weak solutions to (1.1) based on finite difference schemes (5.1).

$$\rho = 0$$

throughout this section. Consequently $\langle A\phi, \phi\rangle = a_s(\phi, \phi)$ defines an equivalent norm on V. For $h > 0$ consider the discretized (in time) variational inequality: Find $y^k - \psi \in \mathcal{C},\ k = 1, \ldots, N$, satisfying

$$\langle \frac{y^k - y^{k-1}}{h} + Ay^k - f^k, y - y^k\rangle \geq 0, \quad y^0 = y_0 \tag{5.1}$$

for all $y - \psi \in \mathcal{C}$, where

$$f^k = \frac{1}{h}\int_{(k-1)h}^{kh} f(t)\, dt$$

and $Nh = T$. Throughout this section we assume that $y_0 \in H$ and $f \in L^2(0, T; V^*)$.

Theorem 5.1. *Assume that $y_0 \in H$ and $f \in L^2(0, T, V^*)$ and that Assumptions (1)–(2) hold. Then there exists a unique solution $\{y^k\}_{k=1}^N$ to* (5.1).

Proof. To establish existence of solutions to (5.1), we proceed by induction with respect to k and assume that existence has been proven up do $k - 1$. To verify the induction step consider the regularized problems

$$\frac{y_c^k - y^{k-1}}{h} + Ay_c^k + c\ \min(0, y_c^k - \psi) - f^k = 0. \tag{5.2}$$

Since

$$y \in H \to \min c\,(0, y - \psi)$$

is Lipschitz continuous and monotone,the operator $B : V \to V^*$ defined by

$$B(y) = \frac{y}{h} + Ay + c\ \min(0, y - \psi)$$

is coercive, monotone and continuous for all $h > 0$. Hence by the theory of maximal monotone operators (5.2) admits a unique solution, cf. e.g. [ItKa], Chapter I.5, [Ba], Chapter II.1. For each $c > 0$ and $k = 1, \ldots N$ we find

$$\frac{1}{2h}(|y_c^k - \psi|_H^2 - |y^{k-1} - \psi|_H^2 + |y_c^k - y^{k-1}|_H^2)$$

$$+\langle A(y_c^k - \psi) + A\psi - f^k, y_c^k - \psi\rangle + c\,|(y_c^k - \psi)^-|_H^2 = 0.$$

Thus the families $|y_c^k - \psi|_V^2$ and $c\,|(y_c^k - \psi)^-|_H^2$ are bounded in $c > 0$ and there exists a subsequence of $\{y_c^k - \psi\}$ that converges to some $y^k - \psi$ weakly in V as $c \to \infty$. As argued in the proof of Theorem 2.2, compare (2.15), $y^k - \psi \in C$ and hence $y^k - \psi \in \mathcal{C}$. Note that

(5.3)

$$(-(y_c^k - \psi)^-, y - y_c^k) = (-(y_c^k - \psi)^-, y - \psi - (y_c^k - \psi)) \le 0 \quad \text{for all } y - \psi \in C,$$

and

(5.4)

$$\begin{aligned}
&\liminf_{c\to\infty} \langle Ay_c^k, y_c^k - y\rangle \\
&= \liminf_{c\to\infty} \big(\langle (Ay_c^k - \psi), y_c^k - \psi\rangle + \langle Ay_c^k, \psi - y\rangle + \langle A\psi, y_c^k - \psi\rangle\ \big) \\
&\ge \langle A(y^k - \psi), y^k - \psi\rangle + \langle Ay^k, \psi - y\rangle + \langle A\psi, y^k - \psi\rangle \\
&= \langle Ay^k, y^k - y\rangle.
\end{aligned}$$

Passing to the limit in (5.2) utilizing (5.3) and (5.4) we obtain

$$\begin{aligned}
&\langle \frac{y^k - y^{k-1}}{h} + Ay^k - f^k, y^k - y\rangle \\
&\le \liminf_{c\to\infty} \big(\langle \frac{y_c^k - y^{k-1}}{h}, y_c^k - y\rangle + \langle Ay_c^k, y_c^k - y\rangle - \langle f^k, y_c^k - y\rangle\big) \le 0,
\end{aligned}$$

and hence y^k satisfies (5.1).

To verify uniqueness, let $\tilde{y}^k$ be another solution to (5.1). Then, from (5.1)

$$\langle \frac{(y^k - \tilde{y}^k) - (y^{k-1} - \tilde{y}^{k-1}}{h}) + A(y^k - \tilde{y}^k), y^k - \tilde{y}^k\rangle \le 0$$

and thus

$$\frac{1}{2h}|y^k - \tilde{y}^k|_H^2 + \langle A(y^k - \tilde{y}^k), y^k - \tilde{y}^k\rangle \le \frac{1}{2h}|y^{k-1} - \tilde{y}^{k-1}|_H^2.$$

Since $y^0 = \tilde{y}^0 = y_0$, this implies that $y^k = \tilde{y}^k$ for all $k \ge 1$.

□

Next we discuss existence and uniqueness of weak solutions to (1.1) by passing to the limit in the piecewise defined functions

$$\text{(5.5)} \qquad y_h^{(1)} = y^k + \frac{t - k\,h}{h}\,(y^{k+1} - y^k), \quad y_h^{(2)} = y^{k+1}, \text{ on } (k\,h, (k+1)\,h],$$

for $k = 0, \dots, N-1$.

Theorem 5.2. *Suppose that the assumptions of Theorem 5.1 hold. Then there exists a unique weak solution y^* of (1.1). Moreover $t \to y^*(t) \in H$ is right-continuous, $y^* \in B(0,T;H)$ and $y_h^{(2)} - \psi \to y^* - \psi$ strongly in $L^2(0;T;V)$.*

Proof. Setting $y = \psi$ in (5.1), we obtain

$$|y^k - \psi|_H^2 - |y^{k-1} - \psi|_H^2 + |y^k - y^{k-1}|_H^2 + h\omega\, |y^k - \psi|_V^2 \le \frac{h}{\omega} |A\psi - f^k|_{V^*}^2.$$

Thus,

$$|y^m - \psi|_H^2 + \sum_{k=\ell+1}^{m} (|y^k - y^{k-1}|_H^2 + \omega h\, |y^k - \psi|_V^2) \le |y^\ell - \psi|_H^2 + \frac{1}{\omega} \sum_{k=\ell+1}^{m} |A\psi - f^k|_{V^*}^2\, h \tag{5.6}$$

for all $0 \le \ell < m \le N$ and consequently

$$\begin{aligned} \int_0^T |y_h^{(1)} - y_h^{(2)}|_H^2\, dt &= \sum_{k=1}^{N} |y^k + \frac{t - kh}{h}(y^{k+1} - y^k) - y^{k+1}|^2 \\ &\le \frac{h}{3} \sum_{k=1}^{N} |y^k - y^{k-1}|_H^2 \to 0, \quad \text{as } h \to 0^+. \end{aligned} \tag{5.7}$$

From (5.6) it follows that $\{y_h^{(1)}\}$ and $\{y_h^{(2)}\}$ are bounded in $L^2(0,T;V)$ uniformly with respect to $h \in (0,1)$. Together with (5.7) this implies the existence of subsequences of $y_h^{(1)}$, $y_h^{(2)}$ (denoted by the same symbols) and $y^*(t) \in L^2(0,T;V)$ such that

$$y_h^{(1)}(t),\ y_h^{(2)}(t) \to y^*(t) \text{ weakly in } L^2(0,T;V) \quad \text{as } h \to 0^+. \tag{5.8}$$

Note that

$$\frac{d}{dt} y_h^{(1)} = \frac{y^{k+1} - y^k}{h} \text{ on } (k\,h, (k+1)\,h].$$

Thus, we have from (5.1) for every $y \in \mathcal{K}$

$$\langle \frac{d}{dt} y + A y_h^{(2)} - f_h, y - y_h^{(2)} \rangle + \langle \frac{d}{dt} y_h^{(1)} - \frac{d}{dt} y, y - y_h^{(2)} \rangle \ge 0, \tag{5.9}$$

a.e. in $(0,T)$. Here

(5.10)
$$\langle \frac{d}{dt}(y_h^{(1)}-y), y-y_h^{(2)}\rangle = \langle \frac{d}{dt}y_h^{(1)}-\frac{d}{dt}y, y-y_h^{(1)}\rangle + \langle \frac{d}{dt}y_h^{(1)}-\frac{d}{dt}y, y_h^{(1)}-y_h^{(2)}\rangle$$

with

$$\int_0^T \langle \frac{d}{dt}y_h^{(1)}-\frac{d}{dt}y, y-y_h^{(1)}\rangle\,dt \le \frac{1}{2}|y(0)-y_0|_H^2 \tag{5.11}$$

and

$$\int_0^T (\frac{d}{dt}y_h^{(1)}, y_h^{(1)}-y_h^{(2)})\,dt = -\frac{1}{2}\sum_{k=1}^N |y^k-y^{k-1}|_H^2. \tag{5.12}$$

Since
$$\int_0^T \langle Ay^*, y^*-y\rangle\,dt \le \liminf_{h\to 0^+}\int_0^T \langle Ay_h^{(2)}, y_h^{(2)}-y\rangle\,dt,$$
which can be argued as in (5.4), it follows from (5.8)–(5.12) that every weak cluster point y^* of $y_h^{(2)}$ satisfies

$$\int_0^T \langle \frac{d}{dt}y+Ay^*-f, y-y^*\rangle\,dt + \frac{1}{2}|y(0)-y_0|_H^2 \ge 0, \tag{5.13}$$

for all $y\in\mathcal{K}$. Hence $y^*\in L^2(0,T;V)$ is a weak solution of (1.1) and $y^*\in B(0,T;H)$.

Moreover, from (5.6)

$$|y^*(t)-\psi|_H^2 \le |y^*(\tau)-\psi|_H^2 + \frac{1}{\omega}\int_\tau^t |A\psi-f(s)|_{V^*}^2\,ds,$$

for all $0\le\tau\le t\le T$. Thus,

$$\limsup_{t\downarrow\tau}|y^*(t)-\psi|_H^2 \le |y^*(\tau)-\psi|_H^2.$$

which implies that $t\to y^*(t)\in H$ is right-continuous.

Let y^* be a weak solution. Setting $y=y_h^{(1)}\in\mathcal{K}$ in (5.13) and $y=y^*(t)$ in (5.9) we have

$$\int_0^T \langle \frac{d}{dt}y_h^{(1)}+Ay^*-f, y_h^{(1)}-y^*\rangle\,dt \ge 0, \tag{5.14}$$

$$(5.15) \qquad \int_0^T \langle \frac{d}{dt} y_h^{(1)} + A y_h^{(2)} - f_h, y^* - y_h^{(2)} \rangle \, dt \geq 0,$$

where we used that

$$\int_0^T \langle \frac{d}{dt}(y_h^{(1)} - y^*), y^* - y_h^{(2)} \rangle \, dt \leq 0,$$

from (5.10) - (5.12).

Summing up (5.14), (5.15), and using (5.12) implies that

$$\int_0^T \left(\langle Ay^*, y_h^{(1)} - y_h^{(2)} \rangle - \langle f, y_h^{(1)} - y^* \rangle - \langle f_h, y^* - y_h^{(2)} \rangle \right) dt$$

$$\geq \frac{1}{2} \sum_{k=1}^N |y^k - y^{k-1}|_H^2 + \int_0^T \langle A(y^* - y_h^{(2)}), y^* - y_h^{(2)} \rangle \, dt.$$

Letting $h \to 0^+$ we obtain $0 \geq \langle A(y^*(t) - \hat{y}(t)), y^*(t) - \hat{y}(t) \rangle$ a.e. on $(0, T)$, for every weak cluster point $\hat{y}$ of $y_h^{(2)}$ in $L^2(0, T; V)$. This implies that the weak solution is unique and that

$$\int_0^T \langle A(y^* - y_h^{(2)}), y^* - y_h^{(2)} \rangle \, dt \to 0$$

as $h \to 0^+$. □

Corollary 5.1. *Let* $y = y(y_0, y)$ *denote the weak solution to* (1.1)*, given* $y_0 \in H$ *and* $f \in L^2(0, T; V^*)$*. Then for all* $t \in [0, T]$

$$|y(y_0, f)(t) - y(\tilde{y}_0, \tilde{f})(t)|_H + \omega \int_0^t |y(y_0, f) - y(\tilde{y}_0, \tilde{f})|_V^2 \, ds$$

$$\leq |y_0 - \tilde{y}_0|_H^2 + \frac{1}{\omega} \int_0^t |f - \tilde{f}|_{V^*}^2 \, ds.$$

Proof. Let y^k and $\tilde{y}^k$ is the solution to (5.1) corresponding to (y_0, f) and $(\tilde{y}_0, \tilde{f})$. It then follows from (5.1) that

$$\langle \frac{(y^k - \tilde{y}^k) - (y^{k-1} - \tilde{y}^{k-1})}{h} + A(y^k - \tilde{y}^k) - (f^k - \tilde{f}^k), y^k - \tilde{y}^k \rangle \leq 0$$

Thus,

$$|y^k - \tilde{y}^k|_H^2 + \omega h\, |y^k - \tilde{y}^k|_V^2 \leq |y^{k-1} - \tilde{y}^{k-1}|_H^2 + \frac{h}{\omega} |f^k - \tilde{f}^k|_{V^*}$$

Summing this in k, we have

$$|y^m - \tilde{y}^m|_H^2 + \omega \sum_{k=1}^m h\, |y^k - \tilde{y}^k|_V^2 \leq |y_0 - \tilde{y}_0|_H^2 + \frac{1}{\omega} \sum_{k=1}^m h\, |f^k - \tilde{f}^k|_{V^*}.$$

which implies the desired estimate by letting $h \to 0^+$. □

Corollary 5.2. *Let $\bar{\lambda} \in H$ satisfy $\bar{\lambda} \leq 0$ and let $y_c \in W(0,T)$ be the solution to*

$$\frac{d}{dt} y_c(t) + A y_c(t) + \min(0, \bar{\lambda} + c\,(y_c - \psi)) = f. \tag{5.16}$$

Then, $y_c \to y^*$ weakly in $L^2(0,T;V)$ and $y_c(T) \to y^*(T)$ weakly in H as $c \to \infty$, where y^* is the unique weak solution to (1.1)*. In addition, if $y^* - \psi \in W(0,T)$, then*

$$|y_c - y^*|_{L^2(0,T;V)} + |y_c - y^*|_{C(0,T;H)} \to 0,$$

as $c \to \infty$.

Proof. Note that

$$(\min(0, \bar{\lambda} + c\,(y_c - \psi)), y_c - \psi) \geq \frac{c}{2}\, |(y_c - \psi)^-|_H^2 - \frac{1}{2c} |\bar{\lambda}|_H^2.$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt} |y_c - \psi|_H^2 + \langle A(y - \psi), y_c - \psi \rangle + \frac{c}{2}\, |(y_c - \psi)^-|_H^2 \leq \langle f - A\psi, y_c - \psi \rangle + \frac{1}{2c} |\bar{\lambda}|^2$$

and

$$\begin{aligned} |y_c(t) - \psi|_H^2 &+ \int_0^t (\omega |y_c - \psi|_V^2 + c\, |(y_c - \psi)^-|_H^2)\, ds \\ &\leq \int_0^t (\frac{1}{\omega} |f - A\psi|_{V^*}^2 + \frac{1}{c} |\bar{\lambda}|^2)\, ds + |y_0 - \psi|_H^2. \end{aligned}$$

Hence $\int_0^T |(y_c-\psi)^-|_H^2\,dt \to 0$ as $c\to 0$ and $\{y_c-\psi\}_{c\geq 1}$ is bounded in $L^2(0,T;V)$. Using the same arguments as in the proof of Theorem 2.2, there exist y^* and a subsequence of $\{y_c-\psi\}_{c\geq 1}$ that converges weakly to $y^*-\psi\in L^2(0,T;V)$, and $y^*-\psi\geq 0$ a.e. in $(0,T)\times\Omega$. For $y(t)\in\mathcal{K}$

$$\int_0^T [\langle \frac{d}{dt}y(t)-\frac{d}{dt}(y(t)-y_c),y(t)-y_c(t)\rangle+\langle Ay_c(t)-f(t),y(t)-y_c(t)\rangle$$

$$+(\min(0,\bar\lambda+c\,(y_c-\psi)),y(t)-\psi-(y_c-\psi))]\,dt=0$$

where

$$(5.17)\quad \int_0^T \langle -\frac{d}{dt}(y(t)-y_c),y(t)-y_c(t)\rangle=\frac{1}{2}(|y(0)-y_0|_H^2-|y(T)-y_c(T)|^2)$$

$$(5.18)\qquad (\min(0,\bar\lambda+c\,(y_c-\psi)),y(t)-\psi-(y_c-\psi))\leq\frac{1}{2c}|\bar\lambda|_H^2.$$

Hence, we have

$$\int_0^T [\langle \tfrac{d}{dt}y(t),y(t)-y_c(t)\rangle+\langle Ay_c(t)-f(t),y(t)-y_c(t)\rangle+\tfrac{1}{2}|y(0)-y_0|_H^2$$

$$\geq\tfrac{1}{2}|y(T)-y_c(T)|_H^2-\tfrac{1}{2c}\int_0^T|\bar\lambda|_H^2\,ds.$$

Letting $c\to\infty$, y^* satisfies (1.4) and thus y^* is the weak solution of (1.1).

Suppose that $y^*-\psi\in W(0,T)$. Then by (5.16),

$$\langle\frac{d}{dt}(y_c-y^*)+A(y_c-y^*)+\frac{d}{dt}y^*+Ay^*-f,y^*-y_c\rangle$$

$$+(\min(0,\bar\lambda+c\,(y_c-\psi)),y^*(t)-\psi-(y_c-\psi))=0.$$

From (5.17)- (5.18), and since $y_c-\psi\to y^*-\psi$ weakly in $L^2(0,T;V)$,

$$\frac{1}{2}\,|y_c(t)-y^*(t)|_H^2+\int_0^t\langle A(y_c-y^*),y_c-y^*\rangle\,ds\to 0,$$

and this convergence is uniform with respect to $t\in[0,T]$. □

6 Optimal Control Problem and Necessary Optimality Condition

Let Q_{ad} be a closed convex subset of a Hilbert space U, where U denotes the parameter space. We assume that the parameter dependent operator $A = A(q)$ satisfies

$$dom(A(q)) = dom(A) = dom(A^*),$$

which is endowed with the graph norm. Throughout this section we assume that for constants $M > 0$ and $\omega > 0$ the following estimates hold for all $q, \bar{q} \in Q_{ad}$:

$$(6.1) \qquad \begin{cases} |A(q)y|_{V^*} \leq M\,|y|_X \text{ for } y \in X \\ \\ \langle A(q)\phi, \phi\rangle_{V^*,V} \geq \omega\,|\phi|_V^2 \text{ for } \phi \in V, \\ \\ |A(q)y - A(\bar{q})y|_{V^*} \leq M\,|q-\bar{q}|_U |y|_X \text{ for all } y \in X. \end{cases}$$

The results of this section can also be obtained under the more general coercivity estimate

$$\langle A(q)\phi, \phi\rangle_{V^*,V} \geq \omega\,|\phi|_V^2 - \rho\,|\phi|_H^2 \text{ for } \phi \in V,$$

for some $\rho \in \mathbb{R}$. Moreover we assume that there exists $A'(\bar{q}) : U \to \mathcal{L}(X, V^*)$ such that

$$(6.2) \quad \lim_{s\to 0^+} \Big|\frac{A(\bar{q}+s(q-\bar{q}))y - A(\bar{q})y}{s} - A'(\bar{q})(q-\bar{q})y\Big|_{V^*} \to 0 \text{ for all } y \in X.$$

with

$$|\langle A'(q)(h)y, \phi\rangle| \leq M|h|_U|y|_X|\phi|_V \text{ for } (h, y, \phi) \in U \times X \times V.$$

Consider the calibration problem:

$$(P) \qquad \min \quad J(q) = \frac{1}{2}\int_\Omega w(x)\,|y(T) - y_d|^2\,dx + \beta\,W(q) \quad \text{over } q \in Q_{ad}$$

subject to y being the weak solution to (1.1) with $A = A(q)$, i.e.

$$(6.3) \qquad \int_0^T \big\langle \tfrac{d}{dt}y(t) + Ay^*(t) - f(t), y(t) - y^*(t)\big\rangle\,dt + \tfrac{1}{2}|y(0) - y_0|_H^2 \geq 0,$$

for all $y-\psi \in \mathcal{K}$, where $y_0 \in H$ and $f \in L^2(0,T;V^*)$, $w \in L^\infty(\Omega), w > 0$ *a.e.*

Here y_d is the target value at $t = T$. The last term in (P) with $\beta > 0$ represents the penalty on the parameter q. We assume that W is a C^2 functional on U. For $q \in Q_{ad}$ let $y(q)$ denote the weak solution with $A = A(q)$. Note also that $y = y(q)$ is the weak solution to (1.1) with $A = A(\bar q)$ and $f = (A(q) - A(\bar q))y(q) - f$, for each $\bar q \in Q_{ad}$. Since $y(q) \in L^2(0,T;V)$ is uniformly bounded over $q \in Q_{ad}$ it follows from (6.1) and Corollary 5.1 that for $q_n \to \bar q$ in U

$$|y(q_n) - y(\bar q)|_{L^2(0,T;V)} + |y(q_n) - y(\bar q)|_{B(0,T:H)} \to 0, \tag{6.4}$$

as $n \to \infty$. Assume that for some $\tilde\eta > 0$

(6.5)

$\tilde Q = \{q \in Q_{ad} : W(q) \le J(q_0) + \tilde\eta\}$ is compact in U for some $q_0 \in Q_{ad}$.

Let $q_n \in Q_{ad}$ be a minimizing sequence for J over Q_{ad}. Then, since $W(q_n) \le J(q_0) + \tilde\eta$ for all n sufficiently large, $\{q_n\} \in \tilde Q$ for all such n. Hence there exists $q^* \in Q_{ad}$ such that$|q_n - q^*|_U \to 0$ as $n \to \infty$ by (6.5). From (6.4) and continuity of W it follows that $(y(q^*), q^*) \in L^2(0,T;V) \times Q_{ad}$ is an optimal solution for (P). We shall derive first order necessary optimality conditions.

Based on the regularized equations already considered in (2.1) we first consider for $c > 0$ the following regularisation of (P):

$$(P_c) \quad \min \quad J_c(q) = \frac{1}{2}\int_\Omega w(x)\,|y_c(T) - y_d|^2\,dx + \beta\,W(q) \quad \text{over } q \in Q_{ad},$$

subject to $y_c(q)$ being the solution to

$$\frac{d}{dt}y_c + A(q)y_c - (\bar\lambda + c(y_c - \psi))^- = f, \quad y(0) = y_0. \tag{6.6}$$

For $\bar q \in Q_{ad}$ let $p_c = p_c(\bar q) \in W(0,T) = L^2(0,T;V) \cap H^1(0,T;V^*)$ be the solution to the adjoint equation

$$-\frac{d}{dt}p_c + A(\bar q)^* p_c + c\chi_c\, p_c = 0, \quad p_c(T) = -w(x)(y_c(T,\bar q) - y_d), \tag{6.7}$$

where χ_c is the characteristic function of the set $\{\bar\lambda + c(y_c(\bar q) - \psi) < 0\}$.

Proposition 6.1. *Assume that* (6.1), (6.2) *hold, and that* q_c *is a solution to* (P_c)*. Then the Gateaux derivative of* $q \in Q_{ad} \to J_c(q)$ *exists and is given by*

(6.8)

$$J_c'(\bar q)h = \int_0^T \langle A'(\bar q)\, h\, y_c(\bar q), p_c\rangle\, dt + \langle W'(\bar q), h\rangle_{U^*,U}, \quad \text{for all } h \in U,\ \bar q \in Q_{ad}.$$

The necessary optimality condition for q_c *is given by*

$$(6.9)\quad \int_0^T \langle A'(q_c)(q-q_c)y_c(q_c), p_c\rangle + \langle W'(q_c), q-q_c\rangle_{U^*,U} \geq 0 \text{ for all } q \in Q_{ad}.$$

Proof. Note that for $q_s = (1-s)\bar q + sq$, with $q,\ \bar q \in Q_{ad},\ s \in (0,1)$,

(6.10)

$$J_c(q_s) - J_c(\bar q) = \int_\Omega w(x)[(y_c(T,\bar q) - y_d\,,\, y_c(T,q_s) - y_c(T,\bar q))$$

$$+\tfrac{1}{2}|y_c(T,q_s) - y_c(T,\bar q)|^2\,]\ dx + s\langle W'(\bar q), q - \bar q\rangle_{U^*,U} + o(s|q-\bar q|_U).$$

Let y_c and $\bar y_c$ denote the solution to (6.6) corresponding to $A = A(q_s)$ and $A = A(\bar q)$, respectively. Since for $y - \psi \in W(0,T),\ p \in W(0,T)$

$$\frac{d}{dt}(y,p)_H = \langle \frac{d}{dt}y, p\rangle + \langle \frac{d}{dt}p, y\rangle, \quad \text{for a.e. } t,$$

we have

$$\frac{d}{dt}(y_c - \bar y_c, p_c)_H + \langle (A(q_s) - A(\bar q))\,\bar y_c + A(q_s)(y_c - \bar y_c), p_c\rangle$$
$$-\langle A(\bar q)^* p_c, y_c - \bar y_c\rangle - \Lambda = 0,$$

where

$$\Lambda_s = ((\bar\lambda + c(y_c - \psi))^- - (\bar\lambda + c(\bar y_c - \psi))^-, p_c) + s(c\chi_c(y_c - \bar y_c), p_c) = 0.$$

Consequently we find

$$(6.11)\qquad \begin{aligned} &(y_c(T) - \bar y_c(T), w\,(\bar y_c(T) - y_d))_H \\ &= \int_0^T [\,\langle (A(q_s) - A(\bar q))\,\bar y_c + A(q_s)(y_c - \bar y_c), p_c\rangle \\ &\qquad + \langle A(\bar q)^* p_c, y_c - \bar y_c\rangle - \Lambda]\, dt. \end{aligned}$$

Since

$$-((\bar{\lambda} + c(y_c - \psi))^- - (\bar{\lambda} + c(\bar{y}_c - \psi))^-, y_c - \bar{y}_c) \geq 0,$$

it follows from (6.6) that

$$\frac{1}{2}\frac{d}{dt}|y_c - \bar{y}_c|_H^2 + \langle A(q_s)(y_c - \bar{y}_c) + (A(q_s) - A(\bar{q}))\bar{y}_c, y_c - \bar{y}_c\rangle \leq 0.$$

By the second inequality in (6.1) therefore

$$\frac{d}{dt}|y_c - \bar{y}_c|_H^2 + \omega|y_c - \bar{y}_c|_V^2 \leq \frac{1}{\omega}|(A(q) - A(\bar{q}))\bar{y}_c|_{V^*}^2$$

and thus for $t \in [0, T]$

$$|y_c(t) - \bar{y}_c(t)|_H^2 + \omega \int_0^t |y_c - \bar{y}_c|_V^2\, ds \leq \frac{1}{\omega} \int_0^t |(A(q_s) - A(\bar{q}))\bar{y}_c|_{V^*}^2.$$

The last inequality in (6.1) implies that

$$\max_{t\in[0,T]} |y_c(t) - \bar{y}_c(t)|_H^2 + \omega \int_0^T |y_c - \bar{y}_c|_V^2\, ds = O(s^2) \tag{6.12}$$

This estimate together with Lebesgue's bounded convergence theorem imply that

$$\frac{1}{s}\int_0^T \Lambda_s(t)\, dt \to 0 \text{ as } s \to 0^+. \tag{6.13}$$

and by (6.2)

$$\int_0^T \langle \frac{1}{s}(A(q_s)\bar{y}_c - A(\bar{q})\bar{y}_c - sA'(\bar{q})\bar{y}_c), p_c\rangle\, dt \to 0 \tag{6.14}$$

as $s \to 0^+$. It thus follows from (6.11)–(6.14) that

$$\lim_{s\to 0^+} \frac{1}{s}(y_c(T) - \bar{y}_c(T), w\,(\bar{y}_c(T) - y_d))_H = \int_0^T \langle A'(\bar{q})\bar{y}_c, p_c\rangle\, dt. \tag{6.15}$$

Combining (6.10), (6.12) and (6.15) we find

$$\lim_{s\to 0^+} \frac{J(q_s) - J(\bar{q})}{s} = \int_0^T \langle A'(\bar{q})(q - \bar{q})\bar{y}_c, p_c\rangle\, dt + \langle W'(\bar{q}), q - \bar{q}\rangle_{U^*,U},$$

where p_c satisfies the adjoint equation (6.7). Hence the Gateaux derivative of $q \in Q_{ad} \to J(q)$ exists and is given by

$$J'(\bar{q})h = \int_0^T \langle A'(\bar{q})\, h\, \bar{y}_c, p_c\rangle\, dt + \langle W'(\bar{q}), h\rangle_{U^*,U}, \quad h \in U.$$

This also implies the first order necessary optimality condition for (P_c):

$$\int_0^T \langle A'(q_c)(q - q_c)\bar{y}_c, p_c\rangle + \langle W'(q_c), q - q_c\rangle_{U^*,U} \geq 0 \quad \text{for all } q \in Q_{ad}.$$

□

As a consequence of the proof, in particular the estimate just about (6.12), we have the following corollary.

Corollary 6.1. *If* (6.1) *and* (6.5) *hold, then* (P_c) *admits a solution* q_c *for each* $c > 0$.

Next, we consider the limit as $c \to \infty$.

Theorem 6.1. *Assume that* (6.1) *and* (6.5) *hold and that there exists an optimal solution* $q^* \in Q_{ad}$ *to* (P) *such that* $y(q^*) - \psi \in W(0,T)$*. Then there exists a weak cluster point in* $L^2(0,T;V) \times U$ *of the family of solutions* $\{(y_c(q_c), q_c)\}$ *to* (P_c) *as* $c \to \infty$*, and every such weak cluster point is a solution to (P).*

Proof. Since $q_c \in Q_{ad}$ is optimal for (P_c),

$$W(q_c) \leq W(q^*) + \frac{1}{2}|y_c(q^*)(T) - y_d|_H^2$$

for each $c > 0$. By Corollary 5.2 and since by assumption $y(q^*) - \psi \in W(0,T)$, we have $y_c(q^*)(T) \to y(q^*)(T)$ in H. Consequently $\limsup_{c>0} W(q_c) \leq J(q^*) \leq J(q_0)$ and hence q_c lie in the compact subset $\tilde{Q}$ for c sufficiently large. Therefore there exists a subsequence (denoted by the same symbol) and $\bar{q} \in Q_{ad}$ such that $|q_c - \bar{q}|_U \to 0$ as $c \to \infty$. As shown in the proof of Proposition 6.1,

$$\tfrac{1}{2}\tfrac{d}{dt}|y_c(q_c) - y_c(\bar{q})|_H^2 + \langle A(q_c)(y_c(q_c) - y_c(\bar{q})), y_c(q_c) - y_c(\bar{q})\rangle$$
$$+\langle (A(\bar{q}) - A(q_c))y_c(\bar{q}), y_c(q_c) - y_c(\bar{q})\rangle \leq 0$$

and thus by (6.1)

$$|y_c(q_c)(t) - y_c(\bar{q})(t)|_H^2 + \omega \int_0^t |y_c(q_c) - y_c(\bar{q})|_V^2 \, ds$$

$$\le \frac{1}{\omega} \int_0^t |(A(\bar{q}) - A(q_c)) y_c(\bar{q})|_{V^*}^2 \, ds \to 0$$

as $c \to \infty$. Since from Corollary 5.2 $y_c(\bar{q})(T) \to y(\bar{q})(T)$ weakly in H, it follows that

$$y_c(q_c)(T) \to y(\bar{q})(T) \text{ weakly in } H.$$

In the following step we again use that $y_c(q^*)(T) \to y(q^*)(T)$ strongly in H. Due to lower semi-continuity of norms, taking $c \to \infty$ in $J_c(q_c) \le J_c(q^*)$, implies $J(\bar{q}) \le J(q^*)$, and hence $\bar{q}$ is optimal. □

To derive an optimality system for (P) additional assumptions are required which are stated next.

$$\begin{aligned} &\text{There exist } r_0 > 1 \text{ and } \beta \in \mathbb{R} \text{ such that} \\ &r\,(A^*(q)p, |p|^{r-2}p) \ge -\beta |p|_{L^r(\Omega)}^r \\ &\text{for all } q \in Q_{ad}, p \in dom(A), \text{ and } r \in (0, r_0], \end{aligned} \tag{6.16}$$

$$\begin{aligned} &\text{there exists } \alpha_0 \text{ such that} \\ &C(\Omega) \subset W^\alpha = [H, dom(A)]_\alpha, \text{ for all } \alpha \in (\alpha_0, 1]. \end{aligned} \tag{6.17}$$

$$\begin{aligned} &\text{For } q \to \bar{q} \text{ weakly in } U, \text{ we have } W'(q) \to W'(\bar{q}) \text{ weakly in } U^* \\ &\text{and } \liminf \langle W'(q), q \rangle_{U^*,U} \ge \langle W'(\bar{q}), \bar{q} \rangle_{U^*,U}. \end{aligned} \tag{6.18}$$

$$\begin{cases} |A'(q)(h)y|_H \le M|h|_U |y|_{dom(A)} \text{for } h \in U, q \in Q_{ad}, y \in dom(A), \\ |(A'(q) - A'(\bar{q}))(h)v|_{V^*} \le M|q - \bar{q}|_U |h|_U |v|_X, \\ \text{for } h \in U, q, \bar{q} \in Q_{ad}, v \in V. \end{cases} \tag{6.19}$$

Theorem 6.2. *Assume that* (6.1), (6.2), (6.5), (6.16)-(6.19), *and assumptions (3)–(4) hold, and that* $y_0 - \psi \in \mathcal{C} \cap dom(A)$, $f \in L^2(0,T;H)$. *Then for every weak cluster point* $(y(q^*), q^*)$ *in* $L^2(0,T;V) \times Q_{ad}$ *of the family* *of solutions* $\{(y_c(q_c), q_c)\}$ *to* (P_c) *there exist associated Lagrange multipliers* $p \in L^2(0,T;V) \cap L^\infty(0,T;H)$, $\lambda^* \in L^2(0,T;H)$ *and* $\mu \in C([0,T] \times \Omega))^*$ *such that for a subsequence* $c_k \to \infty$,

$$\lambda_{c_k} \to \lambda^* \text{ weakly in } L^2(0,T;H)$$

$$c\chi_{c_k} p_{c_k} \to \mu \text{ weakly star in } L^\infty((0,T) \times \bar{\Omega})^*,$$

$$p_{c_k} \to p \text{ weakly in } L^2(0,T;V) \text{ and weakly star in } L^\infty(0,T;H),$$

and we have for all $\phi \in C(Q) \cap W(0,T)$ *with* $\phi(0,\cdot) = 0$

$$\int_0^T \langle A'(q^*)(q - q^*)y^*, p\rangle\, dt + \langle W'(q^*), (q - q^*)\rangle_{U^*,U} \geq 0, \forall q \in Q_{ad} \tag{6.20}$$

$$\frac{d}{dt} y^* + Ay^* + \lambda^* = f, \quad y(0) = y_0, \tag{6.21}$$

$$\lambda^* = \min(0, \lambda^* + (y^* - \psi)) \tag{6.22}$$

$$\begin{aligned}(p, \tfrac{d}{dt}\phi)_{L^2(V,V^*)} + (p, A(q^*)\phi)_{L^2(V,V^*)} + \langle \mu, \phi\rangle_{L^\infty(Q)^*, L^\infty(Q)} \\ = (p(T), \phi(T))_H,\end{aligned} \tag{6.23}$$

$$p(T) = w\,(y^*(T) - y_d), \tag{6.24}$$

$$\langle \mu, y^* - \psi\rangle_{L^\infty(Q)^*, L^\infty(Q)} = 0, \tag{6.25}$$

$$\frac{1}{2}|p(t)|_H^2 + \int_0^T \langle Ap, p\rangle\, ds \leq \frac{1}{2}|p(T)|_H^2, \tag{6.26}$$

where $Q = (0,T) \times \Omega$.

Proof. Under the assumptions of the theorem all conclusions of Proposition 2.1 and Theorem 2.1 apply. Moreover by Corollary 6.1 problem (P_c) admits a solution q_c satisfying

$$J_c(q_c) \leq J_c(q) \text{ for every } q \in Q_{ad}.$$

By Theorem 2.1 we have that $y(q)-\psi \in W(0,T)$ for any $q \in Q_{ad}$. Therefore one can argue as in the proof of Theorem 6.1 that there exists a subsequential weak limit $(y(q^*),q^*) \in L^2(0,T;V)\times U$ with q^* a solution to (P). As in the argument in Proposition 2.1 $q_c \to q^*$ strongly in U. Moreover, since $y_0 \in dom(A)$, we have that $\lambda^* \in L^2(0,T;H)$ and (1.3) can be expressed in the form (6.21) and (6.22) with $y^* \in L^2(0,T;dom(A))$. Note that for fixed y^*, the associated Lagrange multiplier λ^* is unique.

Let $y_c = y_c(q_c) \in L^2(0,T;dom(A))$ be solutions to (2.1) with $A = A(q_c)$ and let $p_c = p_c(q_c) \in W(0,T)$ be the associated adjoint solutions satisfying

$$(6.27) \qquad -\frac{d}{dt}p_c + A(q_c)^*p_c + c\chi_c\, p_c = 0, \quad p_c(T) = -w(x)(y_c(T,q_c)-y_d).$$

Recall from the proof of Theorem 2.1 that

$$(6.28) \qquad \bar\lambda \le \lambda_c = \min(0, \bar\lambda + c\,(y_c-\psi)) \le 0$$

and that $|y_c(T)-\psi|_H^2$ is bounded uniformly in $c \ge 1$, by Corollary 5.2 and (6.1). By (6.28) λ_c converges weakly to λ^* in $L^2(0,T;H)$. Since

$$-\frac{1}{2}\frac{d}{dt}|p_c|_H^2 + \langle A(q_c)p_c, p_c\rangle + (c\chi_c p_c, p_c) = 0$$

we have

$$(6.29) \qquad |p_c(t)|_H^2 + 2\int_t^T (\omega\,|p_c|_V^2 + (c\chi_c\, p_c, p_c))\,ds \le |p_c(T)|_H^2$$

for $t \in [0,T]$, where $p_c(T) = w\,\big(y_c(q_c)(T) - y_d\big)$. Thus $\{p_c\}$ is bounded in $L^2(0,T;V)\cap C(0,T;H)$ and there exists p such that subsequentially p_c converges weakly in $L^2(0,T;V)$ and weakly star in $L^\infty(0,T;H)$ to p. Further by (6.29)

$$c\int_0^T (\chi_c\, p_c, p_c)\,ds \le |p_c(T)|_H^2.$$

Thus, $\chi_c\, p_c \to 0 \in L^2(0,T;H)$ and

$$(6.30) \qquad \int_0^T\int_\Omega |\lambda_c p_c|\,dt \le \int_0^T |(\chi_c p_c, \bar\lambda)|\,dt \le \int_0^T |\chi_c p_c|_H\,|\bar\lambda|_H\,dt \to 0,$$

i.e., $\lambda_c p_c \to 0$ in $L^1((0,T)\times\Omega)$.

For the adjoint equations we find using (6.16)

$$-(\frac{d}{dt}p_c, |p_c|^{r-2}p_c) + \langle A(q_c)^* p_c, |p_c|^{r-2}p_c\rangle + c(\chi_c p_c, |p_c|^{r-2}p_c) = 0,$$

and hence by (6.16)

$$-\frac{d}{dt}|p_c|^r_{L^r} - \beta|p_c|^r_{L^r} + rc(\chi_c, |p_c|^r)_H \le 0.$$

Consequently

$$e^{\beta t}|p_c(t)|^r_{L^r} + c\int_t^T e^{\beta s}(\chi_c, |p_c|^r)_H\, ds \le e^{\beta T}|p_c(T)|^r_{L^r}$$

and therefore

$$|p_c(t)|^r_{L^r} + c\int_t^T (\chi_c, |p_c|^r)_H\, ds \le e^{\beta(T-t)}|p_c(T)|^r_{L^r} \le \hat{C}, \tag{6.31}$$

for a constant $\hat{C}$ independent of c and $r \in (0, r_0]$.

Note that as $c \to \infty$ we have by (6.28) and (6.30), for $\mu_c = c\,\chi_c\,p_c$

(6.32)

$$|(\mu_c, y_c - \psi)_{L^2(U)}| = |(c\chi_c\,p_c, \chi_c\,(y_c - \psi))_{L^2(Q)}| \le \int_0^T |\chi_c p_c|_H\, |\bar{\lambda}|_H\, dt \to 0.$$

Using the same arguments as in Section 2, one can show, using (6.17) and the third requirement in (6.1) that $|y_c(q_c) - y_c(q^*)|_{C(Q)} \to 0$ as $q_c \to q^*$ in Q_{ad}, uniformly in $c > 0$. It follows from Theorem 2.1 that $|y_c(q^*) - y(q^*)|_{C(Q)} \to 0$ as $c \to \infty$ and thus

$$|y_c(q_c) - y(q^*)|_{C(Q)} \to 0 \text{ as } c \to \infty. \tag{6.33}$$

Letting $r \to 1$ in (6.31), we obtain

$$\int_0^T (c\chi_c, |p_c|)\, dt \le e^{\beta T}\, |p_c(T)|_{L^1}. \tag{6.34}$$

Thus, $\mu_c = c\chi_c p_c$ is bounded in $L^1((0,T) \times \Omega)$. Hence there exists μ such that on a subsequence, denoted by the same symbols, $\mu_c \to \mu$ weak star in $(L^\infty((0,T) \times \Omega))^*$ and we have by (6.32)

$$\langle \mu, y^* - \psi\rangle_{L^\infty(Q)^*, L^\infty(Q)} = 0,$$

so that (6.25) is satisfied.

Next, since

$$-\frac{1}{2}\frac{d}{dt}|p_c|_H^2 + \langle Ap_c, p_c\rangle \le 0,$$

we have

$$\frac{1}{2}|p_c(t)|^2 + \int_t^T \langle Ap_c, p_c\rangle\, ds \le \frac{1}{2}|p_c(T)|_H^2,$$

and by the weak lower semi-continuity of norms

$$\frac{1}{2}|p(t)|^2 + \int_t^T \langle Ap, p\rangle\, ds \le \frac{1}{2}|p(T)|_H^2,$$

which proves (6.26).

To argue the necessary optimality condition (6.20) one cannot proceed as in the proof of Proposition 6.1, since p is not in $W(0,T)$, but rather we pass to the limit in

(6.35)
$$\int_0^T \langle A'(q_c)(q-q_c)y_c(q_c), p_c\rangle + \langle W'(q_c), q-q_c\rangle_{U^*,U} \ge 0 \quad \text{for all } q \in Q_{ad}.$$

From (6.18) we have

$$\int_0^T \langle W'(q^*), q-q^*\rangle \ge \int_0^T \liminf_{c\to\infty} \langle W'(q_c), q-q_c\rangle, \tag{6.36}$$

and thus it suffices to show that

$$\lim_{c\to\infty}\int_0^T \big(\langle A'(q_c)(h_c)y_c, p_c\rangle - \langle A'(q^*)(h^*)y^*, p\rangle\big) = 0,$$

where $h_c = q - q_c,\ h^* = q - q^*$. The latter follows from

$$\begin{aligned}
&\textstyle\int_0^T \langle A'(q_c)(h_c)y_c, p_c\rangle - \langle A'(q^*)(h^*)y^*, p\rangle \\
&= \textstyle\int_0^T \big(\, \langle A'(q_c)(h_c)y_c, p_c - p\rangle + \langle (A'(q_c) - A'(q^*))(h_c)y_c, p\rangle \\
&\quad + \langle A'(q^*)(h_c - h^*)y_c, p\rangle + \langle A'(q^*)(h^*)(y_c - y^*), p\,\rangle\big),
\end{aligned}$$

and the fact that these four terms tend to 0 as $c \to \infty$. Here we use (6.19), (6.2) and that $\{y_c\}_{c\ge 1}$ is bounded in $L^2(0,T; dom(A)\cap X)$, $y_c \to y^*$ in $L^2(0,T;V)$, and $p_c \to p$ in $L^2(0,T;V)$.

It remains to argue that (6.23) and (6.24) hold. The terminal condition (6.24) follows from (6.27) and (6.33). Taking the limit in

$$(p_c, \tfrac{d}{dt}\phi)_{L^2(V,V^*)} + (p_c, A(q_c)\phi)_{L^2(V,V^*)} + \langle \mu_c, \phi \rangle_{L^\infty(Q)^*, L^\infty(Q)} \\ = (p_c(T), \phi(T))_H,$$

using (6.19) one obtains (6.23), for $\phi \in C(Q) \cap W(0,T)$ with $\phi(0,\cdot) = 0$.

□

We remark that by construction $\mu_c p_c \geq 0$ a.e for every $c > 0$.

To argue that the first and second adjoint variables λ^* and p associated the the constraint $y \geq \psi$ satisfy $\lambda^* p = 0$ a.e. in Q, additional assumptions are necessary. Let $J_\varepsilon = J_\varepsilon(q^*) = (I - \varepsilon A(q^*)^*)^{-1}$, $\varepsilon > 0$, denote the resolvent of $A(q^*)$.

Theorem 6.3. *In addition to the assumptions of Theorem 6.2 assume that*

$$\text{(6.37)} \qquad (6.16) \textit{ holds with } r_0 > n,$$

$$\text{(6.38)} \qquad J_\varepsilon \in \mathcal{L}(V, V^*) \cap \mathcal{L}(L^{r_0}(\Omega)) \cap \mathcal{L}(L^1(\Omega)) \textit{ for each } \varepsilon > 0,$$

$$\text{(6.39)} \qquad |J_\varepsilon \phi - \phi| \leq \rho_\varepsilon |\phi|, \textit{ for each } \phi \in V, \textit{ with } \lim_{\varepsilon \to 0^+} \rho_\varepsilon = 0.$$

Then $\lambda^* p$ a.e. in Q.

Proof. Let $\{y_{c_k}, p_{c_k}, \lambda_{c_k}\}$ be as in the statement of Theorem 6.2 and let $\tilde{Q}$ be an arbitrary bounded subdomain of Q.

We shall show that $\{p_{c_k}\}$ restricted to $\tilde{Q}$ contains a subsequence $\{p_{c_{k_j}}\}$ that converges strongly to p in $L^2(\tilde{Q})$. This, together with the weak convergence of $\lambda_{c_{k_j}}$ to λ^* in $L^2(Q)$, and hence in $L^2(\tilde{Q})$, implies that $p_{c_{k_j}} \lambda_{c_{k_j}} \to p\lambda^*$ weakly in $L^1(\tilde{Q})$. Since $p_{c_{k_j}} \lambda_{c_{k_j}} \to 0$ strongly to 0 in $L^1(\tilde{Q})$ by (6.30) it follows that $p\lambda^* = 0$ a.e. in $\tilde{Q}$. Since $\tilde{Q}$ is an arbitrary subset of Q this implies the claim.

To argue the existence of a strongly convergent subsequence we recall from the proof of Theorem 6.2 that $\{p_c\}_{c \geq 1}$ is bounded in $L^2(V) \cap$

$L^\infty(0,T;L^{r_0}(\Omega))$ and $\{c\chi_c p_c\}_{c\geq 1}$ is bounded in $L^1(Q)$. We shall use the regularized family of functions

$$\tilde{p}_c(t) = J_\varepsilon p_c(t).$$

By (6.39)

$$\begin{aligned} |p_c - p|_{L^2(\tilde{Q})} &\leq |p_c - J_\varepsilon p_c|_{L^2(\tilde{Q})} + |J_\varepsilon p_c - J_\varepsilon p|_{L^2(\tilde{Q})} + |J_\varepsilon p - p|_{L^2(\tilde{Q})} \\ &\leq \rho_\varepsilon(|p_c|_{L^2(V)} + |p|_{L^2(V)}) + |J_\varepsilon p_c - J_\varepsilon p|_{L^2(\tilde{Q})}. \end{aligned}$$

The first term on the right hand side tends to 0 for $\varepsilon \to 0^+$. We henceforth fix ε sufficiently small and consider the term $|J_\varepsilon p_c - J_\varepsilon p|_{L^2(\tilde{Q})}$. Note that

$$-\frac{d}{dt} J_\varepsilon p_c = J_\varepsilon A(q_c)^* p_c + c J_\varepsilon \chi_c p_c.$$

By (6.38) and (6.1) the set $\{cJ_\varepsilon\chi_c p_c\}_{c\geq 1}$ is bounded in $L^1(\tilde{Q})$. Moreover $\{J_\varepsilon A(q_c)^* p_c\}_{c\geq 1}$ is bounded in $L^2(\tilde{Q})$ and, since $\tilde{Q}$ bounded, also in $L^1(\tilde{Q})$. Hence $\{\tilde{p}_c\}_{c\geq 1}$ is bounded in $W^{1,1}(\tilde{Q})$. Since $\frac{r_0}{r_0-1} < \frac{n}{n-1}$, the space $W^{1,1}(\tilde{Q})$ embeds compactly into $L^{\frac{r_0}{r_0-1}}$, and there exists a subsequence $p_{c_{k_j}}$ of p_{c_k} converging strongly to $J_\varepsilon p$ in $L^{\frac{r_0}{r_0-1}}(\tilde{Q})$. Moreover $\{\tilde{p}_c\}_{c\geq 1}$ is bounded in $L^{r_0}(\tilde{Q})$ by (6.38) so that without loss of generality the subsequence can be chosen such that $\tilde{p}_{c_{k_j}} \rightharpoonup J_\varepsilon p$ weakly in $L^{r_0}(\tilde{Q})$. This implies that

$$|\tilde{p}_{c_{k_j}} - J_\varepsilon p|^2_{L^2(\tilde{Q})} = \langle \tilde{p}_{c_{k_j}} - J_\varepsilon p, \tilde{p}_{c_{k_j}} - J_\varepsilon p \rangle_{L^{\frac{r_0}{r_0-1}}(\tilde{Q}), L^{r_0}(\tilde{Q})} \to 0,$$

where $\langle \cdot, \cdot \rangle_{L^{\frac{r_0}{r_0-1}}(\tilde{Q}), L^{r_0}(\tilde{Q})}$ denotes the duality pairing between $L^{\frac{r_0}{r_0-1}}(\tilde{Q})$ and $L^{r_0}(\tilde{Q})$. $\square$

7 Example: Black-Scholes Model for American Options

We consider the Black-Scholes model for American options, which is a variational inequality of the form

(7.1)

$$-\frac{d}{dt}v(t,S) - (\frac{\sigma^2}{2}S^2\, v_{SS} + rS\, v_S - r\, v + Bv) \geq 0 \quad \perp v(t,S) \geq \psi(S)$$

$$v(T,S) = \psi(S)$$

for a.e. $(t,S) \in (0,T) \times (0,\infty)$, where $\psi(S) = (K-S)^+$ is for the put and $\psi(S) = (S-K)^+$ for the call option. The integral operator B is defined by

$$Bv = -\lambda \int_0^\infty ((z-1)Sv_S + (v(t,S) - v(t,zS))d\nu(z),$$

where $\lambda \geq 0$. Since $v^+ \geq v$,

$$-\int_0^\infty (v(t,zS) - v^+(t,zS))d\nu(z) \geq 0$$

and thus

$$(-Bv, v^+) \geq (-Bv^+, v^+).$$

Thus,

$$Av = -(\frac{\sigma^2}{2}S^2\, v_{SS} + rS\, v_S - r\, v + Bv)$$

satisfies Assumption (3) .

In (7.1) $S \geq 0$ denotes the price, v the value of the share, $r > 0$ is the interest rate, $\sigma > 0$ is the volatility of the market and K is the strike price. Further T is the maturity date and ψ the pay-off function. Note that (7.1) is backward equation with respect to the time variable. Setting $y(t,S) = v(T-t,S)$ we arrive at (1.1) and (7.1) has the following interpretation [G,S] in mathematical finance. The price process S_t is governed by the Ito's stochastic differential equation:

$$dS_t/S_{t_-} = r\, dt + \sigma\, dB_t + (J_t - 1)\, d\pi_t,$$

where B_t denotes a standard Brownian motion, π_t is a counting Poisson process, and $J_t - 1$ is the magnitude of the jump ν and λ is the rate. The value function v is represented by

(7.2) $\quad v(t,S) = \sup_\tau E^{t,x}[e^{-r(\tau-t)}\psi(S_\tau)], \quad$ over all stopping times $\tau \leq T$.

For the log-price process $X_t = log(S_t)$

$$dX_t = r\,dt + \sigma\,B_t + \lambda\,dZ_t$$

the generator $\mathcal{L}$ is given by

$$\mathcal{L}f = \frac{\sigma 2}{2}(\frac{d2}{dx2}f - \frac{d}{dx}f) + \int_{-\infty}^{\infty}(f(x+y) - f(x) - (e^y - 1)\frac{d}{dx}f)\,d\nu(y).$$

For the Carr-Geman-Madan-Yor (CGMY) model $d\nu(y) = k(y)\,dy$ [CGMY], where

$$k(y) = \begin{cases} C\dfrac{e^{-G|y|}}{|y|^{1+\alpha}} & y > 0 \\ \\ C\dfrac{e^{-M|y|}}{|y|^{1+\alpha}} & y < 0 \end{cases}$$

with $\alpha < 2$. It generalizes a jump diffusion model for Z_t by Kou ($\alpha = -1$) and the Variance Gamma process ($\alpha = 0$). Suppose that $G = M$. Then $k(y) = k(-y)$ and we have

$$(\tilde{B}v, \psi) = \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} k(y)(v(x+y) - v(x))\,dy]\psi(x)\,dx$$

$$= \int_{-\infty}^{\infty}[\int_{0}^{\infty} k(y)(v(x+y) - 2v(x) + v(x-y))\,dy]\psi(x)\,dx$$

$$= -\int_{-\infty}^{\infty}[\int_{0}^{\infty} k(y)(v(x+y) - u(x))(\psi(x+y) - \psi(x))\,dy]\,dx.$$

In general, we have $k = k_s + k_a$ with $k_s(y) = k_s(-y),\ k_a(-y) = k_a(y)$ and

$$(\tilde{B}v, \psi) = -\int_{-\infty}^{\infty}[\int_{0}^{\infty} k_s(y)(v(x+y) - u(x))(\psi(x+y) - \psi(x))\,dy]\,dx.$$

$$+\int_{-\infty}^{\infty}[\int_{0}^{\infty}[\int_{0}^{\infty} k_a(y)(v(x+y) - v(x-y))\,dy]\psi(x)\,dx.$$

Hence if $\sigma > 0$ all conditions of (6.1) are satisfied with for example, $q = (\sigma(x), \lambda, \alpha) \in L^{\infty}(\Omega) \times R^{+} \times R$.

References

[CGMY] p.Carr, H. Geman, D. Madan, and M. Yor: The Fine Structure of Asset Returns: An Empirical Investigation, Journal of Business, 75(2002), 305-332.

[Fat] H.O. Fattorini, *Infinite Dimensional Optimization and Control Theory*, Cambridge University Press, Cambridge, 1999.

[GLT] R. Glowinski, J. L. Lions, and R. Tremoliers: *Numerical Analysis of Variational Inequalities*, North Holland, Amsterdam, 1981.

[ItKa] K. Ito and F. Kappel, *Evolution Equations and Approximations*, World Scientific, New Jersey, 2002.

[IK1] K. Ito and K. Kunisch: Parabolic variational inequalities: the Langrange multiplier approach, Journal de Math. Pures et Appl., 85(2006), 415-449.

[Lio] J.-L. Lions, *Quelques Methodes de Resolution de Problemes aux Limites Non Lineares*, Dunod, 1969, Paris.

[Pa] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, 1983.

[Tan] H. Tanabe, *Equations of Evolution*, Pitman, London, 1979.

[Tr] G. M. Troianiello, *Elliptic Differential Equations and Obstacle Problems*, Plenum Press, New York, 1987.