SpezialForschungsBereich F 32

Karl–Franzens Universität Graz
Technische Universität Graz
Medizinische Universität Graz

Boundary element methods for Dirichlet boundary control problems

Gunther Of Thanh Phan Xuan

Olaf Steinbach

SFB-Report No. 2009–003 February 2009

A–8010 GRAZ, HEINRICHSTRASSE 36, AUSTRIA

Supported by the
Austrian Science Fund (FWF)

Der Wissenschaftsfonds.

SFB sponsors:

- **Austrian Science Fund** (FWF)
- **University of Graz**
- **Graz University of Technology**
- **Medical University of Graz**
- **Government of Styria**
- **City of Graz**

Boundary element methods for Dirichlet boundary control problems

Günther Of, Thanh Phan Xuan, Olaf Steinbach

Institut für Numerische Mathematik, TU Graz,
Steyrergasse 30, 8010 Graz, Austria
{of,thanh.phanxuan,o.steinbach}@tugraz.at

Abstract

For the solution of elliptic Dirichlet boundary control problems, we propose and analyze two boundary element approaches. The state equation, the adjoint equation, and the optimality condition are rewritten as systems of boundary integral equations involving the standard boundary integral operators of the Laplace equation and of the Bi–Laplace equation. While the first approach is based on the use of the weakly singular Bi–Laplace boundary integral equation, the additional use of the hypersingular Bi–Laplace boundary integral equation results in a symmetric formulation, which is also symmetric in the discrete case. We prove the unique solvability of both boundary integral approaches and discuss related boundary element discretizations. In particular, we prove stability and related error estimates which are confirmed by a numerical example.

1 Introduction

Optimal control problems of elliptic or parabolic partial differential equations with a Dirichlet boundary control play an important role, for example, in the context of computational fluid mechanics, see, e.g., [1, 6, 10]. A difficulty in the handling of Dirichlet control problems by finite element methods lies in the essential character of Dirichlet boundary conditions. While Neumann or Robin type boundary conditions can be incorporated naturally in the weak formulation of the state equation, given Dirichlet data on the boundary have to be extended into the domain in a suitable way. For a discussion of several finite element approaches for Dirichlet boundary control problems, see, e.g., [2, 4, 8, 11, 14, 15, 16]. In most cases, the Dirichlet control is considered in $L_2(\Gamma)$, but the energy space $H^{1/2}(\Gamma)$ seems to be more natural. In [18], a finite element approach was considered, where the energy norm was realized by using some stabilized hypersingular boundary integral operator.

Since the unknown function in Dirichlet boundary control problems is to be found on the boundary $\Gamma = \partial\Omega$ of the computational domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, the use of boundary integral equations seems to be a natural choice. But to our knowledge, there are only a few results known on the use of boundary integral equations to solve optimal boundary control problems, see, e.g., [5, 23] for problems with point observations. In this paper, we consider the Poisson equation as a model problem, however, this approach can be applied to any elliptic partial differential equation, if a fundamental solution is known. In this case, solutions of partial differential equations can be described by the means of surface and volume potentials. To find the complete Cauchy data, boundary integral equations have to be solved. For an overview on boundary integral equations, see, e.g., [12, 17] and the references given therein. The numerical solution of boundary integral equations results in boundary element methods, see, e.g., [20, 22].

In this paper, we formulate and analyze a boundary element approach to solve Dirichlet boundary control problems where the control is considered in the energy space $H^{1/2}(\Gamma)$. The model problem is described in Section 2, where we also discuss the adjoint problem which characterizes the solution of the reduced minimization problem. In Section 3, we present the representation formulae to describe the solutions of both the primal and adjoint Dirichlet boundary value problems. To find the unknown normal derivatives of the state variable and of the adjoint variable, weakly singular boundary integral equations are formulated. Since the state enters the adjoint boundary value problem as a volume density, an additional volume integral has to be considered. By applying integration by parts, this Newton potential can be reformulated by using boundary potentials of the Bi–Laplace operator. Hence we recall some properties of boundary integral operators for the Bi–Laplace operator in Section 4. In Section 5, we analyze a first boundary integral formulation to solve the Dirichlet boundary control problem, and we discuss stability and error estimates of the related Galerkin boundary element method. Since this boundary element approximation leads to a non–symmetric matrix representation of a self–adjoint operator, we introduce and analyze a symmetric boundary element approach, which includes a second, the so–called hypersingular boundary integral equation, in the optimality condition in Section 6. Again we discuss the related stability and error analysis. Finally, we present a numerical example in Section 7.

2 Dirichlet control problems

As a model problem, we consider the Dirichlet boundary control problem to minimize

$$J(u,z) \,=\, \frac{1}{2}\int\limits_{\Omega}[u(x)-\overline{u}(x)]^2dx + \frac{1}{2}\varrho\|z\|_A^2 \quad \text{for } (u,z)\in H^1(\Omega)\times H^{1/2}(\Gamma) \tag{2.1}$$

subject to the constraint

$$-\Delta u(x) \,=\, f(x) \quad \text{for } x\in\Omega, \quad u(x) \,=\, z(x) \quad \text{for } x\in\Gamma=\partial\Omega, \tag{2.2}$$

where $\overline{u} \in L_2(\Omega)$ is a given target, $f \in L_2(\Omega)$ is a given volume density, $\varrho \in \mathbb{R}_+$ is a fixed parameter, and $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$. Moreover, $\|\cdot\|_A$ is an equivalent norm in $H^{1/2}(\Gamma)$ which is induced by an elliptic, self–adjoint, and bounded operator $A : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, i.e.,

$$\gamma_1^A \|w\|^2_{H^{1/2}(\Gamma)} \leq \langle Aw, w\rangle_\Gamma, \quad \|Aw\|_{H^{-1/2}(\Gamma)} \leq \gamma_2^A \|w\|_{H^{1/2}(\Gamma)} \quad \text{for all } w \in H^{1/2}(\Gamma).$$

For example, we may consider the stabilized hypersingular boundary integral operator $A = \widetilde{D}$, see [19],

$$\langle \widetilde{D}z, w\rangle_\Gamma := \langle Dz, w\rangle_\Gamma + \langle z, 1\rangle_\Gamma \langle w, 1\rangle_\Gamma \quad \text{for all } z, w \in H^{1/2}(\Gamma)$$

where

$$(Dz)(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} U^*(x, y) z(y) ds_y \quad \text{for } x \in \Gamma,$$

and

$$U^*(x, y) = \begin{cases} -\dfrac{1}{2\pi} \log|x - y| & \text{for } n = 2, \\ \dfrac{1}{4\pi} \dfrac{1}{|x - y|} & \text{for } n = 3 \end{cases} \tag{2.3}$$

is the fundamental solution of the Laplace operator [22]. Note that for $\tau \in H^{-1/2}(\Gamma)$ and $w \in H^{1/2}(\Gamma)$

$$\langle \tau, w\rangle_\Gamma = \int_\Gamma \tau(x) w(x) ds_x$$

denotes the related duality pairing.

Let $u_f \in H^1_0(\Omega)$ be the weak solution of the homogeneous Dirichlet boundary value problem

$$-\Delta u_f(x) = f(x) \quad \text{for } x \in \Omega, \quad u_f(x) = 0 \quad \text{for } x \in \Gamma.$$

The solution of the Dirichlet boundary value problem (2.2) is then given by $u = u_z + u_f$, where $u_z \in H^1(\Omega)$ is the unique solution of the Dirichlet boundary value problem

$$-\Delta u_z(x) = 0 \quad \text{for } x \in \Omega, \quad u_z(x) = z(x) \quad \text{for } x \in \Gamma. \tag{2.4}$$

Note that the solution of the Dirichlet boundary value problem (2.4) defines a linear map $u_z = \mathcal{S}z$ with $\mathcal{S} : H^{1/2}(\Gamma) \to H^1(\Omega) \subset L_2(\Omega)$. Then, by using $u = \mathcal{S}z + u_f$, we consider the problem to find the minimizer $z \in H^{1/2}(\Gamma)$ of the reduced cost functional

$$\widetilde{J}(z) = \frac{1}{2} \int_\Omega [(\mathcal{S}z)(x) + u_f(x) - \overline{u}(x)]^2 dx + \frac{1}{2} \varrho \langle Az, z\rangle_\Gamma. \tag{2.5}$$

Since the reduced cost functional $\widetilde{J}(\cdot)$ is convex, the unconstrained minimizer z can be found from the optimality condition

$$\mathcal{S}^*\mathcal{S}z + \mathcal{S}^*(u_f - \overline{u}) + \varrho Az = 0, \tag{2.6}$$

where $\mathcal{S}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ is the adjoint operator of $\mathcal{S} : H^{1/2}(\Gamma) \to L_2(\Omega)$, i.e.,

$$\langle \mathcal{S}^*\psi, \varphi\rangle_\Gamma = \langle \psi, \mathcal{S}\varphi\rangle_\Omega = \int\limits_\Omega \psi(x)(\mathcal{S}\varphi)(x)dx \quad \text{for all } \varphi \in H^{1/2}(\Gamma), \psi \in L_2(\Omega).$$

Note that the operator

$$T_\varrho \,:\, \varrho A + \mathcal{S}^*\mathcal{S} \,:\, H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is bounded and $H^{1/2}(\Gamma)$–elliptic, see, e.g., [18]. Hence, the operator equation (2.6), i.e.,

$$T_\varrho z \,=\, (\varrho A + \mathcal{S}^*\mathcal{S})z \,=\, \mathcal{S}^*(\bar{u} - u_f) =: g \tag{2.7}$$

admits a unique solution $z \in H^{1/2}(\Gamma)$. Inserting the primal variable $u = \mathcal{S}z + u_f$, and introducing the adjoint variable $\tau = \mathcal{S}^*(u - \overline{u}) \in H^{-1/2}(\Gamma)$, we have to solve the coupled problem

$$\tau + \varrho Az = 0, \quad \tau = \mathcal{S}^*(u - \overline{u}), \quad u = \mathcal{S}z + u_f \tag{2.8}$$

instead of (2.7) and (2.6), respectively. Note that for given $z \in H^{1/2}(\Gamma)$ and $f \in L_2(\Omega)$ the application of $u = Sz + u_f$ corresponds to the solution of the Dirichlet boundary value problem (2.2). The application of the adjoint operator $\tau = \mathcal{S}^*(u - \overline{u})$ is characterized by the Neumann datum

$$\tau(x) = -\frac{\partial}{\partial n_x} p(x) \quad \text{for } x \in \Gamma,$$

where p is the unique solution of the adjoint Dirichlet boundary value problem

$$-\Delta p(x) \,=\, u(x) - \bar{u}(x) \quad \text{for } x \in \Omega, \quad p(x) \,=\, 0 \quad \text{for } x \in \Gamma. \tag{2.9}$$

Hence we can rewrite the optimality condition $\tau + \varrho Az = 0$ as

$$\frac{\partial}{\partial n_x} p(x) \,=\, \varrho(Az)(x) \quad \text{for } x \in \Gamma. \tag{2.10}$$

Therefore, we have to solve a coupled system, in particular of the state equation (2.2), of the adjoint boundary value problem (2.9), and of the optimality condition (2.10), to find the minimizer $(u, z) \in H^1(\Omega) \times H^{1/2}(\Gamma)$ of the cost functional (2.1) subject to the constraint (2.2). Since the unknown control $z \in H^{1/2}(\Gamma)$ is considered on the boundary $\Gamma = \partial\Omega$, the use of boundary integral equations to solve both the primal boundary value problem (2.2) and the adjoint boundary value problem (2.9) seems to be a natural choice.

3 Laplace boundary integral equations

3.1 Primal boundary value problem

The solution of the Dirichlet boundary value problem (2.2),

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega, \quad u(x) = z(x) \quad \text{for } x \in \Gamma,$$

is given by the representation formula for $\widetilde{x} \in \Omega$, see, e.g., [22],

$$u(\widetilde{x}) \;=\; \int\limits_\Gamma U^*(\widetilde{x},y)\frac{\partial}{\partial n_y}u(y)ds_y - \int\limits_\Gamma \frac{\partial}{\partial n_y}U^*(\widetilde{x},y)z(y)ds_y + \int\limits_\Omega U^*(\widetilde{x},y)f(y)dy, \qquad (3.1)$$

where $U^*(x,y)$ is the fundamental solution of the Laplace operator as given in (2.3). To find the related Neumann datum $t = \frac{\partial}{\partial n}u \in H^{-1/2}(\Gamma)$ for a given Dirichlet datum $z \in H^{1/2}(\Gamma)$, we consider the representation formula (3.1) for $\Omega \ni \widetilde{x} \to x \in \Gamma$ to obtain the boundary integral equation

$$z(x) = u(x) = \int\limits_\Gamma U^*(x,y)t(y)ds_y + \frac{1}{2}z(x) - \int\limits_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)z(y)ds_y + \int\limits_\Omega U^*(x,y)f(y)dy$$

for almost all $x \in \Gamma$, which can be written as

$$(Vt)(x) \;=\; (\frac{1}{2}I + K)z(x) - (N_0 f)(x) \quad \text{for } x \in \Gamma. \qquad (3.2)$$

Note that

$$(Vt)(x) \;=\; \int\limits_\Gamma U^*(x,y)t(y)ds_y \quad \text{for } x \in \Gamma$$

is the Laplace single layer potential $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ satisfying

$$\|Vt\|_{H^{1/2}(\Gamma)} \;\leq\; c_2^V \, \|t\|_{H^{-1/2}(\Gamma)} \quad \text{for all } t \in H^{-1/2}(\Gamma),$$

and

$$(Kz)(x) \;=\; \int\limits_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)z(y)ds_y \quad \text{for } x \in \Gamma$$

is the Laplace double layer potential $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ satisfying

$$\|(\frac{1}{2}I + K)z\|_{H^{1/2}(\Gamma)} \;\leq\; c_2^K \, \|z\|_{H^{1/2}(\Gamma)} \quad \text{for all } z \in H^{1/2}(\Gamma).$$

Moreover,

$$(N_0 f)(x) \;=\; \int\limits_\Omega U^*(x,y)f(y)dy \quad \text{for } x \in \Gamma$$

is the related Newton potential. Note that the single layer potential V is $H^{-1/2}(\Gamma)$–elliptic, see, e.g., [22], where for $n = 2$ we assume the scaling condition diam $\Omega < 1$ to ensure this:

$$\langle Vt, t\rangle_\Gamma \;\geq\; c_1^V \, \|t\|^2_{H^{-1/2}(\Gamma)} \quad \text{for all } t \in H^{-1/2}(\Gamma).$$

Note that in general we have the mapping properties

$$V : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \quad K : H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma),$$

where $|s| \leq \frac{1}{2}$ in the case of a Lipschitz boundary Γ, see, e.g., [3, 12, 17].

3.2 Adjoint boundary value problem

The solution of the adjoint Dirichlet boundary value problem (2.9),

$$-\Delta p(x) = u(x) - \bar{u}(x) \quad \text{for } x \in \Omega, \quad p(x) = 0 \quad \text{for } x \in \Gamma,$$

is given correspondingly by the representation formula for $\widetilde{x} \in \Omega$,

$$p(\widetilde{x}) \;=\; \int_\Gamma U^*(\widetilde{x}, y)\frac{\partial}{\partial n_y}p(y)ds_y + \int_\Omega U^*(\widetilde{x}, y)[u(y) - \bar{u}(y)]dy. \tag{3.3}$$

As in (3.2), we obtain a boundary integral equation

$$(Vq)(x) = (N_0\bar{u})(x) - (N_0 u)(x) \quad \text{for } x \in \Gamma \tag{3.4}$$

to determine the unknown Neumann datum $q = \frac{\partial}{\partial n}p \in H^{-1/2}(\Gamma)$.

Remark 3.1 *While the boundary integral equation* (3.2) *can be used to determine the unknown Neumann datum $t \in H^{-1/2}(\Gamma)$ of the primal Dirichlet boundary value problem* (2.2)*, the unknown Neumann datum $q \in H^{-1/2}(\Gamma)$ of the adjoint Dirichlet boundary value problem* (2.9) *is given as the solution of the boundary integral equation* (3.4)*. Then, the control $z \in H^{1/2}(\Gamma)$ is determined by the optimality condition* (2.10)*. However, since the solution u of the primal Dirichlet boundary value problem* (2.2) *enters the volume potential $N_0 u$ in the boundary integral equation* (3.4)*, we also need to include the representation formula* (3.1)*. Hence we have to solve a coupled system of boundary and domain integral equations. Instead, we will now describe a system of only boundary integral equations to solve the adjoint boundary value problem* (2.9)*.*

To end up with a system of boundary integral equations only, instead of (3.3), we will introduce a modified representation formula for the adjoint state p as follows. First we note that

$$V^*(x, y) \;=\; \begin{cases} -\dfrac{1}{8\pi}|x-y|^2\big(\log|x-y| - 1\big) & \text{for } n = 2, \\ \dfrac{1}{8\pi}\,|x-y| & \text{for } n = 3 \end{cases} \tag{3.5}$$

is a solution of the Poisson equation

$$\Delta_y V^*(x, y) \;=\; U^*(x, y) \quad \text{for } x \neq y, \tag{3.6}$$

i.e., $V^*(x, y)$ is the fundamental solution of the Bi–Laplacian. Hence we can rewrite the volume integral for u in (3.3), by using Green's second formula, as follows:

$$\begin{aligned}\int_\Omega U^*(\widetilde{x}, y)u(y)dy \;&=\; \int_\Omega [\Delta_y V^*(\widetilde{x}, y)]u(y)dy \\ &= \int_\Gamma \frac{\partial}{\partial n_y}V^*(\widetilde{x}, y)u(y)ds_y - \int_\Gamma V^*(\widetilde{x}, y)\frac{\partial}{\partial n_y}u(y)ds_y + \int_\Omega V^*(\widetilde{x}, y)[\Delta u(y)]dy \\ &= \int_\Gamma \frac{\partial}{\partial n_y}V^*(\widetilde{x}, y)z(y)ds_y - \int_\Gamma V^*(\widetilde{x}, y)t(y)ds_y - \int_\Omega V^*(\widetilde{x}, y)f(y)dy\,.\end{aligned}$$

Therefore, we now obtain from (3.3) the modified representation formula

$$\begin{aligned} p(\widetilde{x}) &= \int_\Gamma U^*(\widetilde{x},y)q(y)ds_y + \int_\Gamma \frac{\partial}{\partial n_y} V^*(\widetilde{x},y)z(y)ds_y - \int_\Gamma V^*(\widetilde{x},y)t(y)ds_y \\ &\quad - \int_\Omega U^*(\widetilde{x},y)\bar{u}(y)dy - \int_\Omega V^*(\widetilde{x},y)f(y)dy \end{aligned} \tag{3.7}$$

for $\widetilde{x} \in \Omega$, where the volume potentials involve given data only, and $q = \frac{\partial}{\partial n}p$ is the unknown Neumann datum which is related to the adjoint Dirichlet boundary value problem (2.9).

The representation formula (3.7) results, when taking the limit $\Omega \ni \widetilde{x} \to x \in \Gamma$, in the boundary integral equation

$$\begin{aligned} 0 = p(x) &= \int_\Gamma U^*(x,y)q(y)ds_y + \int_\Gamma \frac{\partial}{\partial n_y} V^*(x,y)z(y)ds_y - \int_\Gamma V^*(x,y)t(y)ds_y \\ &\quad - \int_\Omega U^*(x,y)\bar{u}(y)dy - \int_\Omega V^*(x,y)f(y)dy \end{aligned}$$

for almost all $x \in \Gamma$, which can be written as

$$(Vq)(x) = (V_1 t)(x) - (K_1 z)(x) + (N_0 \bar{u})(x) + (M_0 f)(x) \quad \text{for } x \in \Gamma. \tag{3.8}$$

Note that

$$(V_1 t)(x) = \int_\Gamma V^*(x,y)t(y)ds_y \quad \text{for } x \in \Gamma$$

is the Bi–Laplace single layer potential $V_1 : H^{-3/2}(\Gamma) \to H^{3/2}(\Gamma)$ satisfying, see, for example, [12, Theorem 5.7.3],

$$\|V_1 t\|_{H^{3/2}(\Gamma)} \le c_2^{V_1} \|t\|_{H^{-3/2}(\Gamma)} \quad \text{for all } t \in H^{-3/2}(\Gamma), \tag{3.9}$$

and

$$(K_1 z)(x) = \int_\Gamma \frac{\partial}{\partial n_y} V^*(x,y)z(y)ds_y \quad \text{for } x \in \Gamma$$

is the Bi–Laplace double layer potential $K_1 : H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma)$ satisfying

$$\|K_1 z\|_{H^{3/2}(\Gamma)} \le c_2^{K_1} \|z\|_{H^{-1/2}(\Gamma)} \quad \text{for all } z \in H^{-1/2}(\Gamma). \tag{3.10}$$

In addition, we have introduced a second Newton potential, which is related to the fundamental solution of the Bi–Laplace operator,

$$(M_0 f)(x) = \int_\Omega V^*(x,y)f(y)dy \quad \text{for } x \in \Gamma.$$

3.3 Optimality system

Now we are in a position to reformulate the primal Dirichlet boundary value problem (2.2), the adjoint Dirichlet boundary value problem (2.9), and the optimality condition (2.10) as a system of boundary integral equations for $x \in \Gamma$,

$$\begin{pmatrix} -V_1 & V & K_1 \\ V & & -\frac{1}{2}I - K \\ & -I & \varrho A \end{pmatrix} \begin{pmatrix} t \\ q \\ z \end{pmatrix} = \begin{pmatrix} N_0\overline{u} + M_0 f \\ -N_0 f \\ 0 \end{pmatrix}. \tag{3.11}$$

To investigate the unique solvability of (3.11), we first consider the associated Schur complement of (3.11). Since the Laplace single layer potential V is $H^{-1/2}(\Gamma)$–elliptic and therefore invertible, we first obtain

$$t = V^{-1}(\frac{1}{2}I + K)z - V^{-1}N_0 f \tag{3.12}$$

from the second equation in (3.11). Inserting this into the first equation of (3.11) gives

$$Vq = V_1V^{-1}(\frac{1}{2}I + K)z - K_1 z + N_0\overline{u} + M_0 f - V_1V^{-1}N_0 f,$$

and therefore

$$q = V^{-1}V_1V^{-1}(\frac{1}{2}I + K)z - V^{-1}K_1 z + V^{-1}N_0\overline{u} + V^{-1}M_0 f - V^{-1}V_1V^{-1}N_0 f\,. \tag{3.13}$$

Hence it remains to solve the Schur complement system

$$T_\varrho z \;=\; g, \tag{3.14}$$

where

$$T_\varrho \;:=\; V^{-1}K_1 - V^{-1}V_1V^{-1}(\frac{1}{2}I + K) + \varrho A \tag{3.15}$$

is the boundary integral representation of the operator T_ϱ as defined in (2.7), and

$$g \;:=\; V^{-1}N_0\overline{u} + V^{-1}M_0 f - V^{-1}V_1V^{-1}N_0 f \tag{3.16}$$

is the related right hand side.

To investigate the unique solvability of the Schur complement boundary integral equation (3.14), we first will recall some mapping properties of boundary integral operators which are related to the Bi–Laplace partial differential equation, see also [13].

4 Bi–Laplace boundary integral equations

In this section, we will consider a representation formula and related boundary integral equations for the Bi–Laplace equation

$$\Delta^2 u(x) \;=\; 0 \quad \text{for } x \in \Omega, \tag{4.1}$$

which can be written as a system,

$$\Delta w(x) \;=\; 0, \quad \Delta u(x) \;=\; w(x) \quad \text{for } x \in \Omega. \tag{4.2}$$

As for the Laplace equation we first find a representation formula for $\widetilde{x} \in \Omega$,

$$w(\widetilde{x}) \;=\; \int\limits_{\Gamma} U^*(\widetilde{x}, y) \frac{\partial}{\partial n_y} w(y) ds_y - \int\limits_{\Gamma} \frac{\partial}{\partial n_y} U^*(\widetilde{x}, y) w(y) ds_y, \tag{4.3}$$

which results in the boundary integral equation

$$w(x) \;=\; (V\tau)(x) + \frac{1}{2} w(x) - (Kw)(x) \quad \text{for } x \in \Gamma. \tag{4.4}$$

Note that

$$w = \Delta u \quad \text{and} \quad \tau = \frac{\partial}{\partial n} w = n \cdot \nabla w = n \cdot \nabla \Delta u$$

are the associated Cauchy data on Γ. When taking the normal derivative of the representation formula (4.3), we get a second, the so–called hypersingular boundary integral equation

$$\tau(x) \;=\; \frac{1}{2}\tau(x) + (K'\tau)(x) + (Dw)(x) \quad \text{for } x \in \Gamma, \tag{4.5}$$

where

$$(K'\tau)(x) \;=\; \int\limits_{\Gamma} \frac{\partial}{\partial n_x} U^*(x, y) \tau(y) ds_y \quad \text{for } x \in \Gamma$$

is the adjoint Laplace double layer potential $K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, and

$$(Dw)(x) \;=\; -\frac{\partial}{\partial n_x} \int\limits_{\Gamma} \frac{\partial}{\partial n_y} U^*(x, y) w(y) ds_y \quad \text{for } x \in \Gamma$$

is the related hypersingular boundary integral operator $D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$.

To obtain a representation formula for the solution u of the Bi–Laplace equation (4.1), we first consider the related Green's first formula

$$\int\limits_{\Omega} \Delta u(y) \Delta v(y) dy = \int\limits_{\Gamma} \frac{\partial}{\partial n_y} u(y) \Delta v(y) ds_y - \int\limits_{\Gamma} \frac{\partial}{\partial n_y} \Delta v(y) u(y) ds_y + \int\limits_{\Omega} [\Delta^2 v(y)] u(y) dy, \tag{4.6}$$

and in the sequel Green's second formula,

$$\begin{aligned} &\int\limits_{\Gamma} \frac{\partial}{\partial n_y} u(y) \Delta v(y) ds_y - \int\limits_{\Gamma} \frac{\partial}{\partial n_y} \Delta v(y) u(y) ds_y + \int\limits_{\Omega} [\Delta^2 v(y)] u(y) dy \\ &\qquad = \int\limits_{\Gamma} \frac{\partial}{\partial n_y} v(y) \Delta u(y) ds_y - \int\limits_{\Gamma} \frac{\partial}{\partial n_y} \Delta u(y) v(y) ds_y + \int\limits_{\Omega} [\Delta^2 u(y)] v(y) dy\,. \end{aligned}$$

When choosing $v(y) = V^*(\widetilde{x}, y)$ for $\widetilde{x} \in \Omega$, i.e., the fundamental solution (3.5) of the Bi–Laplace operator, the solution of the Bi–Laplace partial differential equation (4.1) is given by the representation formula for $\widetilde{x} \in \Omega$ by

$$\begin{aligned} u(\widetilde{x}) &= \int_\Gamma \frac{\partial}{\partial n_y} u(y) \Delta_y V^*(\widetilde{x}, y) ds_y - \int_\Gamma \frac{\partial}{\partial n_y} \Delta_y V^*(\widetilde{x}, y) u(y) ds_y \\ &\quad - \int_\Gamma \frac{\partial}{\partial n_y} V^*(\widetilde{x}, y) \Delta u(y) ds_y + \int_\Gamma \frac{\partial}{\partial n_y} \Delta u(y) V^*(\widetilde{x}, y) ds_y \,. \end{aligned}$$

By using (3.6), this can be written as

$$\begin{aligned} u(\widetilde{x}) &= \int_\Gamma U^*(\widetilde{x}, y) t(y) ds_y - \int_\Gamma \frac{\partial}{\partial n_y} U^*(\widetilde{x}, y) u(y) ds_y \\ &\quad - \int_\Gamma \frac{\partial}{\partial n_y} V^*(\widetilde{x}, y) w(y) ds_y + \int_\Gamma V^*(\widetilde{x}, y) \tau(y) ds_y \,. \end{aligned} \tag{4.7}$$

Hence we obtain the boundary integral equation

$$u(x) = (Vt)(x) + \frac{1}{2} u(x) - (Ku)(x) - (K_1 w)(x) + (V_1 \tau)(x) \tag{4.8}$$

for almost all $x \in \Gamma$. Moreover, when taking the normal derivative of the representation formula (4.7), this gives another boundary integral equation for $x \in \Gamma$,

$$t(x) = \frac{1}{2} t(x) + (K't)(x) + (Du)(x) + (D_1 w)(x) + (K_1' \tau)(x), \tag{4.9}$$

where

$$(K_1' \tau)(x) = \int_\Gamma \frac{\partial}{\partial n_x} V^*(x, y) \tau(y) ds_y \quad \text{for } x \in \Gamma$$

is the adjoint Bi–Laplace double layer potential $K_1' : H^{-3/2}(\Gamma) \to H^{1/2}(\Gamma)$, and

$$(D_1 w)(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} V^*(x, y) w(y) ds_y \quad \text{for } x \in \Gamma$$

is the Bi–Laplace hypersingular boundary integral operator $D_1 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$.

The boundary integral equations (4.4), (4.5), (4.8), and (4.9) can now be written as a system, including the so–called Calderon projection $\mathcal{C}$,

$$\begin{pmatrix} u \\ t \\ w \\ \tau \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V & -K_1 & V_1 \\ D & \frac{1}{2}I + K' & D_1 & K_1' \\ & & \frac{1}{2}I - K & V \\ & & D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} u \\ t \\ w \\ \tau \end{pmatrix} . \tag{4.10}$$

Lemma 4.1 *The Calderon projection* $\mathcal{C}$ *as defined in* (4.10) *is a projection, i.e.,* $\mathcal{C}^2 = \mathcal{C}$.

Proof. The proof follows as in the case of the Laplace equation [17, 22], for the Bi–Laplace equation see also [13]. ■

From the projection property as stated in Lemma 4.1 we obtain some well–known relations of all boundary integral operators which were introduced for both the Laplace and the Bi–Laplace equation.

Lemma 4.2 *For all boundary integral operators there hold the relations*

$$KV = VK', \quad DK = K'D, \quad VD = \frac{1}{4}I - K^2, \quad DV = \frac{1}{4}I - K'^2 \tag{4.11}$$

and

$$\begin{aligned} K_1V - VK_1' &= V_1K' - KV_1, &(4.12)\\ K_1'D - DK_1 &= D_1K - K'D_1, &(4.13)\\ VD_1 + V_1D + KK_1 + K_1K &= 0, &(4.14)\\ DV_1 + D_1V + K'K_1' + K_1'K' &= 0. &(4.15) \end{aligned}$$

Proof. The relations of (4.11) for the Laplace operator are well–known, see, e.g., [22], for the Bi–Laplace operator, see also [13]. ■

To prove the ellipticity of the Schur complement boundary integral operator T_ϱ as defined in (3.15), we need the following result:

Lemma 4.3 *For any* $t \in H^{-1/2}(\Gamma)$ *there holds the equality*

$$\|\widetilde{V}t\|^2_{L_2(\Omega)} = \langle K_1Vt, t\rangle_\Gamma - \langle V_1(\frac{1}{2}I + K')t, t\rangle_\Gamma \tag{4.16}$$

where

$$(\widetilde{V}t)(x) = \int_\Gamma U^*(x,y)t(y)ds_y \quad \textit{for } x \in \Omega\,.$$

Proof. For $x \in \Omega$ and $t \in H^{-1/2}(\Gamma)$, we define the Bi–Laplace single layer potential

$$u_t(x) = (\widetilde{V}_1t)(x) = \int_\Gamma V^*(x,y)t(y)ds_y$$

which is a solution of the Bi–Laplace differential equation (4.1). Then, the related Cauchy data are given by

$$u_t(x) = (V_1t)(x), \quad \frac{\partial}{\partial n_x}u_t(x) = (K_1't)(x) \quad \text{for } x \in \Gamma.$$

On the other hand, for $x \in \Omega$

$$w_t(x) = \Delta_x u_t(x) = \Delta_x \int\limits_\Gamma V^*(x,y)t(y)ds_y = \int\limits_\Gamma U^*(x,y)t(y)ds_y = (\widetilde{V}t)(x)$$

is a solution of the Laplace equation. Hence, the related Cauchy data are given by

$$w_t(x) \;=\; (Vt)(x), \quad \frac{\partial}{\partial n_x} w_t(x) = \frac{1}{2}t(x) + (K't)(x) \quad \text{for } x \in \Gamma.$$

Now, for $u = v = u_t$, Green's first formula (4.6) reads

$$\int\limits_\Omega [\Delta u_t(x)]^2 dx \;=\; \int\limits_\Gamma \frac{\partial}{\partial n_x} u_t(x) \Delta u_t(x) ds_x - \int\limits_\Gamma \frac{\partial}{\partial n_x} \Delta u_t(x) u_t(x) ds_x,$$

and therefore we conclude

$$\begin{aligned}
\int\limits_\Omega [w_t(x)]^2 dx \;&=\; \int\limits_\Gamma \frac{\partial}{\partial n_x} u_t(x) w_t(x) ds_x - \int\limits_\Gamma \frac{\partial}{\partial n_x} w_t(x) u_t(x) ds_x \\
&=\; \int\limits_\Gamma (K_1't)(x)(Vt)(x) ds_x - \int\limits_\Gamma [\frac{1}{2}t(x) + (K't)(x)](V_1t)(x) ds_x \\
&=\; \langle K_1't, Vt\rangle_\Gamma - \langle \frac{1}{2}t + K't, V_1t\rangle_\Gamma \\
&=\; \langle t, K_1Vt\rangle_\Gamma - \langle V_1(\frac{1}{2}I + K')t, t\rangle_\Gamma.
\end{aligned}$$

The assertion follows with $w_t = \widetilde{V}t$. ■

5 Non–symmetric boundary integral formulation

Now we able to prove the unique solvability of the Schur complement boundary integral equation (3.14), where the operator T_ϱ is defined by (3.15).

Theorem 5.1 *The composed boundary integral operator*

$$T_\varrho \;:=\; \varrho A + V^{-1}K_1 - V^{-1}V_1V^{-1}(\frac{1}{2}I + K) \;:\; H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is self–adjoint, bounded and $H^{1/2}(\Gamma)$–elliptic, i.e.,

$$\langle T_\varrho z, z\rangle_\Gamma \;\geq\; c_1^T \,\|z\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

Proof. The mapping properties of $T_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ follow from the boundedness of all used boundary integral operators [17, 20, 22]. In addition, we use the compact embedding of $H^{3/2}(\Gamma)$ in $H^{1/2}(\Gamma)$.

Next we will show the self–adjointness of T_ϱ. For $u, v \in H^{1/2}(\Gamma)$ we have

$$\begin{aligned}
\langle T_\varrho u, v\rangle_\Gamma &= \langle \varrho A u, v\rangle_\Gamma + \langle V^{-1}K_1 u, v\rangle_\Gamma - \frac{1}{2}\langle V^{-1}V_1V^{-1}u, v\rangle_\Gamma - \langle V^{-1}V_1V^{-1}Ku, v\rangle_\Gamma \\
&= \langle u, \varrho A v\rangle_\Gamma + \langle u, K_1'V^{-1}v\rangle_\Gamma - \frac{1}{2}\langle u, V^{-1}V_1V^{-1}v\rangle_\Gamma - \langle u, K'V^{-1}V_1V^{-1}v\rangle_\Gamma \\
&= \langle u, \varrho A v\rangle_\Gamma - \frac{1}{2}\langle u, V^{-1}V_1V^{-1}v\rangle_\Gamma + \langle u, [K_1'V^{-1} - K'V^{-1}V_1V^{-1}]v\rangle_\Gamma.
\end{aligned}$$

Now, we conclude by using the relations (4.11) and (4.12)

$$\begin{aligned}
K_1'V^{-1} - K'V^{-1}V_1V^{-1} &= K_1'V^{-1} - V^{-1}KV_1V^{-1} \;=\; V^{-1}[VK_1' - KV_1]V^{-1} \\
&= V^{-1}[K_1V - V_1K']V^{-1} \;=\; V^{-1}K_1 - V^{-1}V_1K'V^{-1} \\
&= V^{-1}K_1 - V^{-1}V_1V^{-1}K\,.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\langle T_\varrho u, v\rangle_\Gamma &= \langle u, \varrho A v\rangle_\Gamma - \frac{1}{2}\langle u, V^{-1}V_1V^{-1}v\rangle_\Gamma + \langle u, [V^{-1}K_1 - V^{-1}V_1V^{-1}K]v\rangle_\Gamma \\
&= \langle u, [\varrho A + V^{-1}K_1 - V^{-1}V_1V^{-1}(\frac{1}{2}I + K)]v\rangle_\Gamma \;=\; \langle u, T_\varrho v\rangle_\Gamma,
\end{aligned}$$

i.e., T_ϱ is self–adjoint.

Moreover, for $z \in H^{1/2}(\Gamma)$ we have, by using (4.11), $t = V^{-1}z$, and by Lemma 4.3,

$$\begin{aligned}
\langle T_\varrho z, z\rangle_\Gamma &= \varrho\langle Az, z\rangle_\Gamma + \langle V^{-1}K_1 z, z\rangle_\Gamma - \langle V^{-1}V_1V^{-1}(\frac{1}{2}I + K)z, z\rangle_\Gamma \\
&= \varrho\langle Az, z\rangle_\Gamma + \langle K_1VV^{-1}z, V^{-1}z\rangle_\Gamma - \langle V_1(\frac{1}{2}I + K')V^{-1}z, V^{-1}z\rangle_\Gamma \\
&= \varrho\langle Az, z\rangle_\Gamma + \langle K_1Vt, t\rangle_\Gamma - \langle V_1(\frac{1}{2}I + K')t, t\rangle_\Gamma \\
&= \varrho\langle Az, z\rangle_\Gamma + \|\widetilde{V}t\|^2_{L_2(\Omega)} \\
&\geq \varrho\,\|z\|_A^2,
\end{aligned}$$

i.e., the $H^{1/2}(\Gamma)$–ellipticity of T_ϱ, since $\|\cdot\|_A$ defines an equivalent norm in $H^{1/2}(\Gamma)$. ■

Due to Theorem 5.1, we conclude the unique solvability of the Schur complement boundary integral equation (3.14) by applying the Lax–Milgram lemma and therefore of the coupled system (3.11).

5.1 Galerkin boundary element discretization

For the Galerkin discretization of (3.14) based on the boundary integral representation (3.15), let

$$S_H^1(\Gamma) = \mathrm{span}\{\varphi_i\}_{i=1}^M \subset H^{1/2}(\Gamma)$$

be some boundary element space of, e.g., piecewise linear and continuous basis functions φ_i, which are defined with respect to a globally quasi–uniform and shape regular boundary element mesh of mesh size H. The Galerkin discretization of the Schur complement system (3.14) is to find $z_H \in S_H^1(\Gamma)$ such that

$$\langle T_\varrho z_H, v_H\rangle_\Gamma = \langle g, v_H\rangle_\Gamma \quad \text{for all } v_H \in S_H^1(\Gamma). \tag{5.1}$$

While the Galerkin variational formulation (5.1) admits a unique solution z_H due to Cea's lemma satisfying the error estimate

$$\|z - z_H\|_{H^{1/2}(\Gamma)} \le c \inf_{v_H \in S_H^1(\Gamma)} \|z - v_H\|_{H^{1/2}(\Gamma)}, \tag{5.2}$$

the composed boundary integral operator

$$T_\varrho := \varrho A + V^{-1}K_1 - V^{-1}V_1V^{-1}(\frac{1}{2}I + K)$$

does not allow a direct boundary element discretization in general. Instead, we may introduce an appropriate boundary element approximation $\widetilde{T}_\varrho$ as follows.

5.2 Boundary element approximation of T_ϱ

For an arbitrary but fixed $z \in H^{1/2}(\Gamma)$, the application of $T_\varrho z$ reads

$$T_\varrho z = \varrho Az + V^{-1}K_1 z - V^{-1}V_1V^{-1}(\frac{1}{2}I + K)z = \varrho Az + q_z,$$

where $q_z, t_z \in H^{-1/2}(\Gamma)$ are the unique solutions of the boundary integral equations

$$Vq_z = K_1 z - V_1 t_z, \quad Vt_z = (\frac{1}{2}I + K)z. \tag{5.3}$$

For a Galerkin approximation of (5.3), let

$$S_h^0(\Gamma) = \text{span}\{\psi_k\}_{k=1}^N \subset H^{-1/2}(\Gamma)$$

be another boundary element space of, e.g., piecewise constant basis functions ψ_k, which are defined with respect to a second globally quasi–uniform and shape regular boundary element mesh of mesh size h. Now, $t_{z,h} \in S_h^0(\Gamma)$ is the unique solution of the Galerkin formulation

$$\langle Vt_{z,h}, \tau_h\rangle_\Gamma = \langle(\frac{1}{2}I + K)z, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.4}$$

Moreover, $\widetilde{q}_{z,h} \in S_h^0(\Gamma)$ is the unique solution of the Galerkin formulation

$$\langle V\widetilde{q}_{z,h}, \tau_h\rangle_\Gamma = \langle K_1 z - V_1 t_{z,h}, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.5}$$

Hence we can define an approximation $\widetilde{T}_\varrho$ of the operator T_ϱ by

$$\widetilde{T}_\varrho z := \varrho Az + \widetilde{q}_{z,h}\,. \tag{5.6}$$

Lemma 5.2 *The approximate operator* $\widetilde{T}_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *as defined in* (5.6) *is bounded, i.e.,*

$$\|\widetilde{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \le c_2^{\widetilde{T}_\varrho} \|z\|_{H^{1/2}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

Proof. From the Galerkin formulation (5.4) we first find, by choosing $\tau_h = t_{z,h}$ and by using the $H^{-1/2}(\Gamma)$–ellipticity of the single layer potential,

$$\|t_{z,h}\|_{H^{-1/2}(\Gamma)} \le \frac{1}{c_1^V} \|(\frac{1}{2}I + K)z\|_{H^{1/2}(\Gamma)} \le \frac{c_2^K}{c_1^V} \|z\|_{H^{1/2}(\Gamma)}.$$

From (5.5), we now find for $\tau_h = \widetilde{q}_{z,h}$, by using $H^{3/2}(\Gamma) \subset H^{1/2}(\Gamma)$ and (3.9), (3.10),

$$\begin{aligned}
\|\widetilde{q}_{z,h}\|_{H^{-1/2}(\Gamma)} &\le \frac{1}{c_1^V} \|K_1 z - V_1 t_{z,h}\|_{H^{1/2}(\Gamma)} \\
&\le \frac{1}{c_1^V} \|K_1 z - V_1 t_{z,h}\|_{H^{3/2}(\Gamma)} \\
&\le \frac{1}{c_1^V} \left[c_2^{K_1} \|z\|_{H^{-1/2}(\Gamma)} + c_2^{V_1} \|t_{z,h}\|_{H^{-3/2}(\Gamma)} \right].
\end{aligned}$$

The assertion now follows from $H^{1/2}(\Gamma) \subset H^{-1/2}(\Gamma)$ and $H^{-1/2}(\Gamma) \subset H^{-3/2}(\Gamma)$. ■

Lemma 5.3 *Let* $T_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *be given by* (3.15), *and let* $\widetilde{T}_\varrho$ *be defined by* (5.6). *Then there holds the error estimate*

$$\|T_\varrho z - \widetilde{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \le \frac{c_2^V}{c_1^V} \inf_{\tau_h \in S_h^0(\Gamma)} \|q_z - \tau_h\|_{H^{-1/2}(\Gamma)} + \frac{c_2^{V_1}}{c_1^V} \|t_z - t_{z,h}\|_{H^{-3/2}(\Gamma)}, \tag{5.7}$$

where $q_z, t_z \in H^{-1/2}(\Gamma)$ *are defined as in* (5.3), *and* $t_{z,h} \in S_h^0(\Gamma)$ *is the unique solution of the Galerkin variational problem* (5.4).

Proof. For an arbitrary chosen but fixed $z \in H^{1/2}(\Gamma)$ we have, by definition,

$$T_\varrho z = \varrho A z + q_z, \quad q_z = V^{-1}[K_1 z - V_1 t_z], \quad t_z = V^{-1}(\frac{1}{2}I + K)z.$$

In particular, $t_z \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle V t_z, \tau \rangle_\Gamma = \langle (\frac{1}{2}I + K)z, \tau \rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma),$$

and $q_z \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle V q_z, \tau \rangle_\Gamma = \langle K_1 z, \tau \rangle_\Gamma - \langle V_1 t_z, \tau \rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma).$$

By using definition (5.6), we also have

$$\widetilde{T}_\varrho z = \varrho A z + \widetilde{q}_{z,h},$$

where $\widetilde{q}_{z,h}$ is the unique solution of the Galerkin variational problem

$$\langle V\widetilde{q}_{z,h}, \tau_h\rangle_\Gamma \;=\; \langle K_1 z, \tau_h\rangle_\Gamma - \langle V_1 t_{z,h}, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma),$$

and $t_{z,h} \in S_h^0(\Gamma)$ is the unique solution of the variational problem

$$\langle V t_{z,h}, \tau_h\rangle_\Gamma \;=\; \langle (\frac{1}{2}I + K)z, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma).$$

By applying Cea's lemma, we first conclude the error estimate

$$\|t_z - t_{z,h}\|_{H^{-1/2}(\Gamma)} \;\leq\; \frac{c_2^V}{c_1^V} \inf_{\tau_h \in S_h^0(\Gamma)} \|t_z - \tau_h\|_{H^{-1/2}(\Gamma)}\,.$$

Let us further define $q_{z,h} \in S_h^0(\Gamma)$ as the unique solution of the variational problem

$$\langle V q_{z,h}, \tau_h\rangle_\Gamma \;=\; \langle K_1 z, \tau_h\rangle_\Gamma - \langle V_1 t_z, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.8}$$

Again, by using Cea's lemma we have

$$\|q_z - q_{z,h}\|_{H^{-1/2}(\Gamma)} \;\leq\; \frac{c_2^V}{c_1^V} \inf_{\tau_h \in S_h^0(\Gamma)} \|q_z - \tau_h\|_{H^{-1/2}(\Gamma)}.$$

By subtracting (5.5) from (5.8) we obtain the perturbed Galerkin orthogonality

$$\langle V(q_{z,h} - \widetilde{q}_{z,h}), \tau_h\rangle_\Gamma \;=\; \langle V_1(t_{z,h} - t_z), \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma),$$

from which we further conclude the estimate

$$\begin{aligned}\|q_{z,h} - \widetilde{q}_{z,h}\|_{H^{-1/2}(\Gamma)} \;&\leq\; \frac{1}{c_1^V}\|V_1(t_z - t_{z,h})\|_{H^{1/2}(\Gamma)} \\ &\leq\; \frac{1}{c_1^V}\|V_1(t_z - t_{z,h})\|_{H^{3/2}(\Gamma)} \;\leq\; \frac{c_2^{V_1}}{c_1^V}\,\|t_z - t_{z,h}\|_{H^{-3/2}(\Gamma)}.\end{aligned}$$

Hence we find, by applying the triangle inequality,

$$\|q_z - \widetilde{q}_{z,h}\|_{H^{-1/2}(\Gamma)} \;\leq\; \frac{c_2^V}{c_1^V} \inf_{\tau_h \in S_h^0(\Gamma)} \|q_z - \tau_h\|_{H^{-1/2}(\Gamma)} + \frac{c_2^{V_1}}{c_1^V}\,\|t_z - t_{z,h}\|_{H^{-3/2}(\Gamma)},$$

and the assertion follows from $T_\varrho z - \widetilde{T}_\varrho z = q_z - \widetilde{q}_{z,h}$. ■

By using the approximation property of the trial space $S_h^0(\Gamma)$ and the Aubin–Nitsche trick, we conclude an error estimate from (5.7) when assuming some regularity of q_z and t_z, respectively.

Corollary 5.4 *Assume $q_z, t_z \in H^s_{\rm pw}(\Gamma)$ for some $s \in [0,1]$. Then there holds the error estimate*

$$\|T_\varrho z - \widetilde{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \;\leq\; c_1\, h^{s+\frac{1}{2}}\, \|q_z\|_{H^s_{\rm pw}(\Gamma)} + c_2\, h^{s+\frac{3}{2}}\, \|t_z\|_{H^s_{\rm pw}(\Gamma)}. \tag{5.9}$$

5.3 Boundary element approximation of g

As in (5.6), we may also define a boundary element approximation of the right hand side g as defined in (3.16)

$$g = V^{-1}N_0\bar{u} + V^{-1}M_0 f - V^{-1}V_1V^{-1}N_0 f.$$

In particular, $g \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle Vg, \tau\rangle_\Gamma \;=\; \langle N_0\bar{u} + M_0 f - V_1V^{-1}N_0 f, \tau\rangle_\Gamma \;=\; \langle N_0\bar{u} + M_0 f, \tau\rangle_\Gamma - \langle V_1 t_f, \tau\rangle_\Gamma$$

for all $\tau \in H^{-1/2}(\Gamma)$, where $t_f = V^{-1}N_0 f \in H^{-1/2}(\Gamma)$ solves the variational problem

$$\langle Vt_f, \tau\rangle_\Gamma \;=\; \langle N_0 f, \tau\rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma).$$

Hence we can define a boundary element approximation $\widetilde{g}_h \in S_h^0(\Gamma)$ as the unique solution of the Galerkin variational problem

$$\langle V\widetilde{g}_h, \tau_h\rangle_\Gamma \;=\; \langle N_0\bar{u} + M_0 f, \tau_h\rangle_\Gamma - \langle V_1 t_{f,h}, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma), \tag{5.10}$$

where $t_{f,h} \in S_h^0(\Gamma)$ is the unique solution the Galerkin problem

$$\langle Vt_{f,h}, \tau_h\rangle_\Gamma \;=\; \langle N_0 f, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.11}$$

Lemma 5.5 *Let g be the right hand side as defined by* (3.16)*, and let $\widetilde{g}_h$ be the boundary element approximation as defined in* (5.10)*. Then there holds the error estimate*

$$\|g - \widetilde{g}_h\|_{H^{-1/2}(\Gamma)} \le \frac{c_2^V}{c_1^V} \inf_{\tau_h \in S_h^0(\Gamma)} \|g - \tau_h\|_{H^{-1/2}(\Gamma)} + \frac{c_2^{V_1}}{c_1^V} \|t_f - t_{f,h}\|_{H^{-3/2}(\Gamma)}. \tag{5.12}$$

Proof. In addition to (5.10), let us consider the Galerkin formulation to find $g_h \in S_h^0(\Gamma)$ such that

$$\langle Vg_h, \tau_h\rangle_\Gamma \;=\; \langle N_0\bar{u} + M_0 f, \tau_h\rangle_\Gamma - \langle V_1 t_f, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.13}$$

Again, by using Cea's lemma, we obtain

$$\|g - g_h\|_{H^{-1/2}(\Gamma)} \le \frac{c_2^V}{c_1^V} \inf_{\tau_h \in S_h^0(\Gamma)} \|g - \tau_h\|_{H^{-1/2}(\Gamma)}.$$

Subtracting (5.10) from (5.13) gives the perturbed Galerkin orthogonality

$$\langle V(g_h - \widetilde{g}_h), \tau_h\rangle_\Gamma \;=\; -\langle V_1(t_f - t_{f,h}), \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma).$$

For $\tau_h = g_h - \widetilde{g}_h$ and by using the $H^{-1/2}(\Gamma)$–ellipticity of the single layer potential V and the estimate (3.9), we further obtain

$$\|g_h - \widetilde{g}_h\|_{H^{-1/2}(\Gamma)} \le \frac{1}{c_1^V} \|V_1(t_f - t_{f,h})\|_{H^{1/2}(\Gamma)} \le \frac{c_2^{V_1}}{c_1^V} \|t_f - t_{f,h}\|_{H^{-3/2}(\Gamma)}.$$

The assertion finally follows from the triangle inequality. ■

By using the approximation property of the trial space $S_h^0(\Gamma)$ and the Aubin–Nitsche trick, we conclude an error estimate from (5.12) when assuming some regularity of g and t_f, respectively.

Corollary 5.6 *Assume $g, t_f \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [0,1]$. Then there holds the error estimate*

$$\|g - \widetilde{g}_h\|_{H^{-1/2}(\Gamma)} \,\le\, c_1\, h^{s+\frac{1}{2}}\, \|g\|_{H^s_{\text{pw}}(\Gamma)} + c_2\, h^{s+\frac{3}{2}}\, \|t_f\|_{H^s_{\text{pw}}(\Gamma)}. \tag{5.14}$$

5.4 Perturbed Galerkin variational problem

Instead of the Galerkin variational problem (5.1), we now consider a perturbed Galerkin formulation to find $\widetilde{z}_H \in S^1_H(\Gamma)$ such that

$$\langle \widetilde{T}_\varrho \widetilde{z}_H, v_H\rangle_\Gamma \,=\, \langle \widetilde{g}_h, v_H\rangle_\Gamma \quad \text{for all } v_H \in S^1_H(\Gamma). \tag{5.15}$$

By combining the boundary element approximations (5.4) and (5.5) with (5.10) and (5.11), it is sufficient to consider the Galerkin boundary element formulation of (3.11): Find $(t_h, q_h, \widetilde{z}_H) \in S^0_h(\Gamma) \times S^0_h(\Gamma) \times S^1_H(\Gamma)$ such that

$$\begin{aligned}
-\langle V_1 t_h, w_h\rangle_\Gamma + \langle V q_h, w_h\rangle_\Gamma + \langle K_1 \widetilde{z}_H, w_h\rangle_\Gamma &= \langle N_0 \bar{u} + M_0 f, w_h\rangle_\Gamma, &(5.16)\\
\langle V t_h, \tau_h\rangle_\Gamma - \langle (\frac{1}{2} I + K)\widetilde{z}_H, \tau_h\rangle_\Gamma &= -\langle N_0 f, \tau_h\rangle_\Gamma, &(5.17)\\
-\langle q_h, v_H\rangle_\Gamma + \varrho\langle A\widetilde{z}_H, v_H\rangle_\Gamma &= 0 &(5.18)
\end{aligned}$$

is satisfied for all $(w_h, \tau_h, v_H) \in S^0_h(\Gamma) \times S^0_h(\Gamma) \times S^1_H(\Gamma)$. The Galerkin formulation (5.16)–(5.18) is equivalent to a system of linear equations

$$\begin{pmatrix} -V_{1,h} & V_h & K_{1,h} \\ V_h & & -(\frac{1}{2} M_h + K_h) \\ & -M_h^\top & \varrho A_H \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{q} \\ \underline{\widetilde{z}} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \underline{0} \end{pmatrix}, \tag{5.19}$$

where

$$\begin{aligned}
V_h[\ell,k] &= \langle V\psi_k, \psi_\ell\rangle_\Gamma, & K_h[\ell,i] &= \langle K\varphi_i, \psi_\ell\rangle_\Gamma,\\
V_{1,h}[\ell,k] &= \langle V_1\psi_k, \psi_\ell\rangle_\Gamma, & K_{1,h}[\ell,i] &= \langle K_1\varphi_i, \psi_\ell\rangle_\Gamma,\\
A_H[j,i] &= \langle A\varphi_i, \varphi_j\rangle_\Gamma, & M_h[\ell,i] &= \langle \varphi_i, \psi_\ell\rangle_\Gamma,
\end{aligned}$$

and

$$f_{1,\ell} = \langle N_0 \bar{u} + M_0 f, \psi_\ell\rangle_\Gamma, \quad f_{2,\ell} = -\langle N_0 f, \psi_\ell\rangle_\Gamma$$

for $k, \ell = 1, \ldots, N$ and $i, j = 1, \ldots, M$.

Since the Laplace single layer potential V is $H^{-1/2}(\Gamma)$–elliptic, the related Galerkin matrix V_h is positive definite and therefore invertible. Hence, we can resolve the second equation in (5.19) to obtain

$$\underline{t} \,=\, V_h^{-1}(\frac{1}{2} M_h + K_h)\underline{\widetilde{z}} + V_h^{-1}\underline{f}_2.$$

Inserting this into the first equation of (5.19) gives

$$V_h \underline{q} = V_{1,h} V_h^{-1}(\frac{1}{2} M_h + K_h)\underline{\widetilde{z}} + V_{1,h} V_h^{-1}\underline{f}_2 - K_{1,h}\underline{\widetilde{z}} + \underline{f}_1$$

and therefore

$$\underline{q} = V_h^{-1} V_{1,h} V_h^{-1} (\frac{1}{2} M_h + K_h) \underline{\widetilde{z}} + V_h^{-1} V_{1,h} V_h^{-1} \underline{f}_2 - V_h^{-1} K_{1,h} \underline{z} + V_h^{-1} \underline{f}_1 .$$

Hence it remains to solve the Schur complement system

$$\left[\varrho A_H - M_h^\top V_h^{-1} V_{1,h} V_h^{-1} (\frac{1}{2} M_h + K_h) + M_h^\top V_h^{-1} K_{1,h} \right] \underline{\widetilde{z}} = M_h^\top V_h^{-1} \left[\underline{f}_1 + V_{1,h} V_h^{-1} \underline{f}_2 \right] \tag{5.20}$$

where

$$\widetilde{T}_{\varrho,H} = \varrho A_H - M_h^\top V_h^{-1} V_{1,h} V_h^{-1} (\frac{1}{2} M_h + K_h) + M_h^\top V_h^{-1} K_{1,h} \tag{5.21}$$

defines a non–symmetric Galerkin boundary element approximation of the self–adjoint Schur complement boundary integral operator T_ϱ.

Theorem 5.7 *The approximate Schur complement $\widetilde{T}_{\varrho,H}$ as defined in* (5.21) *is positive definite, i.e.,*

$$(\widetilde{T}_{\varrho,H} \underline{z}, \underline{z}) \;\geq\; \frac{1}{2} c_1^T \, \|z_H\|^2_{H^{1/2}(\Gamma)}$$

for all $\underline{z} \in \mathbb{R}^M \leftrightarrow z_H \in S_H^1(\Gamma)$, if $h \leq c_0 H$ is sufficiently small.

Proof. For an arbitrary chosen but fixed $\underline{z} \in \mathbb{R}^M$ let $z_H \in S_H^1(\Gamma)$ be the associated boundary element function. Then we have

$$\begin{aligned} (\widetilde{T}_{\varrho,H} \underline{z}, \underline{z}) &= \langle \widetilde{T}_\varrho z_H, z_H \rangle_\Gamma \\ &= \langle T_\varrho z_H, z_H \rangle_\Gamma - \langle (T_\varrho - \widetilde{T}_\varrho) z_H, z_H \rangle_\Gamma \\ &\geq c_1^T \, \|z_H\|^2_{H^{1/2}(\Gamma)} - \|(T_\varrho - \widetilde{T}_\varrho) z_H\|_{H^{-1/2}(\Gamma)} \|z_H\|_{H^{1/2}(\Gamma)} . \end{aligned}$$

Since $z_H \in S_H^1(\Gamma)$ is a continuous function, we have $z_H \in H^1(\Gamma)$. Hence we find

$$t_{z_H} = V^{-1} (\frac{1}{2} I + K) z_H \in L_2(\Gamma),$$

and

$$q_{z_H} = V^{-1} [K_1 z_H - V_1 t_{z_H}] \in L_2(\Gamma)$$

according to (5.4) and (5.5). Therefore we can apply the error estimate (5.9) for $s = 0$ to obtain

$$\|T_\varrho z_H - \widetilde{T}_\varrho z_H\|_{H^{-1/2}(\Gamma)} \;\leq\; c_1 \, h^{1/2} \, \|q_{z_H}\|_{L_2(\Gamma)} + c_2 \, h^{3/2} \, \|t_{z_H}\|_{L_2(\Gamma)} \;\leq\; c_3 \, h^{1/2} \, \|z_H\|_{H^1(\Gamma)} .$$

Now, by applying the inverse inequality for $S_H^1(\Gamma)$,

$$\|z_H\|_{H^1(\Gamma)} \;\leq\; c_I \, H^{-1/2} \, \|z_H\|_{H^{1/2}(\Gamma)} ,$$

we obtain

$$\|T_\varrho z_H - \widetilde{T}_\varrho z_H\|_{H^{-1/2}(\Gamma)} \le c_3 c_I \left(\frac{h}{H}\right)^{1/2} \|z_H\|_{H^{1/2}(\Gamma)}.$$

Hence we finally obtain

$$(\widetilde{T}_{\varrho,H}\underline{z},\underline{z}) \ \ge \ \left[c_1^T - c_3 c_I \left(\frac{h}{H}\right)^{1/2}\right] \|z_H\|^2_{H^{1/2}(\Gamma)} \ge \frac{1}{2} c_1^T \|z_H\|^2_{H^{1/2}(\Gamma)},$$

if

$$c_3 c_I \left(\frac{h}{H}\right)^{1/2} \le \frac{1}{2} c_1^T$$

is satisfied. ■

Note that Theorem 5.7 ensures the unique solvability of the linear system (5.20) and therefore of the perturbed variational problem (5.15). Since the approximate operator $\widetilde{T}_\varrho$ is $S_H^1(\Gamma)$–elliptic, an error estimate for the approximate solution $\widetilde{z}_H$ of the perturbed Galerkin variational problem (5.15) follows from the Strang lemma, see, e.g., [22, Theorem 8.2, Theorem 8.3], which reads for our problem as:

Theorem 5.8 *Let z be the unique solution of the operator equation* (3.14)*. Let $h \le c_0 H$ be satisfied such that the approximate Schur complement $\widetilde{T}_{\varrho,H}$ as defined in* (5.21) *is positive definite, and let $\widetilde{z}_H \in S_H^1(\Gamma)$ be the unique solution of the perturbed Galerkin variational formulation* (5.15)*. Then there holds the error estimate*

$$\|z - \widetilde{z}_H\|_{H^{1/2}(\Gamma)} \le c_1 \inf_{v_H \in S_H^1(\Gamma)} \|z - v_H\|_{H^{1/2}(\Gamma)} + c_2 \|(T_\varrho - \widetilde{T}_\varrho)z\|_{H^{-1/2}(\Gamma)} + c_3 \|g - \widetilde{g}_h\|_{H^{-1/2}(\Gamma)}. \tag{5.22}$$

Corollary 5.9 *When combining the error estimate* (5.22) *with the approximation property of the ansatz space $S_H^1(\Gamma)$, and with the error estimates* (5.9) *and* (5.14)*, we finally obtain the error estimate*

$$\begin{aligned}\|z - \widetilde{z}_H\|_{H^{1/2}(\Gamma)} \ \le \ & c_1 H^{3/2} |z|_{H^2(\Gamma)} + c_2 h^{3/2} \|q_z\|_{H^1_{pw}(\Gamma)} + c_3 h^{5/2} \|t_z\|_{H^1_{pw}(\Gamma)} \\ & + c_4 h^{3/2} \|g\|_{H^1_{pw}(\Gamma)} + c_5 h^{5/2} \|t_f\|_{H^1_{pw}(\Gamma)}\end{aligned}$$

when assuming $z \in H^2(\Gamma)$, and $q_z, t_z, g, t_f \in H^1_{pw}(\Gamma)$, respectively. For $h \le c_0 H$ we can expect the convergence rate 1.5 *when measuring the error in the energy norm*

$$\|z - \widetilde{z}_H\|_{H^{1/2}(\Gamma)} \le c(z, \overline{u}, f)\, H^{3/2}. \tag{5.23}$$

Moreover, we are also able to derive an error estimate in $L_2(\Gamma)$, i.e.,

$$\|z - \widetilde{z}_H\|_{L_2(\Gamma)} \le c(z, \overline{u}, f)\, H^2. \tag{5.24}$$

when applying the Aubin–Nitsche trick [22].

Remark 5.1 *The error estimates* (5.23) *and* (5.24) *provide optimal convergence rates when approximating the control z by using piecewise linear basis functions. However, we have to assume $h \leq c_0 H$ to ensure the unique solvability of the perturbed Galerkin formulation* (5.15)*, where the constant c_0 is in general unknown. Moreover, the matrix $\widetilde{T}_{\varrho,H}$ as given in* (5.21) *defines a non–symmetric approximation of the exact symmetric stiffness matrix $T_{\varrho,H}$ as used in* (5.1)*. Hence we are interested in deriving a symmetric boundary element method which is stable without any additional constraints in the choice of the boundary element trial spaces.*

6 Symmetric boundary integral formulation

The boundary integral formulation of the primal boundary value problem (2.2) is given by (3.2), while the adjoint boundary value problem (2.9) corresponds to the modified boundary integral equation (3.8). In what follows, we will rewrite the optimality condition (2.10) by using a hypersingular boundary integral equation for the adjoint problem to obtain a symmetric boundary integral formulation for the coupled problem.

Since the adjoint variable p, as defined in the representation formula (3.7), is a solution of the adjoint Dirichlet boundary value problem (2.9), the normal derivative

$$q(x) \;=\; \lim_{\Omega \ni \widetilde{x} \to x \in \Gamma} n_x \cdot \nabla_{\widetilde{x}} p(\widetilde{x}) \;=\; \frac{\partial}{\partial n} p(x) \quad \text{for } x \in \Gamma$$

is well defined. When computing the normal derivative of the representation formula (3.7), this gives a second boundary integral equation for $x \in \Gamma$

$$q(x) \;=\; (\frac{1}{2}I + K')q(x) - (D_1 z)(x) - (K_1' t)(x) - (N_1 \bar{u})(x) - (M_1 f)(x), \tag{6.1}$$

where we introduce Newton potentials for $x \in \Gamma$

$$(N_1 \bar{u})(x) \;=\; \lim_{\Omega \ni \widetilde{x} \to x \in \Gamma} n_x \cdot \nabla_{\widetilde{x}} \int_\Omega U^*(\widetilde{x}, y) \bar{u}(y) dy \quad \text{for } x \in \Gamma$$

and

$$(M_1 f)(x) \;=\; \lim_{\Omega \ni \widetilde{x} \to x \in \Gamma} n_x \cdot \nabla_{\widetilde{x}} \int_\Omega V^*(\widetilde{x}, y) f(y) dy \quad \text{for } x \in \Gamma$$

in addition to the boundary integral operators used in (4.10). Combining the optimality condition (2.10) and the boundary integral equation (6.1) gives a boundary integral equation for $x \in \Gamma$,

$$\varrho (Az)(x) \;=\; (\frac{1}{2}I + K')q(x) - (D_1 z)(x) - (K_1' t)(x) - (N_1 \bar{u})(x) - (M_1 f)(x). \tag{6.2}$$

Now, to find the yet unknown triple $(z,t,q) \in H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ we solve the system of boundary integral equations (3.2), (3.8), and (6.2) which can be written as

$$\begin{pmatrix} -V_1 & V & K_1 \\ V & & -\frac{1}{2}I - K \\ K_1' & -\frac{1}{2}I - K' & \varrho A + D_1 \end{pmatrix} \begin{pmatrix} t \\ q \\ z \end{pmatrix} = \begin{pmatrix} N_0\bar{u} + M_0 f \\ -N_0 f \\ -N_1\bar{u} - M_1 f \end{pmatrix}. \tag{6.3}$$

To investigate the unique solvability of (6.3), we consider the related Schur complement. As in (3.12) and (3.13) we obtain

$$t \;=\; V^{-1}(\frac{1}{2}I + K)z - V^{-1}N_0 f$$

and

$$q = V^{-1}V_1V^{-1}(\frac{1}{2}I+K)z - V^{-1}K_1 z + V^{-1}N_0\bar{u} + V^{-1}M_0 f - V^{-1}V_1V^{-1}N_0 f.$$

Hence it remains to solve the Schur complement system

$$\begin{aligned}\Big[\varrho A + D_1 + K_1'V^{-1}(\frac{1}{2}I+K) + (\frac{1}{2}I+K')V^{-1}K_1 - (\frac{1}{2}I+K')V^{-1}V_1V^{-1}(\frac{1}{2}I+K)\Big]\, z \\ = K_1'V^{-1}N_0 f - N_1\bar{u} - M_1 f + (\frac{1}{2}I+K')V^{-1}\left[N_0\bar{u} + M_0 f - V_1V^{-1}N_0 f\right]. \end{aligned} \tag{6.4}$$

Note that (6.4) corresponds to a symmetric boundary integral formulation of the operator equation (2.7) representing the optimality condition.

Theorem 6.1 *The composed boundary integral operator*

$$T_\varrho = \varrho A + D_1 - (\frac{1}{2}I+K')V^{-1}V_1V^{-1}(\frac{1}{2}I+K) + K_1'V^{-1}(\frac{1}{2}I+K) + (\frac{1}{2}I+K')V^{-1}K_1 \tag{6.5}$$

is self–adjoint, bounded, i.e., $T_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, *and* $H^{1/2}(\Gamma)$*–elliptic.*

Proof. While the self–adjointness of T_ϱ in the symmetric representation (6.5) is obvious, the boundedness and ellipticity estimates follow as in the proof of Theorem 5.1. In particular, the Schur complement operators T_ϱ in the symmetric representation (6.5) and in the non–symmetric representation (3.15) coincide. Indeed, by using (4.11) and (4.12) we obtain

$$\begin{aligned} T_\varrho &= \varrho A + D_1 - (\frac{1}{2}I+K')V^{-1}V_1V^{-1}(\frac{1}{2}I+K) + K_1'V^{-1}(\frac{1}{2}I+K) + (\frac{1}{2}I+K')V^{-1}K_1 \\ &= \varrho A + D_1 + \Big[K_1' - (\frac{1}{2}I+K')V^{-1}V_1\Big]V^{-1}(\frac{1}{2}I+K) + (\frac{1}{2}I+K')V^{-1}K_1 \\ &= \varrho A + D_1 + V^{-1}\Big[VK_1' - KV_1 - \frac{1}{2}V_1\Big]V^{-1}(\frac{1}{2}I+K) + (\frac{1}{2}I+K')V^{-1}K_1 \\ &= \varrho A + D_1 + V^{-1}\Big[K_1V - V_1K' - \frac{1}{2}V_1\Big]V^{-1}(\frac{1}{2}I+K) + (\frac{1}{2}I+K')V^{-1}K_1 \\ &= \varrho A + D_1 + V^{-1}K_1(\frac{1}{2}I+K) - V^{-1}V_1(\frac{1}{2}I+K')V^{-1}(\frac{1}{2}I+K) + (\frac{1}{2}I+K')V^{-1}K_1. \end{aligned}$$

Due to the representation of the Laplace Steklov–Poincaré operator, see, e.g., [22],

$$S = V^{-1}(\frac{1}{2}I + K) = D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K),$$

we further conclude

$$(\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K) = V^{-1}(\frac{1}{2}I + K) - D.$$

Therefore, by using (4.11) and (4.14) we have

$$\begin{aligned} T_\varrho &= \varrho A + D_1 + V^{-1}K_1(\frac{1}{2}I + K) - V^{-1}V_1 \left[V^{-1}(\frac{1}{2}I + K) - D \right] + V^{-1}(\frac{1}{2}I + K)K_1 \\ &= \varrho A + V^{-1} \left[VD_1 + V_1 D + K_1(\frac{1}{2}I + K) - V_1 V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K)K_1 \right] \\ &= \varrho A + V^{-1} \left[-KK_1 - K_1 K + K_1(\frac{1}{2}I + K) - V_1 V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K)K_1 \right] \\ &= \varrho A + V^{-1} \left[K_1 - V_1 V^{-1}(\frac{1}{2}I + K) \right], \end{aligned}$$

and we finally obtain the non–symmetric representation (3.15). Therefore, the ellipticity of T_ϱ follows as in Theorem 5.1. ■

Due to the $H^{1/2}(\Gamma)$–ellipticity of the symmetric representation (6.5) of T_ϱ, we can conclude the unique solvability of the Schur complement boundary integral equation (6.4), and therefore of the coupled system (6.3).

6.1 Galerkin boundary element discretization

In what follows, we will consider a boundary element discretization of the boundary integral equation system (6.3). Again, let

$$S_h^0(\Gamma) = \text{span}\{\psi_k\}_{k=1}^N \subset H^{-1/2}(\Gamma), \quad S_h^1(\Gamma) = \text{span}\{\varphi_i\}_{i=1}^M \subset H^{1/2}(\Gamma)$$

be some boundary element spaces of piecewise constant and piecewise linear basis functions ψ_k and φ_i, which are defined with respect to some admissible boundary element mesh of mesh size h. The Galerkin boundary element formulation of (6.3) then reads to find $(t_h, q_h, \widehat{z}_h) \in S_h^0(\Gamma) \times S_h^0(\Gamma) \times S_h^1(\Gamma)$ such that

$$-\langle V_1 t_h, w_h\rangle_\Gamma + \langle V q_h, w_h\rangle_\Gamma + \langle K_1 \widehat{z}_h, w_h\rangle_\Gamma = \langle N_0 \bar{u} + M_0 f, w_h\rangle_\Gamma, \quad (6.6)$$

$$\langle V t_h, \tau_h\rangle_\Gamma - \langle(\frac{1}{2}I + K)\widehat{z}_h, \tau_h\rangle_\Gamma = -\langle N_0 f, \tau_h\rangle_\Gamma, \quad (6.7)$$

$$\langle K_1' t_h, v_h\rangle_\Gamma - \langle(\frac{1}{2}I + K')q_h, v_h\rangle_\Gamma + \langle(\varrho A + D_1)\widehat{z}_h, v_h\rangle_\Gamma = -\langle N_1 \bar{u} + M_1 f, v_h\rangle_\Gamma \quad (6.8)$$

is satisfied for all $(w_h, \tau_h, v_h) \in S_h^0(\Gamma) \times S_h^0(\Gamma) \times S_h^1(\Gamma)$. The Galerkin formulation (6.6)–(6.8) is equivalent to a system of linear equations,

$$\begin{pmatrix} -V_{1,h} & V_h & K_{1,h} \\ V_h & & -(\frac{1}{2}M_h + K_h) \\ K_{1,h}^\top & -(\frac{1}{2}M_h^\top + K_h^\top) & \varrho A_h + D_{1,h} \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{q} \\ \underline{\widehat{z}} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \underline{f}_3 \end{pmatrix}, \tag{6.9}$$

where we used, in addition to those entries of the linear system (5.19),

$$D_{1,h}[j,i] = \langle D_1 \varphi_i, \varphi_j \rangle_\Gamma, \quad f_{3,j} = -\langle N_1 \bar{u} + M_1 f, \varphi_j \rangle_\Gamma \quad \text{for } i,j = 1, \ldots, M.$$

To investigate the unique solvability of the linear system (6.9), we consider the invertibility of the related Schur complement. In particular, the second equation in (6.9) gives

$$\underline{t} = V_h^{-1}(\frac{1}{2}M_h + K_h)\underline{\widehat{z}} + V_h^{-1}\underline{f}_2\,,$$

and we obtain from the first equation

$$\underline{q} = V_h^{-1}[V_{1,h}V_h^{-1}(\frac{1}{2}M_h + K_h) - K_{1,h}]\underline{\widehat{z}} + V_h^{-1}\underline{f}_1 + V_h^{-1}V_{1,h}V_h^{-1}\underline{f}_2\,.$$

Hence, by inserting these results into the third equation of (6.9), we finally end up with the Schur complement system of the symmetric boundary integral formulation

$$\widehat{T}_{\varrho,h}\underline{\widehat{z}} = \underline{f}, \tag{6.10}$$

where the Schur complement is given by

$$\begin{aligned} \widehat{T}_{\varrho,h} \;=\; & \varrho A_h + D_{1,h} + K_{1,h}^\top V_h^{-1}(\frac{1}{2}M_h + K_h) + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}K_{1,h} \\ & -(\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}V_{1,h}V_h^{-1}(\frac{1}{2}M_h + K_h), \end{aligned} \tag{6.11}$$

and the right hand side is

$$\underline{f} = \underline{f}_3 - K_{1,h}^\top V_h^{-1}\underline{f}_2 + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}\underline{f}_1 + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}V_{1,h}V_h^{-1}\underline{f}_2.$$

Lemma 6.2 *The symmetric matrix*

$$\begin{aligned} \widehat{T}_h \;=\; \widehat{T}_{\varrho,h} - \varrho A_h \;=\; & D_{1,h} + K_{1,h}^\top V_h^{-1}(\frac{1}{2}M_h + K_h) + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}K_{1,h} \\ & -(\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}V_{1,h}V_h^{-1}(\frac{1}{2}M_h + K_h) \end{aligned}$$

is positive semi–definite, i.e., all eigenvalues of $\widehat{T}_h$ are non–negative,

$$(\widehat{T}_h\underline{z}, \underline{z}) \;\geq\; 0 \quad \textit{for all } \underline{z} \in \mathbb{R}^M.$$

Proof. We consider the generalized eigenvalue problem

$$\widehat{T}_h \underline{z} = \mu \left[\widetilde{S}_h + (\frac{1}{2}M_h^\top + K_h^\top) V_h^{-1} (\frac{1}{2}M_h + K_h) \right] \underline{z}, \tag{6.12}$$

where the stabilized discrete Steklov–Poincaré operator

$$\widetilde{S}_h = \widetilde{D}_h + (\frac{1}{2}M_h^\top + K_h^\top) V_h^{-1} (\frac{1}{2}M_h + K_h)$$

is symmetric and positive definite.

Since the eigenvalue problem (6.12) can be written as

$$\begin{aligned} &\begin{pmatrix} (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1} & I \end{pmatrix} \begin{pmatrix} -V_{1,h} & K_{1,h} \\ K_{1,h}^\top & D_{1,h} \end{pmatrix} \begin{pmatrix} V_h^{-1}(\frac{1}{2}M_h + K_h) \\ I \end{pmatrix} \underline{z} \\ &\qquad = \mu \begin{pmatrix} (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1} & I \end{pmatrix} \begin{pmatrix} V_h & \\ & \widetilde{S}_h \end{pmatrix} \begin{pmatrix} V_h^{-1}(\frac{1}{2}M_h + K_h) \\ I \end{pmatrix} \underline{z}, \end{aligned}$$

it is sufficient to consider the generalized eigenvalue problem

$$\begin{pmatrix} -V_{1,h} & K_{1,h} \\ K_{1,h}^\top & D_{1,h} \end{pmatrix} \begin{pmatrix} \underline{w} \\ \underline{z} \end{pmatrix} = \mu \begin{pmatrix} V_h \underline{w} \\ \widetilde{S}_h \underline{z} \end{pmatrix}, \tag{6.13}$$

where

$$\underline{w} = V_h^{-1} (\frac{1}{2}M_h + K_h) \underline{z}\,.$$

From (6.13), we then conclude

$$\begin{aligned} (K_{1,h}\underline{z}, \underline{w}) - (V_{1,h}\underline{w}, \underline{w}) &= \mu(V_h \underline{w}, \underline{w}), \\ (K_{1,h}^\top \underline{w}, \underline{z}) + (D_{1,h}\underline{z}, \underline{z}) &= \mu(\widetilde{S}_h \underline{z}, \underline{z}), \end{aligned}$$

and by taking the difference we obtain

$$(D_{1,h}\underline{z}, \underline{z}) + (V_{1,h}\underline{w}, \underline{w}) = \mu[(\widetilde{S}_h \underline{z}, \underline{z}) - (V_h \underline{w}, \underline{w})] = \mu(\widetilde{D}_h \underline{z}, \underline{z}).$$

Since the Galerkin matrices $D_{1,h}$ and $V_{1,h}$ of the Bi–Laplace boundary integral operators D_1 and V_1 are positive semi–definite and positive definite, respectively, and since the stabilized Laplace hypersingular integral operator $\widetilde{D}_h$ is positive definite, we conclude

$$\mu \geq 0\,.$$

Hence, we finally obtain

$$(\widehat{T}_h \underline{z}, \underline{z}) \geq \mu_{\min} \left[(\widetilde{S}_h \underline{z}, \underline{z}) + (V_h^{-1}(\frac{1}{2}M_h + K_h)\underline{z}, (\frac{1}{2}M_h + K_h)\underline{z}) \right] \geq 0$$

for all $\underline{z} \in \mathbb{R}^M$. ∎

As a corollary of Lemma 6.2, we find the positive definiteness of the symmetric Schur complement matrix $\widehat{T}_{\varrho,h}$ as defined in (6.11).

Corollary 6.3 *The approximate Schur complement* $\widehat{T}_{\varrho,h}$ *as defined in* (6.11) *is positive definite, i.e.,*

$$(\widehat{T}_{\varrho,h}\underline{z},\underline{z}) \;\geq\; \varrho(A_h\underline{z},\underline{z}) \;=\; \varrho\langle Az_h, z_h\rangle_\Gamma \geq \varrho\gamma_1^A \, \|z_h\|^2_{H^{1/2}(\Gamma)}$$

for all $\underline{z} \in \mathbb{R}^M \leftrightarrow z_h \in S_h^1(\Gamma)$.

Hence, we can ensure the unique solvability of the Schur complement system (6.10) and therefore of the system (6.9) as well as of the Galerkin variational problem (6.6)–(6.8). To find an error estimate for the approximate solution $\widehat{z}_h \in S_h^1(\Gamma)$, as for the non–symmetric boundary element formulation, we will consider a perturbed variational problem of the operator equation (6.4) leading to the Schur complement system (6.10).

6.2 Symmetric boundary element approximation of T_ϱ

For an arbitrary but fixed given $z \in H^{1/2}(\Gamma)$ the application of $T_\varrho z$ reads, by using the symmetric representation (6.5),

$$T_\varrho z \;=\; \varrho A z + D_1 z + K_1' t_z - (\frac{1}{2}I + K')q_z,$$

where $q_z, t_z \in H^{-1/2}(\Gamma)$ are the unique solutions of the boundary integral equations (5.3). Hence, by using the unique solutions $\widetilde{q}_{z,h}, t_{z,h} \in S_h^0(\Gamma)$ of the related Galerkin variational formulations (5.4) and (5.5) we can define the approximation

$$\widehat{T}_\varrho z \;:=\; \varrho A z + D_1 z + K_1' t_{z,h} - (\frac{1}{2}I + K')\widetilde{q}_{z,h}. \qquad (6.14)$$

Lemma 6.4 *The approximate operator* $\widehat{T}_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *as defined in* (6.14) *is bounded, i.e.,*

$$\|\widehat{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \;\leq\; c_2^{\widehat{T}_\varrho}\, \|z\|_{H^{1/2}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

Moreover, there holds the error estimate

$$\|T_\varrho z - \widehat{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \leq c_1 \inf_{\tau_h \in S_h^0(\Gamma)} \|q_z - \tau_h\|_{H^{-1/2}(\Gamma)} + c_2\, \|t_z - t_{z,h}\|_{H^{-3/2}(\Gamma)}. \qquad (6.15)$$

Proof. The proof follows as for the boundary element approximation of the non–symmetric formulation, see Lemma 5.2 and Lemma 5.3. ■

By using the approximation property of the trial space $S_h^0(\Gamma)$ and the Aubin–Nitsche trick, we then conclude an error estimate from (6.15) when assuming some regularity of q_z and t_z, respectively.

Corollary 6.5 *Assume* $q_z, t_z \in H^s_{\rm pw}(\Gamma)$ *for some* $s \in [0,1]$. *Then there holds the error estimate*

$$\|T_\varrho z - \widehat{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \;\leq\; c_1\, h^{s+\frac{1}{2}}\, \|q_z\|_{H^s_{\rm pw}(\Gamma)} + c_2\, h^{s+\frac{3}{2}}\, \|t_z\|_{H^s_{\rm pw}(\Gamma)}. \qquad (6.16)$$

6.3 Boundary element approximation of g

As in the approximation (6.14), we can define a boundary element approximation of the related right hand side, see (6.4),

$$\begin{aligned} g &= K_1'V^{-1}N_0 f - N_1\overline{u} - M_1 f + (\frac{1}{2}I + K')V^{-1}[N_0\overline{u} + M_0 f - V_1 V^{-1}N_0 f] \\ &= K_1' t_f - N_1\overline{u} - M_1 f + (\frac{1}{2}I + K')q_f, \end{aligned}$$

where $t_f = V^{-1}N_0 f \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle V t_f, \tau\rangle_\Gamma = \langle N_0 f, \tau\rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma),$$

and $q_f = V^{-1}[N_0\overline{u} + M_0 f - V_1 t_f] \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle V q_f, \tau\rangle_\Gamma = \langle N_0\overline{u} + M_0 f - V_1 t_f, \tau\rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma).$$

Hence we can define an approximation

$$\widehat{g} := K_1' t_{f,h} - N_1\overline{u} - M_1 f + (\frac{1}{2}I + K')\widetilde{q}_{f,h}, \tag{6.17}$$

where $\widetilde{q}_{f,h} \in S_h^0(\Gamma)$ is the unique solution of the Galerkin variational problem

$$\langle V\widetilde{q}_{f,h}, \tau_h\rangle_\Gamma = \langle N_0\overline{u} + M_0 f - V_1 t_{f,h}, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma),$$

and $t_{f,h} \in S_h^0(\Gamma)$ is the unique solution of the Galerkin variational problem

$$\langle V t_{f,h}, \tau_h\rangle_\Gamma = \langle N_0 f, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma).$$

As in (5.14), we conclude the error estimate

$$\|g - \widehat{g}\|_{H^{-1/2}(\Gamma)} \le c_1\, h^{s+\frac{1}{2}}\, \|q_f\|_{H^s_{\rm pw}(\Gamma)} + c_2\, h^{s+\frac{3}{2}}\, \|t_f\|_{H^s_{\rm pw}(\Gamma)} \tag{6.18}$$

when assuming $q_f, t_f \in H^s_{\rm pw}(\Gamma)$ for some $s \in [0,1]$.

6.4 Perturbed Galerkin variational problem

We now consider a perturbed Galerkin formulation instead of the operator equation (6.4) to find $z_h \in S_h^1(\Gamma)$ such that

$$\langle \widehat{T}_\varrho z_h, v_h\rangle_\Gamma = \langle \widehat{g}, v_h\rangle_\Gamma \quad \text{for all } v_h \in S_h^1(\Gamma). \tag{6.19}$$

Note that the Galerkin discretization of the perturbed variational problem (6.19) results in the linear system (6.10). Now we are in a position to formulate an error estimate for the approximate solution z_h by applying the Strang lemma.

Theorem 6.6 *Let $z \in H^{1/2}(\Gamma)$ be the unique solution of the operator equation* (6.4)*, and let $z_h \in S_h^1(\Gamma)$ be the unique solution of the perturbed Galerkin variational problem* (6.19)*. Then there holds the error estimate*

$$\|z-\widehat{z}_h\|_{H^{1/2}(\Gamma)} \leq c_1 \inf_{v_h \in S_h^1(\Gamma)} \|z-v_h\|_{H^{1/2}(\Gamma)} + c_2 \|(T_\varrho - \widehat{T}_\varrho)z\|_{H^{-1/2}(\Gamma)} + c_3 \|g-\widehat{g}\|_{H^{-1/2}(\Gamma)}. \quad (6.20)$$

Corollary 6.7 *When combining the error estimate* (6.20) *with the approximation property of the ansatz space $S_h^1(\Gamma)$, and with the error estimates* (6.16) *and* (6.18)*, we finally obtain the error estimate*

$$\begin{aligned}\|z-\widehat{z}_h\|_{H^{1/2}(\Gamma)} \;\leq\;& c_1\, h^{3/2}\, |z|_{H^2(\Gamma)} + c_2\, h^{3/2}\, \|q_z\|_{H^1_{pw}(\Gamma)} + c_3\, h^{5/2}\, \|t_z\|_{H^1_{pw}(\Gamma)} \\ & + c_4\, h^{3/2}\, \|q_f\|_{H^1_{pw}(\Gamma)} + c_5\, h^{5/2}\, \|t_f\|_{H^1_{pw}(\Gamma)}\end{aligned}$$

when assuming $z \in H^2(\Gamma)$, and $q_z, t_z, q_f, t_f \in H^1_{pw}(\Gamma)$, respectively. Hence we can expect the convergence order 1.5 *when measuring the error in the energy norm,*

$$\|z-\widehat{z}_h\|_{H^{1/2}(\Gamma)} \;\leq\; c(z,\overline{u},f)\, h^{3/2}\,. \quad (6.21)$$

Moreover, applying the Aubin–Nitsche trick [22] *we are also able to derive an error estimate in $L_2(\Gamma)$, i.e.,*

$$\|z-\widehat{z}_h\|_{L_2(\Gamma)} \;\leq\; c(z,\overline{u},f)\, h^2\,. \quad (6.22)$$

7 Numerical results

We consider the Dirichlet boundary control problem (2.1) and (2.2) for the domain $\Omega = (0,\frac{1}{2})^2 \subset \mathbb{R}^2$ where

$$\overline{u}(x) = -\left(4+\frac{1}{\varrho}\right)[x_1(1-2x_1)+x_2(1-2x_2)]\,, \quad f(x) = -\frac{8}{\varrho}, \quad \varrho = 0.01\,.$$

For the boundary element discretization, we introduce a uniform triangulation of $\Gamma = \partial\Omega$ on several levels where the mesh size is $h_L = 2^{-(L+1)}$. Since the minimizer of (2.1) is not known in this case, we use the boundary element solution z_h of the 9th level as reference solution. The boundary element discretization is done by using the trial space $S_h^0(\Gamma)$ of piecewise constant basis functions, and $S_h^1(\Gamma)$ of piecewise linear and continuous functions. In particular we use the same boundary element mesh to approximate the control z by a piecewise linear approximation, and piecewise constant approximations for the fluxes t and q. Note that we have $h = H$ in this case, and therefore we can not ensure the $S_h^1(\Gamma)$–ellipticity of the non–symmetric boundary element approximation, see Theorem 5.7. However, the numerical example shows stability.
In Table 1, we present the errors for the control z in the $L_2(\Gamma)$ norm and the estimated order of convergence (eoc). These results correspond to the error estimate (5.24) of the non–symmetric boundary element approximation, and to the error estimate (6.22) of the

	Non–symmetric BEM (5.20)		Symmetric BEM (6.10)		FEM [18]	
L	$\Vert\widetilde{z}_{h_L} - \widetilde{z}_{h_9}\Vert_{L_2(\Gamma)}$	eoc	$\Vert\widehat{z}_{h_L} - \widehat{z}_{h_9}\Vert_{L_2(\Gamma)}$	eoc	$\Vert z^{\mathrm{FEM}}_{h_L} - z^{\mathrm{FEM}}_{h_9}\Vert_{L_2(\Gamma)}$	eoc
2	4.52 –1		2.25 –0		3.89 –1	
3	1.28 –1	1.82	4.66 –1	2.27	1.07 –1	1.86
4	3.54 –2	1.85	8.84 –2	2.39	2.81 –2	1.93
5	9.03 –3	1.97	1.63 –2	2.44	7.28 –3	1.95
6	2.18 –3	2.05	3.02 –3	2.43	1.87 –3	1.96
7	5.16 –4	2.08	5.73 –4	2.40	4.69 –4	2.00
8	1.39 –4	1.89	1.24 –4	2.20	1.06 –4	2.15

Table 1: Comparison of BEM/FEM errors of the Dirichlet control.

symmetric boundary element approximation. In addition, we give the error of the related finite element solution, see [18]. The results show a quadratic order of convergence, which confirm the theoretical estimates. Note that in the finite element approach only a convergence order of 1.5 can be proved [18].

In Table 2, we present the errors for the flux t of the primal boundary value problem, again in the $L_2(\Gamma)$ norm. Since the computation of t corresponds to the solution of a Dirichlet boundary value problem with approximated Dirichlet data, we can expect and observe a linear order of convergence when using piecewise constant basis functions, see, e.g., [22].

	Non–symmetric BEM (5.20)		Symmetric BEM (6.10)	
L	$\Vert t_{h_L} - t_{h_9}\Vert_{L_2(\Gamma)}$	eoc	$\Vert t_{h_L} - t_{h_9}\Vert_{L_2(\Gamma)}$	eoc
2	33.51		39.87	
3	24.03	0.48	28.67	0.48
4	14.80	0.70	16.22	0.82
5	8.44	0.81	8.88	0.87
6	4.59	0.88	4.73	0.91
7	2.40	0.93	2.45	0.95
8	1.17	1.04	1.19	1.05

Table 2: Comparison of non–symmetric/symmetric BEM.

Acknowledgement

This work has been supported by the Austrian Science Fund (FWF) under the Grant SFB Mathematical Optimization and Applications in Biomedical Sciences, Subproject Fast Finite Element and Boundary Element Methods for Optimality Systems. The authors would like to thank K. Kunisch for helpful discussions.

References

[1] R. Becker: Mesh adaption for Dirichlet flow control via Nitsche's method. Comm. Numer. Methods Engrg. 18 (2002) 669–680.

[2] E. Casas, J. P. Raymond: Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations. SIAM J. Control Optim. 45 (2006) 1586–1611.

[3] M. Costabel: Boundary integral operators on Lipschitz domains: Elementary results. SIAM J. Math. Anal. 19 (1988) 613–626.

[4] K. Deckelnick, A. Günther, M. Hinze: Finite element approximation of Dirichlet boundary control for elliptic PDEs on two– and three–dimensional curved domains. Preprint SPP1253–08–05. DFG Priority Program 1253 Optimization with Partial Differential Equations, 2008.

[5] Z. Ding: Optimal boundary controls of linear elastostatics on Lipschitz domains with point observations. J. Dynam. Control Syst. 7 (2001) 181–207.

[6] A. V. Fursikov, M. D. Gunzburger, L. S. Hou: Boundary value problems and optimal boundary control for the Navier–Stokes system: the two–dimensional case. SIAM J. Control Optim. 36 (1998) 852–894.

[7] M. D. Gunzburger, L. S. Hou, T. Swobodny: Analysis and finite element approximation of optimal control problems for the stationary Navier–Stokes equations with Dirichlet controls. Math. Model. Numer. Anal. 25 (1991) 711–748.

[8] W. Hackbusch: On the fast solving of parabolic boundary control problems. SIAM J. Control Optim. 17 (1979) 231–244.

[9] M. Hinze: A variational discretization concept in control constrained optimation: the linear–quadratic case. Comput. Optim. Appl. 30 (2005) 45–61.

[10] M. Hinze, K. Kunisch: Second order methods for boundary control of the instationary Navier–Stokes system. Z. Angew. Math. Mech. 84 (2004) 171–187.

[11] M. Hinze, R. Pinnau, M. Ulbrich, S. Ulbrich: Optimization with PDE Constraints. Mathematical Modelling: Theory and Applications, vol. 23, Springer, Heidelberg, 2009.

[12] G. C. Hsiao, W. L. Wendland: Boundary Integral Equations. Springer, Heidelberg, 2008.

[13] B. N. Khoromskij, G. Schmidt: Boundary integral equations for the biharmonic Dirichlet problem on non–smooth domains. J. Integral Equations Appl. 11 (1999) 217–253.

[14] K. Kunisch, B. Vexler: Constrained Dirichlet boundary control in L^2 for a class of evolution equations. SIAM J. Control Optim. 46 (2007) 1726–1753.

[15] A. Kunoth: Fast iterative solution of saddle point problems in optimal control based on wavelets. Comput. Optim. Appl. 22 (2002) 225–259.

[16] S. May, R. Rannacher, B. Vexler: Error analysis for a finite element approximation of elliptic Dirichlet boundary control problems. Lehrstuhl für Angewandte Mathematik, Universität Heidelberg, Preprint 05/2008.

[17] W. McLean: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, 2000.

[18] G. Of, T. X. Phan, O. Steinbach: An energy space finite element approach for elliptic Dirichlet boundary control problems. Berichte aus dem Institut für Numerische Mathematik, Bericht 2009/1, TU Graz, 2009.

[19] G. Of, O. Steinbach: A fast multipole boundary element method for a modified hypersingular boundary integral equation. In: Analysis and Simulation of Multifield Problems (W. L. Wendland, M. Efendiev eds.), Lecture Notes in Applied and Computational Mechanics, vol. 12, Springer, Heidelberg, pp. 163–169, 2003.

[20] S. A. Sauter, C. Schwab: Randelementmethoden. Analyse, Numerik und Implementierung schneller Algorithmen. B. G. Teubner, Wiesbaden, 2004.

[21] O. Steinbach: Mixed approximations for boundary elements. SIAM J. Numer. Anal. 38 (2000) 401–413.

[22] O. Steinbach: Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements. Springer, New York, 2008.

[23] P. You, Z. Ding, J. Zhou: Optimal boundary control of the Stokes fluids with point velocity observations. SIAM J. Control Optim. 36 (1998) 981–1004.