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G. Of T. X. Phan O. Steinbach

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G. Of, T. X. Phan, O. Steinbach

Institut für Numerische Mathematik, TU Graz,
Steyrergasse 30, 8010 Graz, Austria
{of,thanh.phanxuan,o.steinbach}@tugraz.at

Abstract

In this paper we present a finite element analysis for a Dirichlet boundary control problem where the Dirichlet control is considered in the energy space $H^{1/2}(\Gamma)$. As an equivalent norm in $H^{1/2}(\Gamma)$ we use a norm which is induced by a stabilized hypersingular boundary integral operator. The analysis is based on the mapping properties of the solution operators related to the primal and adjoint boundary value problems, and their finite element approximations. Some numerical results are given.

1 Introduction

In this paper, we focus on the a priori error analysis of the finite element approximation of an elliptic Dirichlet boundary control problem. Our approach is based on the use of the energy space $H^{1/2}(\Gamma)$ as control space, where an equivalent norm is realized by using a stabilized hypersingular boundary integral operator [19]. Since the solution of the state equation, i.e., of an elliptic Dirichlet boundary value problem, defines an appropriate solution operator, the minimizer of the reduced cost functional is characterized by the unique solution of an operator equation in the energy space $H^{1/2}(\Gamma)$. The Galerkin discretization of this operator equation is based on the use of a finite element approximation of the solution operator related to the elliptic Dirichlet boundary value problem. The unique solvability of the resulting perturbed Galerkin formulation and related a priori error estimates are then based on the discrete ellipticity of the approximated operator, and on the use of the Strang lemma [5]. Although we do not consider additional constraints on the Dirichlet control, this approach can be generalized in a straightforward manner. We consider the Poisson equation as a model problem, however, this approach can be applied to any elliptic partial differential equation.

Optimal control problems of elliptic or parabolic partial differential equations with a Dirichlet boundary control play an important role, for example, in the context of computational fluid mechanics, see, e.g., [4, 9, 12]. For an overview on a priori error estimates for finite element approximations of optimal control problems, see, for example, [3, 13], and the references given therein. For more advanced estimates, in particular for problems with distributed control, see also [2, 17].

One difficulty in the handling of Dirichlet control problems lies in the essential character of Dirichlet boundary conditions. While Neumann or Robin type boundary conditions can be incorporated naturally in the weak formulation of the state equation, given Dirichlet data on the boundary have to be extended onto the domain in a suitable way. For Neumann boundary control problems, see, e.g., [6]; for a discussion of several variational formulations for Dirichlet boundary control problems, see [14]. In [11], a rather general concept is given to solve optimal control problems and is applied to Dirichlet boundary control problems in [8]. If the Dirichlet control is considered in $L_2(\Gamma)$, a very week variational formulation of the state equation has to be used. This was investigated in [7] first, see also [3, 10, 15, 16], or [23] in the case of a finite dimensional Dirichlet control.

Instead of a very weak variational formulation of the state equation, we consider standard finite element approximations of both the primal and adjoint boundary value problems in this paper. Due to the use of an induced energy norm in the cost functional the optimality condition results in an operator equation in $H^{1/2}(\Gamma)$. Since the optimality condition involves the normal derivative of the adjoint variable, we use Green's first formula to obtain a variational formulation of the operator equation to be solved. Based on the mapping properties of the continuous solution operators of the primal and adjoint boundary value problems, we introduce related finite element approximations.

The paper is organized as follows: In Section 2, we describe the considered Dirichlet boundary control problem, the primal boundary value problem, and the reduced cost functional as well as the related adjoint boundary value problem. In addition, the weak formulation of the optimality condition is analyzed. The finite element approximations of both the primal and adjoint boundary value problems are given in Section 3, where we also describe the incorporation of the Dirichlet boundary control as a Dirichlet boundary condition in the primal problem. In Section 4, we give the details of the finite element implementation which leads finally to a block system of linear equations to be solved. The main results are given in Section 5, where we discuss stability and a priori error estimates of the proposed approach. Some numerical results are finally given in Section 6.

For an overview on the used Sobolev spaces in the domain and on the boundary, see, for example, [1, 18, 21, 22].

2 Dirichlet control problems

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$. As a model problem, we consider the Dirichlet boundary control problem to minimize the cost functional

$$J(u, z) = \frac{1}{2} \int\limits_{\Omega} [u(x) - \overline{u}(x)]^2 dx + \frac{1}{2} \varrho \|z\|_A^2 \qquad (2.1)$$

subject to the constraint

$$-\Delta u(x) \;=\; f(x) \quad \text{for } x \in \Omega, \quad u(x) \;=\; z(x) \quad \text{for } x \in \Gamma, \tag{2.2}$$

where $z \in H^{1/2}(\Gamma)$ is the control to be determined, while $f \in L_2(\Omega)$ and $\overline{u} \in L_2(\Omega)$ are given, and $\varrho \in \mathbb{R}_+$ is a fixed parameter. Moreover, $\|\cdot\|_A^2 = \langle A\cdot,\cdot\rangle_\Gamma$ is an equivalent norm in $H^{1/2}(\Gamma)$ which is induced by an elliptic, self–adjoint, and bounded operator $A : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, i.e.,

$$\gamma_1^A \,\|w\|^2_{H^{1/2}(\Gamma)} \le \langle Aw, w\rangle_\Gamma, \quad \|Aw\|_{H^{-1/2}(\Gamma)} \le \gamma_2^A \,\|w\|_{H^{1/2}(\Gamma)} \quad \text{for all } w \in H^{1/2}(\Gamma).$$

For example, we will consider the stabilized hypersingular boundary integral operator $A = \widetilde{D}$, see [19],

$$\langle \widetilde{D}z, w\rangle_\Gamma := \langle Dz, w\rangle_\Gamma + \langle z, 1\rangle_\Gamma \langle w, 1\rangle_\Gamma \quad \text{for all } z, w \in H^{1/2}(\Gamma),$$

where

$$(Dz)(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) z(y) ds_y \quad \text{for } x \in \Gamma$$

is the Laplace hypersingular integral operator, and

$$U^*(x,y) = \begin{cases} -\dfrac{1}{2\pi} \log|x-y| & \text{for } n = 2, \\ \dfrac{1}{4\pi} \dfrac{1}{|x-y|} & \text{for } n = 3 \end{cases}$$

is the fundamental solution of the Laplace operator [21]. Note that for $\tau \in H^{-1/2}(\Gamma)$ and $w \in H^{1/2}(\Gamma)$

$$\langle \tau, w\rangle_\Gamma = \int_\Gamma \tau(x) w(x) ds_x$$

denotes the related duality pairing.

Assumption 2.1 *While we may assume Ω to be a Lipschitz domain to ensure all mapping properties as required to prove the unique solvability of the continuous formulation, for the finite element analysis we will assume that Ω is either a convex Lipschitz domain, or nonconvex but with smooth boundary. Note that this assumption can be weakened when considering the Aubin–Nitsche lemma in appropriate function spaces.*

2.1 Primal boundary value problem

To rewrite the Dirichlet boundary control problem (2.1) and (2.2) by using a reduced cost functional, we introduce a linear solution operator describing the application of the constraint (2.2). Therefore we consider a homogeneous partial differential equation with the control as Dirichlet boundary condition, and an inhomogeneous partial differential equation with zero Dirichlet boundary conditions to describe a particular solution. Let u_f be the weak solution of the homogeneous Dirichlet boundary value problem

$$-\Delta u_f(x) \;=\; f(x) \quad \text{for } x \in \Omega, \quad u_f(x) \;=\; 0 \quad \text{for } x \in \Gamma, \tag{2.3}$$

i.e., $u_f \in H^1_0(\Omega)$ is the unique solution of the variational problem

$$\int\limits_\Omega \nabla u_f(x) \cdot \nabla v(x) dx \;=\; \int\limits_\Omega f(x) v(x) dx \quad \text{for all } v \in H^1_0(\Omega). \tag{2.4}$$

Note that the bilinear form

$$a(u,v) = \int\limits_\Omega \nabla u(x) \cdot \nabla v(x) dx$$

is bounded for all $u, v \in H^1(\Omega)$ and $H^1_0(\Omega)$–elliptic, i.e.,

$$|a(u,v)| \;\leq\; c_2^A \, \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \text{for all } u, v \in H^1(\Omega), \tag{2.5}$$

and

$$a(v,v) \geq c_1^A \, \|v\|^2_{H^1(\Omega)} \quad \text{for all } v \in H^1_0(\Omega). \tag{2.6}$$

Hence we conclude the unique solvability of the variational problem (2.4) by the Lax–Milgram lemma, see, e.g., [5], or [21, Theorem 3.4]. In particular, we obtain the estimate

$$\|u_f\|_{H^1(\Omega)} \;\leq\; \frac{1}{c_1^A} \, \|f\|_{H^{-1}(\Omega)}.$$

Obviously, we also have $u_f \in L_2(\Omega)$ satisfying

$$\|u_f\|_{L_2(\Omega)} \;\leq\; \|u_f\|_{H^1(\Omega)} \;\leq\; \frac{1}{c_1^A} \, \|f\|_{H^{-1}(\Omega)}. \tag{2.7}$$

The solution of the Dirichlet boundary value problem (2.2) is then given by $u = u_z + u_f$, where $u_z \in H^1(\Omega)$ is the unique solution of the Dirichlet boundary value problem

$$-\Delta u_z(x) \;=\; 0 \quad \text{for } x \in \Omega, \quad u_z(x) \;=\; z(x) \quad \text{for } x \in \Gamma. \tag{2.8}$$

Due to the inverse trace theorem, see, e.g., [18, 22] or [21, Theorem 2.22], there exists a bounded extension operator $\mathcal{E} : H^{1/2}(\Gamma) \to H^1(\Omega)$, i.e., for $z \in H^{1/2}(\Gamma)$ there exists $\mathcal{E}z \in H^1(\Omega)$ satisfying $\mathcal{E}z_{|\Gamma} = z$ and

$$\|\mathcal{E}z\|_{H^1(\Omega)} \;\leq\; c_{IT} \, \|z\|_{H^{1/2}(\Gamma)}. \tag{2.9}$$

By writing $u_z = u_0 + \mathcal{E}z$, the variational formulation of the Dirichlet boundary value problem (2.8) is to find $u_0 \in H^1_0(\Omega)$ such that

$$\int\limits_\Omega \nabla u_0(x) \cdot \nabla v(x) dx \;=\; -\int\limits_\Omega \nabla \mathcal{E}z(x) \cdot \nabla v(x) dx \quad \text{for all } v \in H^1_0(\Omega). \tag{2.10}$$

Again, the variational problem (2.10) admits a unique solution $u_0 \in H^1_0(\Omega)$ satisfying

$$\|u_0\|_{H^1(\Omega)} \leq \frac{c_2^A}{c_1^A}\,\|\mathcal{E}z\|_{H^1(\Omega)}.$$

We first have

$$\|u_z\|_{H^1(\Omega)} = \|u_0 + \mathcal{E}z\|_{H^1(\Omega)} \leq \|u_0\|_{H^1(\Omega)} + \|\mathcal{E}z\|_{H^1(\Omega)} \leq \left(\frac{c_2^A}{c_1^A} + 1\right)\|\mathcal{E}z\|_{H^1(\Omega)},$$

and by using the inverse trace theorem (2.9) we further obtain

$$\|u_z\|_{H^1(\Omega)} \leq \left(\frac{c_2^A}{c_1^A} + 1\right) c_{IT}\,\|z\|_{H^{1/2}(\Gamma)}.$$

We also have the obvious estimate

$$\|u_z\|_{L_2(\Omega)} \leq \|u_z\|_{H^1(\Omega)}\,.$$

Hence we may introduce the bounded solution operator $\mathcal{S} : H^{1/2}(\Gamma) \to L_2(\Omega)$, $u_z = \mathcal{S}z$, satisfying

$$\|\mathcal{S}z\|_{L_2(\Omega)} \leq c_2^{\mathcal{S}}\,\|z\|_{H^{1/2}(\Gamma)} \quad \text{for all } z \in H^{1/2}(\Gamma), \quad c_2^{\mathcal{S}} = \left(\frac{c_2^A}{c_1^A} + 1\right) c_{IT}\,. \tag{2.11}$$

Therefore we can write the solution operator of the primal boundary value problem (2.2) as

$$u = \mathcal{S}z + u_f\,. \tag{2.12}$$

2.2 Reduced cost functional and adjoint boundary value problem

By using (2.12), i.e., $u = \mathcal{S}z + u_f$, we can write the cost functional (2.1), which is to be minimized subject to the constraint (2.2), as the reduced cost functional

$$\begin{aligned}
\widetilde{J}(z) &= \frac{1}{2}\int_\Omega [(\mathcal{S}z)(x) + u_f(x) - \overline{u}(x)]^2 dx + \frac{1}{2}\varrho\langle Az, z\rangle_\Gamma \\
&= \frac{1}{2}\langle \mathcal{S}z + u_f - \overline{u}, \mathcal{S}z + u_f - \overline{u}\rangle_\Omega + \frac{1}{2}\varrho\langle Az, z\rangle_\Gamma \\
&= \frac{1}{2}\langle \mathcal{S}^*\mathcal{S}z, z\rangle_\Gamma + \langle \mathcal{S}^*(u_f - \overline{u}), z\rangle_\Gamma + \frac{1}{2}\|u_f - \overline{u}\|^2_{L_2(\Omega)} + \frac{1}{2}\varrho\langle Az, z\rangle_\Gamma
\end{aligned} \tag{2.13}$$

where $\mathcal{S}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ is the adjoint operator of $\mathcal{S} : H^{1/2}(\Gamma) \to L_2(\Omega)$, i.e.,

$$\langle \mathcal{S}^*\psi, \varphi\rangle_\Gamma = \langle \psi, \mathcal{S}\varphi\rangle_\Omega = \int_\Omega (\mathcal{S}\varphi)(x)\psi(x)dx \quad \text{for all } \varphi \in H^{1/2}(\Gamma), \psi \in L_2(\Omega). \tag{2.14}$$

The application of the adjoint operator, $\tau = \mathcal{S}^*\psi$, $\psi \in L_2(\Omega)$, is given by

$$\tau(x) \,=\, -\frac{\partial}{\partial n_x} p(x) \quad \text{for } x \in \Gamma, \tag{2.15}$$

where p is the unique solution of the adjoint Dirichlet boundary value problem

$$-\Delta p(x) \,=\, \psi(x) \quad \text{for } x \in \Omega, \quad p(x) \,=\, 0 \quad \text{for } x \in \Gamma, \tag{2.16}$$

i.e., $p \in H^1_0(\Omega)$ is the unique solution of the variational problem

$$\int\limits_\Omega \nabla p(x) \cdot \nabla v(x) dx \,=\, \int\limits_\Omega \psi(x) v(x) dx \quad \text{for all } v \in H^1_0(\Omega). \tag{2.17}$$

Again, we conclude the unique solvability of the variational problem (2.17) by the Lax–Milgram lemma, in particular we obtain

$$\|p\|_{H^1(\Omega)} \,\leq\, \frac{1}{c_1^A} \, \|\psi\|_{H^{-1}(\Omega)}. \tag{2.18}$$

Lemma 2.2 *The adjoint operator $\mathcal{S}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ as defined in (2.15) is bounded, i.e.,*

$$\|\mathcal{S}^*\psi\|_{H^{-1/2}(\Gamma)} \,\leq\, c_2^{\mathcal{S}} \, \|\psi\|_{L_2(\Omega)} \quad \textit{for all } \psi \in L_2(\Omega), \quad c_2^{\mathcal{S}} = \left(\frac{c_2^A}{c_1^A} + 1 \right) c_{IT}. \tag{2.19}$$

Proof. Since the normal derivative (2.15) is formulated in $H^{-1/2}(\Gamma)$, we consider the weak formulation of (2.15) by choosing a test function $w \in H^{1/2}(\Gamma)$. By using Green's first formula, we obtain

$$\langle \tau, w \rangle_\Gamma \,=\, -\int\limits_\Omega \nabla p(x) \cdot \nabla \mathcal{E} w(x) dx + \int\limits_\Omega \psi(x) \mathcal{E} w(x) dx. \tag{2.20}$$

We conclude, by using duality, due to (2.5) and (2.18),

$$\begin{aligned}
\|\tau\|_{H^{-1/2}(\Gamma)} \;&=\; \sup_{0 \neq w \in H^{1/2}(\Gamma)} \frac{\langle w, \tau \rangle_\Gamma}{\|w\|_{H^{1/2}(\Gamma)}} \\
&=\; \sup_{0 \neq w \in H^{1/2}(\Gamma)} \frac{-\langle \nabla p, \nabla \mathcal{E} w \rangle_\Omega + \langle \psi, \mathcal{E} w \rangle_\Omega}{\|w\|_{H^{1/2}(\Gamma)}} \\
&\leq\; \sup_{0 \neq w \in H^{1/2}(\Gamma)} \frac{c_2^A \|p\|_{H^1(\Omega)} \|\mathcal{E} w\|_{H^1(\Omega)} + \|\psi\|_{\widetilde{H}^{-1}(\Omega)} \|\mathcal{E} w\|_{H^1(\Omega)}}{\|w\|_{H^{1/2}(\Gamma)}} \\
&\leq\; c_{IT} \left[c_2^A \|p\|_{H^1(\Omega)} + \|\psi\|_{\widetilde{H}^{-1}(\Omega)} \right] \\
&\leq\; c_{IT} \left[\frac{c_2^A}{c_1^A} \|\psi\|_{H^{-1}(\Omega)} + \|\psi\|_{\widetilde{H}^{-1}(\Omega)} \right] \,\leq\, c_{IT} \left(\frac{c_2^A}{c_1^A} + 1 \right) \, \|\psi\|_{L_2(\Omega)},
\end{aligned}$$

where we used

$$\|\psi\|_{H^{-1}(\Omega)} = \sup_{0\neq v\in H^1_0(\Omega)} \frac{\langle \psi, v\rangle_\Omega}{\|v\|_{H^1(\Omega)}} \leq \sup_{0\neq v\in H^1(\Omega)} \frac{\langle \psi, v\rangle_\Omega}{\|v\|_{H^1(\Omega)}} = \|\psi\|_{\widetilde{H}^{-1}(\Omega)} \leq \|\psi\|_{L_2(\Omega)}.$$

■

Obviously, the bound (2.19) also follows from the estimate (2.11) by using (2.14), but (2.20) will be used later to derive a variational formulation of the optimality condition.

2.3 Optimality condition

Since the reduced cost functional $\widetilde{J}(\cdot)$, as given in (2.13), is convex, the unconstrained minimizer z can be found from the optimality condition

$$\mathcal{S}^*\mathcal{S}z + \mathcal{S}^*(u_f - \bar{u}) + \varrho A z \,=\, 0,$$

i.e., we have to solve the operator equation

$$T_\varrho z \,:=\, (\mathcal{S}^*\mathcal{S} + \varrho A)z \,=\, \mathcal{S}^*(\bar{u} - u_f) \,=:\, g\,. \tag{2.21}$$

The related variational problem reads: Find $z \in H^{1/2}(\Gamma)$ such that

$$\langle T_\varrho z, w\rangle_\Gamma = \langle \mathcal{S}^*\mathcal{S}z, w\rangle_\Gamma + \varrho\langle Az, w\rangle_\Gamma = \langle \mathcal{S}^*(\overline{u} - u_f), w\rangle_\Gamma = \langle g, w\rangle_\Gamma \tag{2.22}$$

is satisfied for all $w \in H^{1/2}(\Gamma)$.

Lemma 2.3 *The operator $T_\varrho = \varrho A + \mathcal{S}^*\mathcal{S} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is bounded and $H^{1/2}(\Gamma)$–elliptic, i.e.,*

$$\|T_\varrho z\|_{H^{-1/2}(\Gamma)} \,\leq\, c_2^T\, \|z\|_{H^{1/2}(\Gamma)}, \quad \langle T_\varrho z, z\rangle_\Gamma \,\geq\, c_1^T\, \|z\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

Proof. By using the boundedness (2.19) and (2.11) we obtain

$$\begin{aligned}
\|T_\varrho z\|_{H^{-1/2}(\Gamma)} &= \|(\mathcal{S}^*\mathcal{S} + \varrho A)z\|_{H^{-1/2}(\Gamma)} \\
&\leq \|\mathcal{S}^*\mathcal{S}z\|_{H^{-1/2}(\Gamma)} + \varrho\, \|Az\|_{H^{-1/2}(\Gamma)} \\
&\leq c_2^{\mathcal{S}}\, \|\mathcal{S}z\|_{L_2(\Omega)} + \varrho\, \|Az\|_{H^{-1/2}(\Gamma)} \\
&\leq [c_2^{\mathcal{S}}]^2\, \|z\|_{H^{1/2}(\Gamma)} + \varrho\gamma_2^A\, \|z\|_{H^{1/2}(\Gamma)} \,=\, c_2^T\, \|z\|_{H^{1/2}(\Gamma)}
\end{aligned}$$

for all $z \in H^{1/2}(\Gamma)$, where $c_2^T = [c_2^{\mathcal{S}}]^2 + \varrho\gamma_2^A$. Moreover, we get

$$\begin{aligned}
\langle T_\varrho z, z\rangle_\Gamma &= \varrho\langle Az, z\rangle_\Gamma + \langle \mathcal{S}^*\mathcal{S}z, z\rangle_\Gamma \,=\, \varrho\langle Az, z\rangle_\Gamma + \langle \mathcal{S}z, \mathcal{S}z\rangle_\Omega \\
&\geq \varrho\gamma_1^A\|z\|^2_{H^{1/2}(\Gamma)} + \|\mathcal{S}z\|^2_{L_2(\Omega)} \geq c_1^T\, \|z\|^2_{H^{1/2}(\Gamma)}
\end{aligned}$$

for all $z \in H^{1/2}(\Gamma)$, where $c_1^T = \varrho\gamma_1^A$. ■

The unique solvability of the variational problem (2.22) of the optimality condition follows by the Lax–Milgram lemma, i.e., by using (2.19) and (2.7) we have

$$\begin{aligned} \|z\|_{H^{1/2}(\Gamma)} &\leq \frac{1}{c_1^T} \|\mathcal{S}^*(\overline{u}-u_f)\|_{H^{-1/2}(\Gamma)} \\ &\leq \frac{c_2^{\mathcal{S}}}{c_1^T} \|\overline{u}-u_f\|_{L_2(\Omega)} \\ &\leq \frac{c_2^{\mathcal{S}}}{c_1^T} \left[\|\overline{u}\|_{L_2(\Omega)} + \frac{1}{c_1^A} \|f\|_{H^{-1}(\Omega)} \right]. \end{aligned} \tag{2.23}$$

3 Finite element approximations

Although we can consider a finite element approximation of the variational problem (2.22), a practical implementation would require the discretization of the composed solution operator $\mathcal{S}^*\mathcal{S}$, which is not possible in general. Hence, we first introduce finite element approximations to solve the primal variational problem (2.10) and the adjoint problem (2.17), and define an approximate operator $\widetilde{T}_\varrho$ to be used in a perturbed variational formulation.

3.1 Primal boundary value problem

First, we consider a finite element approximation of the primal Dirichlet boundary value problem (2.8), see also (2.10), to find $u_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u_0(x) \cdot \nabla v(x) dx \;=\; - \int_\Omega \nabla \mathcal{E} z(x) \cdot \nabla v(x) dx \quad \text{for all } v \in H_0^1(\Omega). \tag{3.1}$$

Let

$$S_{h,0}^1(\Omega) = \text{span}\{\varphi_k\}_{k=1}^{M_\Omega} \subset H_0^1(\Omega)$$

be a finite element space of piecewise linear and continuous basis functions φ_k, which are defined with respect to some admissible domain triangulation of mesh size h. The finite element approximation of (3.1) is to find $u_{0,h} \in S_{h,0}^1(\Omega)$ such that

$$\int_\Omega \nabla u_{0,h}(x) \cdot \nabla v_h(x) dx \;=\; - \int_\Omega \nabla \mathcal{E} z(x) \cdot \nabla v_h(x) dx \quad \text{for all } v_h \in S_{h,0}^1(\Omega). \tag{3.2}$$

By subtracting (3.2) from (3.1), we obtain the Galerkin orthogonality

$$\int_\Omega \nabla [u_0(x) - u_{0,h}(x)] \cdot \nabla v_h(x) dx \;=\; 0 \quad \text{for all } v_h \in S_{h,0}^1(\Omega).$$

Then, by means of Cea's lemma, see, e.g., [5] or [21, Theorem 8.1], and the approximation property of $S_{h,0}^1(\Omega)$, see, e.g., [21, Theorem 9.10], we conclude the unique solvability of the

Galerkin formulation (3.2) as well as the error estimate

$$\|u_0 - u_{0,h}\|_{H^1(\Omega)} \,\leq\, \frac{c_2^A}{c_1^A} \inf_{v_h \in S^1_{h,0}(\Omega)} \|u_0 - v_h\|_{H^1(\Omega)} \,\leq\, c\, h\, |u_0|_{H^2(\Omega)} \tag{3.3}$$

when assuming $u_0 \in H^2(\Omega)$. This assumption is satisfied, if Ω is either a convex Lipschitz domain, or nonconvex but with a smooth boundary $\Gamma = \partial\Omega$, see also Assumption 2.1. In both cases, by applying the Aubin–Nitsche trick, see, e.g., [5] or [21, Theorem 11.1], we also obtain an error estimate in $L_2(\Omega)$,

$$\|u_0 - u_{0,h}\|_{L_2(\Omega)} \,\leq\, c\, h^2\, |u_0|_{H^2(\Omega)}. \tag{3.4}$$

However, instead of the variational problem (3.2), we will consider a perturbed variational problem to find an approximate solution $\widetilde{u}_{0,h} \in S^1_{h,0}(\Omega)$ such that

$$\int\limits_\Omega \nabla \widetilde{u}_{0,h}(x) \cdot \nabla v_h(x) dx \,=\, -\int\limits_\Omega \nabla Q_h \mathcal{E} z(x) \cdot \nabla v_h(x) dx \quad \text{for all } v_h \in S^1_{h,0}(\Omega), \tag{3.5}$$

where $Q_h : H^1(\Omega) \to S^1_h(\Omega)$ denotes a quasi–interpolation operator [20] into the space of piecewise linear and continuous basis functions,

$$S^1_h(\Omega) = \operatorname{span}\{\varphi_k\}_{k=1}^M = S^1_{h,0}(\Omega) \cup \operatorname{span}\{\varphi_k\}_{k=M_\Omega+1}^M \subset H^1(\Omega). \tag{3.6}$$

Note that there hold the stability estimate [20]

$$\|Q_h v\|_{H^1(\Omega)} \,\leq\, c_2^Q\, \|v\|_{H^1(\Omega)} \quad \text{for all } v \in H^1(\Omega), \tag{3.7}$$

and the error estimate

$$\|v - Q_h v\|_{H^s(\Omega)} \,\leq\, c\, h^{2-s}\, \|v\|_{H^2(\Omega)}, \tag{3.8}$$

when assuming $v \in H^2(\Omega)$ and $s = 0, 1$.

Since the perturbed variational problem (3.5) admits a unique solution $\widetilde{u}_{0,h} \in S^1_{h,0}(\Omega)$, an approximation $\widetilde{S} z$ of the solution operator $\mathcal{S} z = u_0 + \mathcal{E} z$ can be defined by

$$\widetilde{S} z \,:=\, \widetilde{u}_{0,h} + Q_h \mathcal{E} z, \tag{3.9}$$

where $\widetilde{u}_{0,h}$ is the unique solution of the perturbed variational problem (3.5). By choosing $v_h = \widetilde{u}_{0,h} \in S^1_{h,0}(\Omega)$ in (3.5), and by using (2.5) and (2.6), we obtain

$$\|\widetilde{u}_{0,h}\|_{H^1(\Omega)} \,\leq\, \frac{c_2^A}{c_1^A}\, \|Q_h \mathcal{E} z\|_{H^1(\Omega)},$$

and therefore

$$\|\widetilde{S} z\|_{H^1(\Omega)} = \|\widetilde{u}_{0,h} + Q_h \mathcal{E} z\|_{H^1(\Omega)} \leq \left(\frac{c_2^A}{c_1^A} + 1\right) \|Q_h \mathcal{E} z\|_{H^1(\Omega)} \leq \left(\frac{c_2^A}{c_1^A} + 1\right) c_2^Q\, \|\mathcal{E} z\|_{H^1(\Omega)}.$$

Hence we conclude the stability estimate

$$\|\widetilde{S}z\|_{L_2(\Omega)} \leq c_2^{\widetilde{S}} \|z\|_{H^{1/2}(\Gamma)} \quad \text{for all } z \in H^{1/2}(\Gamma) \tag{3.10}$$

which corresponds to the stability estimate (2.11).

To obtain an estimate for the approximation error $\|\mathcal{S}z - \widetilde{\mathcal{S}}z\|_{L_2(\Omega)}$, we apply the Strang lemma, see, e.g., [5] or [21, Theorem 8.2]. By subtracting the perturbed Galerkin formulation (3.5) from the variational formulation (3.2), we obtain the perturbed Galerkin orthogonality

$$\int_\Omega [\nabla u_{0,h}(x) - \nabla \widetilde{u}_{0,h}(x)] \cdot \nabla v_h(x) dx = \int_\Omega \nabla [Q_h \mathcal{E}z(x) - \mathcal{E}z(x)] \cdot \nabla v_h(x) dx \quad \text{for all } v_h \in S_{h,0}^1(\Omega).$$

From (2.6) and (2.5) we then conclude

$$\begin{aligned} c_1^A \|u_{0,h} - \widetilde{u}_{0,h}\|^2_{H^1(\Omega)} &\leq \int_\Omega [\nabla u_{0,h}(x) - \nabla \widetilde{u}_{0,h}(x)] \cdot [\nabla u_{0,h}(x) - \nabla \widetilde{u}_{0,h}(x)] dx \\ &= \int_\Omega [\nabla Q_h \mathcal{E}z(x) - \mathcal{E}z(x)] \cdot [\nabla u_{0,h}(x) - \nabla \widetilde{u}_{0,h}(x)] dx \\ &\leq c_2^A \|Q_h \mathcal{E}z - \mathcal{E}z\|_{H^1(\Omega)} \|u_{0,h} - \widetilde{u}_{0,h}\|_{H^1(\Omega)} \end{aligned}$$

and therefore

$$\|u_{0,h} - \widetilde{u}_{0,h}\|_{H^1(\Omega)} \leq \frac{c_2^A}{c_1^A} \|Q_h \mathcal{E}z - \mathcal{E}z\|_{H^1(\Omega)}.$$

Then, by using the error estimates (3.3) and (3.8), we further obtain the error estimate

$$\begin{aligned} \|u_0 - \widetilde{u}_{0,h}\|_{H^1(\Omega)} &\leq \|u_0 - u_{0,h}\|_{H^1(\Omega)} + \|u_{0,h} - \widetilde{u}_{0,h}\|_{H^1(\Omega)} \\ &\leq \|u_0 - u_{0,h}\|_{H^1(\Omega)} + \frac{c_2^A}{c_1^A} \|Q_h \mathcal{E}z - \mathcal{E}z\|_{H^1(\Omega)} \\ &\leq c_1 \, h \, |u_0|_{H^2(\Omega)} + c_2 \, h \, \|\mathcal{E}z\|_{H^2(\Omega)} \\ &\leq c \, h \, \left[|u_0|_{H^2(\Omega)} + \|z\|_{H^{3/2}(\Gamma)} \right] \end{aligned} \tag{3.11}$$

when assuming $u_0 \in H^2(\Omega)$ and $z \in H^{3/2}(\Gamma)$. Moreover, by applying the Aubin–Nitsche trick we obtain an error estimate in $L_2(\Omega)$, see, e.g., [21, Lemma 11.3],

$$\|u_0 - \widetilde{u}_{0,h}\|_{L_2(\Omega)} \leq c \, h^2 \left[|u_0|_{H^2(\Omega)} + \|z\|_{H^{3/2}(\Gamma)} \right]. \tag{3.12}$$

Lemma 3.1 *For $z \in H^{3/2}(\Gamma)$ let $\widetilde{\mathcal{S}}z$ be the finite element approximation as defined in (3.9). Let Ω be either a convex Lipschitz domain, or nonconvex but with smooth boundary. Then there holds the approximation error estimate*

$$\|\mathcal{S}z - \widetilde{\mathcal{S}}z\|_{L_2(\Omega)} \leq c \, h^2 \left[|u_0|_{H^2(\Omega)} + \|z\|_{H^{3/2}(\Gamma)} \right]. \tag{3.13}$$

Proof. The assertion follows from

$$\begin{aligned}\|\mathcal{S}z - \widetilde{\mathcal{S}}z\|_{L_2(\Omega)} &= \|(u_0 + \mathcal{E}z) - (\widetilde{u}_{0,h} + Q_h\mathcal{E}z)\|_{L_2(\Omega)} \\ &\le \|u_0 - \widetilde{u}_{0,h}\|_{L_2(\Omega)} + \|(I - Q_h)\mathcal{E}z\|_{L_2(\Omega)}\end{aligned}$$

by using the L_2 error estimate (3.12) of the perturbed finite element solution $\widetilde{u}_{0,h}$ and the L_2 error estimate (3.8) of the linear quasi interpolation operator Q_h. ■

In a similar way we can define an approximate solution of the variational formulation (2.4), i.e., $u_{f,h} \in S^1_{h,0}(\Omega)$ is the unique solution of the Galerkin variational formulation

$$\int_\Omega \nabla u_{f,h}(x) \cdot \nabla v_h(x) dx = \int_\Omega f(x) v_h(x) dx \quad \text{for all } v_h \in S^1_{h,0}(\Omega) \tag{3.14}$$

satisfying the error estimate

$$\|u_f - u_{f,h}\|_{H^1(\Omega)} \le \frac{c_2^A}{c_1^A} \inf_{v_h \in S^1_{h,0}(\Omega)} \|u_f - v_h\|_{H^1(\Omega)} \le c\, h\, |u_f|_{H^2(\Omega)}$$

when assuming $u_f \in H^2(\Omega)$. Moreover, the Aubin–Nitsche trick finally gives

$$\|u_f - u_{f,h}\|_{L_2(\Omega)} \le c\, h^2\, |u_f|_{H^2(\Omega)}\,. \tag{3.15}$$

3.2 Adjoint boundary value problem

Next we consider a finite element approximation of the adjoint solution operator $\tau = \mathcal{S}^*\psi$ as defined in (2.15), i.e., of

$$\tau(x) = -\frac{\partial}{\partial n_x} p(x) \quad \text{for } x \in \Gamma,$$

where $p \in H^1_0(\Omega)$ is the unique solution of the variational problem (2.17),

$$\int_\Omega \nabla p(x) \cdot \nabla v(x) dx = \int_\Omega \psi(x) v(x) dx \quad \text{for all } v \in H^1_0(\Omega). \tag{3.16}$$

Recall, that $\tau \in H^{-1/2}(\Gamma)$ solves the variational problem (2.20),

$$\langle \tau, w\rangle_\Gamma = -\int_\Omega \nabla p(x) \cdot \nabla \mathcal{E}w(x) dx + \int_\Omega \psi(x) \mathcal{E}w(x) dx \quad \text{for all } w \in H^{1/2}(\Gamma).$$

The finite element approximation of (3.16) is to find $p_h \in S^1_{h,0}(\Omega)$ such that

$$\int_\Omega \nabla p_h(x) \cdot \nabla v_h(x) dx = \int_\Omega \psi(x) v_h(x) dx \quad \text{for all } v_h \in S^1_{h,0}(\Omega). \tag{3.17}$$

Again, we conclude the unique solvability of the Galerkin formulation (3.17) by means of Cea's lemma, as well as the quasi–optimal error estimate

$$\|p - p_h\|_{H^1(\Omega)} \,\leq\, c\,h\,|p|_{H^2(\Omega)} \,\leq\, c\,h\,\|\psi\|_{L_2(\Omega)}, \tag{3.18}$$

when assuming that Ω is either a convex Lipschitz domain, or nonconvex but with a smooth boundary (Assumption 2.1). Now we are able to define an approximation $\widetilde{\tau} = \widetilde{\mathcal{S}}^*\psi$ of $\tau = \mathcal{S}^*\psi$, $\psi \in L_2(\Omega)$, by

$$\langle \widetilde{\tau}, w\rangle_\Gamma = -\int\limits_\Omega \nabla p_h(x) \cdot \nabla \mathcal{E}w(x)dx + \int\limits_\Omega \psi(x)\mathcal{E}w(x)dx \quad \text{for all } w \in H^{1/2}(\Gamma). \tag{3.19}$$

Lemma 3.2 *For $\psi \in L_2(\Omega)$ let $\widetilde{\tau} = \widetilde{\mathcal{S}}^*\psi$ be the approximation as defined in* (3.19)*. Then, $\widetilde{S}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ is bounded, and there holds the error estimate*

$$\|(\mathcal{S}^* - \widetilde{\mathcal{S}}^*)\psi\|_{H^{-1/2}(\Gamma)} \,=\, \|\tau - \widetilde{\tau}\|_{H^{-1/2}(\Gamma)} \,\leq\, c\,h\,\|\psi\|_{L_2(\Omega)}\,. \tag{3.20}$$

Proof. By using duality, the estimate (2.5), and the inverse trace theorem (2.9), we obtain

$$\begin{aligned}
\|\widetilde{\tau}\|_{H^{-1/2}(\Gamma)} &= \sup_{0\neq w\in H^{1/2}(\Gamma)} \frac{\langle \widetilde{\tau}, w\rangle_\Gamma}{\|w\|_{H^{1/2}(\Gamma)}} \\
&= \sup_{0\neq w\in H^{1/2}(\Gamma)} \frac{-\langle \nabla p_h, \nabla\mathcal{E}w\rangle_\Omega + \langle \psi, \mathcal{E}w\rangle_\Omega}{\|w\|_{H^{1/2}(\Gamma)}} \\
&\leq \sup_{0\neq w\in H^{1/2}(\Gamma)} \frac{c_2^A \|p_h\|_{H^1(\Omega)}\|\mathcal{E}w\|_{H^1(\Omega)} + \|\psi\|_{\widetilde{H}^{-1}(\Omega)}\|\mathcal{E}w\|_{H^1(\Omega)}}{\|w\|_{H^{1/2}(\Gamma)}} \\
&\leq c\,\Big[\|p_h\|_{H^1(\Omega)} + \|\psi\|_{\widetilde{H}^{-1}(\Omega)}\Big]\,.
\end{aligned}$$

From the Galerkin formulation (3.17) we also find

$$\|p_h\|_{H^1(\Omega)} \,\leq\, \frac{1}{c_1^A}\,\|\psi\|_{H^{-1}(\Omega)},$$

and the boundedness of $\widetilde{\mathcal{S}}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ follows from the (compact) embedding of $L_2(\Omega)$ in $H^{-1}(\Omega)$, see also the proof of Lemma 2.2. In an analogue manner we also have

$$\begin{aligned}
\|\tau - \widetilde{\tau}\|_{H^{-1/2}(\Gamma)} &= \sup_{0\neq w\in H^{1/2}(\Gamma)} \frac{\langle \tau - \widetilde{\tau}, w\rangle_\Gamma}{\|w\|_{H^{1/2}(\Gamma)}} \\
&= \sup_{0\neq w\in H^{1/2}(\Gamma)} \frac{\langle \nabla p_h - \nabla p, \nabla\mathcal{E}w\rangle_\Omega}{\|w\|_{H^{1/2}(\Gamma)}} \\
&\leq \sup_{0\neq w\in H^{1/2}(\Gamma)} \frac{c_2^A\|p - p_h\|_{H^1(\Omega)}\|\mathcal{E}w\|_{H^1(\Omega)}}{\|w\|_{H^{1/2}(\Gamma)}} \,\leq\, c_2^A c_{IT}\,\|p - p_h\|_{H^1(\Omega)},
\end{aligned}$$

and the error estimate (3.20) now follows from (3.18). ■

Remark 3.1 *The use of the finite element approximation p_h to define the approximate normal derivative $\widetilde{\tau}$ results in the error estimate* (3.20)*. In particular, we obtain a linear convergence in the energy space $H^{-1/2}(\Gamma)$. However, a direct approximation of τ by using piecewise constant basis functions would result in an approximation order of* 1.5*. Indeed, in our numerical results, see Table 1, we observe a higher order convergence than predicted in the theory. Note that a similar error behavior can also be observed when considering a finite element approach with Lagrange multipliers to include Dirichlet boundary conditions in a weak sense, see, e.g.,* [21, Section 11.3].

Now we are in a position to define a finite element approximation of the operator $T_\varrho = \varrho A + \mathcal{S}^*\mathcal{S}$,

$$\widetilde{T}_\varrho := \varrho A + \widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}} \,:\, H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma). \tag{3.21}$$

Lemma 3.3 *For $z \in H^{3/2}(\Gamma)$ let $u_z = u_0 + \mathcal{E}z \in H^2(\Omega)$ be the unique solution of the homogeneous Dirichlet boundary value problem* (2.8)*, i.e., let Assumption* 2.1 *be satisfied. Then there holds the error estimate*

$$\|T_\varrho z - \widetilde{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \,\leq\, c_1\, h\, \|z\|_{H^{1/2}(\Gamma)} + c_2\, h^2\, \left[|u_0|_{H^2(\Omega)} + \|z\|_{H^{3/2}(\Gamma)} \right]. \tag{3.22}$$

Proof. By Lemma 3.2 we get

$$\begin{aligned}
\|T_\varrho z - \widetilde{T}_\varrho z\|_{H^{-1/2}(\Gamma)} &= \|\mathcal{S}^*\mathcal{S}z - \widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}}z\|_{H^{-1/2}(\Gamma)} \\
&\leq \|(\mathcal{S}^* - \widetilde{\mathcal{S}}^*)\mathcal{S}z\|_{H^{-1/2}(\Gamma)} + \|\widetilde{\mathcal{S}}^*(\mathcal{S} - \widetilde{\mathcal{S}})z\|_{H^{-1/2}(\Gamma)} \\
&\leq c_1\, h\, \|\mathcal{S}z\|_{L_2(\Omega)} + c_2\, \|(\mathcal{S} - \widetilde{\mathcal{S}})z\|_{L_2(\Omega)}\,.
\end{aligned}$$

The assertion follows by using the boundedness (2.11) and the error estimate (3.13). ∎

In the same way we may also define a finite element approximation of the right hand side $g : \mathcal{S}^*\bar{u} - u_f$,

$$\widetilde{g} \,:=\, \widetilde{\mathcal{S}}^*[\bar{u} - u_{f,h}] \in H^{-1/2}(\Gamma), \tag{3.23}$$

where $u_{f,h}$ is the finite element solution of the variational problem (3.14).

Lemma 3.4 *Let $u_f \in H^2(\Omega)$ be the unique solution of the boundary value problem* (2.3)*, i.e., let Assumption* 2.1 *be satisfied. Then there holds the error estimate*

$$\|g - \widetilde{g}\|_{H^{-1/2}(\Gamma)} \,\leq\, c_1\, h\, \left[\|\bar{u}\|_{L_2(\Omega)} + \|u_f\|_{L_2(\Omega)} \right] + c_2\, h^2\, |u_f|_{H^2(\Omega)}. \tag{3.24}$$

Proof. By the triangle inequality and by using Lemma 3.2 we obtain

$$\begin{aligned}
\|g - \widetilde{g}\|_{H^{-1/2}(\Gamma)} &= \|\mathcal{S}^*[\bar{u} - u_f] - \widetilde{\mathcal{S}}^*[\bar{u} - u_{f,h}]\|_{H^{-1/2}(\Gamma)} \\
&\leq \|(\mathcal{S}^* - \widetilde{\mathcal{S}}^*)\bar{u}\|_{H^{-1/2}(\Gamma)} + \|(\widetilde{\mathcal{S}}^* - \mathcal{S}^*)u_f\|_{H^{-1/2}(\Gamma)} + \|\widetilde{\mathcal{S}}^*(u_{f,h} - u_f)\|_{H^{-1/2}(\Gamma)} \\
&\leq c_1\, h\, \left[\|\bar{u}\|_{L_2(\Omega)} + \|u_f\|_{L_2(\Omega)} \right] + c_2\, \|u_f - u_{f,h}\|_{L_2(\Omega)},
\end{aligned}$$

and the assertion follows from the error estimate (3.15) when assuming $u_f \in H^2(\Omega)$. ∎

4 Implementation of the finite element approximation

Instead of the operator equation (2.21) we now consider the perturbed equation to find $\widetilde{z} \in H^{1/2}(\Gamma)$ such that

$$\widetilde{T}_\varrho \widetilde{z} = \widetilde{g}\,, \tag{4.1}$$

i.e., $\widetilde{z} \in H^{1/2}(\Gamma)$ is the solution of the variational problem

$$\langle \widetilde{T}_\varrho \widetilde{z}, w\rangle_\Gamma = \langle \widetilde{\mathcal{S}}^* \widetilde{\mathcal{S}} \widetilde{z}, w\rangle_\Gamma + \varrho \langle A\widetilde{z}, w\rangle_\Gamma = \langle \widetilde{\mathcal{S}}^*(\overline{u} - u_{f,h}), w\rangle_\Gamma = \langle \widetilde{g}, w\rangle_\Gamma \tag{4.2}$$

for all $w \in H^{1/2}(\Gamma)$. In this section we introduce a finite element discretization of the perturbed variational formulation (4.2). The numerical analysis, i.e., the related stability and error analysis will be given in Section 5.

Let

$$S_h^1(\Gamma) = S_h^1(\Omega)_{|\Gamma} = \mathrm{span}\{\phi_\ell\}_{\ell=1}^{M_\Gamma} \subset H^{1/2}(\Gamma)$$

be the finite element space of piecewise linear and continuous basis functions ϕ_ℓ which are the boundary traces of the domain basis functions $\varphi_{M_\Omega+\ell}$ as defined in (3.6). Then, the Galerkin formulation of the perturbed variational problem (4.2) is to find $\widetilde{z}_h \in S_h^1(\Gamma)$ such that

$$\langle \widetilde{T}_\varrho \widetilde{z}_h, w_h\rangle_\Gamma = \langle \widetilde{\mathcal{S}}^* \widetilde{\mathcal{S}} \widetilde{z}_h, w_h\rangle_\Gamma + \varrho \langle A\widetilde{z}_h, w_h\rangle_\Gamma = \langle \widetilde{\mathcal{S}}^*(\overline{u} - u_{f,h}), w_h\rangle_\Gamma \tag{4.3}$$

is satisfied for all $w_h \in S_h^1(\Gamma)$.

Before we analyze the Galerkin formulation (4.3), we first describe the matrix representation of (4.3). Let $\widetilde{z}_h \in S_h^1(\Gamma)$ be given as

$$\widetilde{z}_h(x) = \sum_{\ell=1}^{M_\Gamma} \widetilde{z}_\ell \phi_\ell(x) \quad \text{for } x \in \Gamma.$$

For the application of $\widetilde{\mathcal{S}}\widetilde{z}_h = \widetilde{u}_{0,h} + Q_h \mathcal{E} \widetilde{z}_h$ we obtain $\widetilde{u}_{0,h} \in S_{h,0}^1(\Omega)$, according to (3.5), as the unique solution of the variational problem

$$\int_\Omega \nabla \widetilde{u}_{0,h}(x) \cdot \nabla \varphi_j(x) dx = -\int_\Omega \nabla Q_h \mathcal{E} \widetilde{z}_h(x) \cdot \nabla \varphi_j(x) dx \quad \text{for } j = 1, \ldots, M_\Omega. \tag{4.4}$$

By using

$$\widetilde{u}_{0,h}(x) = \sum_{k=1}^{M_\Omega} u_{0,k} \varphi_k(x), \quad Q_h \mathcal{E} \widetilde{z}_h(x) = \sum_{k=1}^{M_\Omega} (\mathcal{E}\widetilde{z}_h)(x_k) \varphi_k(x) + \sum_{\ell=1}^{M_\Gamma} \widetilde{z}_\ell \varphi_{M_\Omega+\ell}(x),$$

and by defining the coefficients

$$u_{z,k} := u_{0,k} + (\mathcal{E}\widetilde{z}_h)(x_k) \quad \text{for } k = 1, \ldots, M_\Omega,$$

we then obtain the Galerkin equations

$$\sum_{k=1}^{M_\Omega} u_{z,k} \int_\Omega \nabla\varphi_k(x)\cdot\nabla\varphi_j(x)dx = -\sum_{\ell=1}^{M_\Gamma} \widetilde{z}_\ell \int_\Omega \nabla\varphi_{M_\Omega+\ell}(x)\cdot\nabla\varphi_j(x)dx \quad \text{for } j=1,\ldots,M_\Omega,$$

or the system of linear equations

$$K_{II}\underline{u}_z = -K_{CI}\underline{\widetilde{z}},$$

where

$$K_{II}[j,k] = \int_\Omega \nabla\varphi_k(x)\cdot\nabla\varphi_j(x)dx, \quad K_{CI}[j,\ell] = \int_\Omega \nabla\varphi_{M_\Omega+\ell}\cdot\nabla\varphi_j(x)dx$$

for $k,j=1,\ldots,M_\Omega$, $\ell=1,\ldots,M_\Gamma$.

Next we consider the application of $\widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}}\widetilde{z}_h$ within the Galerkin formulation (4.3). In particular, for the test function $w_h=\phi_j$ we have to evaluate, due to (3.19),

$$\begin{aligned}\langle\widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}}\widetilde{z}_h,\phi_j\rangle_\Gamma &= \langle\widetilde{\mathcal{S}}^*(\widetilde{u}_{0,h}+Q_h\mathcal{E}\widetilde{z}_h),\phi_j\rangle_\Gamma \\ &= -\int_\Omega \nabla p_h(x)\cdot\nabla\varphi_{M_\Omega+j}(x)dx + \int_\Omega [\widetilde{u}_{0,h}(x)+Q_h\mathcal{E}\widetilde{z}_h(x)]\varphi_{M_\Omega+j}(x)dx\end{aligned}$$

for $j=1,\ldots,M_\Gamma$, where $p_h\in S^1_{h,0}(\Omega)$ is the unique solution of the variational problem

$$\int_\Omega \nabla p_h(x)\cdot\nabla v_h(x)dx = \int_\Omega [\widetilde{u}_{0,h}(x)+Q_h\mathcal{E}\widetilde{z}_h(x)]v_h(x)dx \quad \text{for all } v_h\in S^1_{h,0}(\Omega).$$

By choosing $v_h=\varphi_j$ for $j=1,\ldots,M_\Omega$, the latter is equivalent to

$$\sum_{k=1}^{M_\Omega} p_k \int_\Omega \nabla\varphi_k(x)\cdot\nabla\varphi_j(x)dx = \sum_{k=1}^{M_\Omega} u_{z,k}\int_\Omega \varphi_k(x)\varphi_j(x)dx + \sum_{\ell=1}^{M_\Gamma}\widetilde{z}_\ell\int_\Omega \varphi_{M_\Omega+\ell}(x)\varphi_j(x)dx,$$

or

$$K_{II}\underline{p} = M_{II}\underline{u}_z + M_{CI}\underline{\widetilde{z}}\,,$$

where

$$M_{II}[j,k] = \int_\Omega \varphi_k(x)\varphi_j(x)dx, \quad M_{CI}[j,\ell] = \int_\Omega \varphi_{M_\Omega+\ell}(x)\varphi_j(x)dx$$

for $k,j=1,\ldots,M_\Omega$, $\ell=1,\ldots,M_\Gamma$. Hence we can write

$$\begin{aligned}\langle\widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}}\widetilde{z}_h,\phi_j\rangle_\Gamma &= -\sum_{k=1}^{M_\Omega} p_k\int_\Omega \nabla\varphi_k(x)\cdot\nabla\varphi_{M_\Omega+j}(x)dx + \sum_{k=1}^{M_\Omega} u_{z,k}\int_\Omega \varphi_k(x)\varphi_{M_\Omega+j}(x)dx \\ &\quad + \sum_{\ell=1}^{M_\Gamma}\widetilde{z}_\ell\int_\Omega \varphi_{M_\Omega+\ell}(x)\varphi_{M_\Omega+j}(x)dx \\ &= -\sum_{k=1}^{M_\Omega} p_k K_{IC}[j,k] + \sum_{k=1}^{M_\Omega} u_{z,k}M_{IC}[j,k] + \sum_{\ell=1}^{M_\Gamma}\widetilde{z}_\ell M_{CC}[j,\ell]\,.\end{aligned}$$

Therefore, the Galerkin discretization of the approximate operator $\widetilde{T}_\varrho$ results in the matrix representation

$$\widetilde{T}_{\varrho,h}\underline{\widetilde{z}} = -K_{IC}\underline{p} + M_{IC}\underline{u}_z + M_{CC}\underline{\widetilde{z}} + \varrho A_h\underline{\widetilde{z}}$$

where

$$K_{II}\underline{u}_z = -K_{CI}\underline{\widetilde{z}}, \quad K_{II}\underline{p} = M_{II}\underline{u}_z + M_{CI}\underline{\widetilde{z}}$$

and

$$M_{CC}[j,\ell] = \int\limits_\Omega \varphi_{M_\Omega+\ell}(x)\varphi_{M_\Omega+j}(x)dx, \quad A_h[j,\ell] = \langle A\phi_\ell, \phi_j\rangle_\Gamma, \quad j,\ell = 1,\ldots,M_\Gamma.$$

Since the finite element stiffness matrix K_{II} is invertible, we finally obtain

$$\widetilde{T}_{\varrho,h}\underline{\widetilde{z}} = \left[K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}K_{CI} - K_{IC}K_{II}^{-1}M_{CI} - M_{IC}K_{II}^{-1}K_{CI} + M_{CC} + \varrho A_h\right]\underline{\widetilde{z}}. \quad (4.5)$$

Next we describe the discrete representation of the right hand side in the perturbed variational formulation (4.3). For $j = 1,\ldots,M_\Gamma$ we have, due to (3.19),

$$\langle \widetilde{\mathcal{S}}^*[\overline{u} - u_{f,h}], \phi_j\rangle_\Gamma = -\int\limits_\Omega \nabla q_h(x)\cdot\nabla\varphi_{M_\Omega+j}(x)dx + \int\limits_\Omega [\overline{u}(x) - u_{f,h}(x)]\varphi_{M_\Omega+j}(x)dx,$$

where $q_h \in S^1_{h,0}(\Omega)$ is the unique solution of the variational problem

$$\int\limits_\Omega \nabla q_h(x)\cdot\nabla v_h(x)dx = \int\limits_\Omega [\overline{u}(x) - u_{f,h}(x)]v_h(x)dx \quad \text{for all } v_h \in S^1_{h,0}(\Omega).$$

Since $u_{f,h} \in S^1_{h,0}(\Omega)$ is the unique solution of the Galerkin formulation (3.14), we find

$$K_{II}\underline{u}_f = \underline{f}_I, \quad \underline{u}_f = K_{II}^{-1}\underline{f}_I,$$

where

$$u_{f,h}(x) = \sum_{k=1}^{M_\Omega} u_{f,k}\varphi_k(x), \quad f_{I,j} = \int\limits_\Omega f(x)\varphi_j(x)dx \quad \text{for } j = 1,\ldots,M_\Omega.$$

Hence we obtain

$$K_{II}\underline{q} = \underline{g}_I - M_{II}\underline{u}_f = \underline{g}_I - M_{II}K_{II}^{-1}\underline{f}_I, \quad \underline{q} = K_{II}^{-1}\underline{g}_I - K_{II}^{-1}M_{II}K_{II}^{-1}\underline{f}_I,$$

where

$$g_{I,j} = \int\limits_\Omega \overline{u}(x)\varphi_j(x)dx \quad \text{for } j = 1,\ldots,M_\Omega.$$

By using

$$g_{C,j} = \int\limits_\Omega \overline{u}(x)\varphi_{M_\Omega+j}(x)dx \quad \text{for } j = 1,\ldots,M_\Gamma$$

we then conclude

$$\langle \widetilde{\mathcal{S}}^*[\overline{u} - u_{f,h}], \phi_j\rangle_\Gamma = -\sum_{k=1}^{M_\Omega} q_k K_{IC}[j,k] + g_{C,j} - \sum_{k=1}^{M_\Omega} u_{f,k} M_{IC}[j,k],$$

and the right hand side of the linear system associated with (4.3) is given by

$$\underline{g}_C - K_{IC}\underline{p} - M_{IC}\underline{u}_f \;=\; \underline{g}_C - K_{IC}K_{II}^{-1}\underline{g}_I + K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}\underline{f}_I - M_{IC}K_{II}^{-1}\underline{f}_I\,.$$

Therefore we have to solve the linear system

$$\begin{aligned}\left[K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}K_{CI} - K_{IC}K_{II}^{-1}M_{CI} - M_{IC}K_{II}^{-1}K_{CI} + M_{CC} + \varrho A_h\right]\underline{\widetilde{z}}& \\ = \underline{g}_C - K_{IC}K_{II}^{-1}\underline{g}_I + K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}\underline{f}_I - M_{IC}K_{II}^{-1}\underline{f}_I&.\end{aligned} \tag{4.6}$$

Note that (4.6) is the Schur complement of a coupled system, i.e., by setting

$$\underline{u}_I = K_{II}^{-1}(\underline{f}_I - K_{CI}\underline{\widetilde{z}}), \quad \underline{p} = K_{II}^{-1}(M_{II}\underline{u}_I + M_{CI}\underline{\widetilde{z}} - \underline{g}_I),$$

we obtain

$$\begin{pmatrix} -M_{II} & K_{II} & -M_{CI} \\ K_{II} & & K_{CI} \\ M_{IC} & -K_{IC} & M_{CC} + \varrho A_h \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{p} \\ \underline{\widetilde{z}} \end{pmatrix} = \begin{pmatrix} -\underline{g}_I \\ \underline{f}_I \\ \underline{g}_C \end{pmatrix}. \tag{4.7}$$

Note that (4.7) would also result from a direct finite element approximation of the adjoint Dirichlet boundary value problem (2.16), of the primal Dirichlet boundary value problem (2.2), and of the optimality condition (2.15). However, we will use the approach as presented above to proceed with a related stability and error analysis .

5 Stability and error analysis

For the numerical analysis of the perturbed variational formulation (4.3) we will apply the Strang lemma, see, e.g., [5, 21]. For this, we first need to establish the $S_h^1(\Gamma)$–ellipticity of the approximate operator $\widetilde{T}_\varrho$.

Lemma 5.1 *The approximate operator $\widetilde{T}_\varrho$ as defined in* (4.2) *is $S_h^1(\Gamma)$–elliptic. In particular, the matrix*

$$\widetilde{T}_{\varrho,h} = K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}K_{CI} - K_{IC}K_{II}^{-1}M_{CI} - M_{IC}K_{II}^{-1}K_{CI} + M_{CC} + \varrho A_h$$

is positive definite, i.e., for all $z_h \in S_h^1(\Gamma) \leftrightarrow \underline{z} \in \mathbb{R}^{M_\Gamma}$ there holds

$$\langle \widetilde{T}_\varrho z_h, z_h\rangle_\Gamma = (\widetilde{T}_{\varrho,h}\underline{z},\underline{z}) \;\geq\; \varrho(A_h\underline{z},\underline{z}) \;=\; \varrho\,\|z_h\|_A^2 \;\geq\; c_1^T\,\|z_h\|^2_{H^{1/2}(\Gamma)}.$$

Proof. For $\underline{z} \in \mathbb{R}^{M_\Gamma}$ and by defining

$$\underline{u} = -K_{II}^{-1} K_{CI} \underline{z}$$

we have

$$\begin{aligned}
(\widetilde{T}_{\varrho,h}\underline{z}, \underline{z}) &= ([K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}K_{CI} - K_{IC}K_{II}^{-1}M_{CI} - M_{IC}K_{II}^{-1}K_{CI} + M_{CC} + \varrho A_h]\underline{z}, \underline{z}) \\
&= (M_{II}K_{II}^{-1}K_{CI}\underline{z}, K_{II}^{-1}K_{CI}\underline{z}) - (M_{CI}\underline{z}, K_{II}^{-1}K_{CI}\underline{z}) - (M_{IC}K_{II}^{-1}K_{CI}\underline{z}, \underline{z}) \\
&\qquad + (M_{CC}\underline{z}, \underline{z}) + \varrho(A_h\underline{z}, \underline{z}) \\
&= (M_{II}\underline{u}, \underline{u}) + (M_{CI}\underline{z}, \underline{u}) + (M_{IC}\underline{u}, \underline{z}) + (M_{CC}\underline{z}, \underline{z}) + \varrho(A_h\underline{z}, \underline{z}) \\
&= \left(\begin{pmatrix} M_{II} & M_{CI} \\ M_{IC} & M_{CC} \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix}, \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix} \right) + \varrho(A_h\underline{z}, \underline{z}) \;\geq\; \varrho(A_h\underline{z}, \underline{z}),
\end{aligned}$$

since the mass matrix

$$M_h = \begin{pmatrix} M_{II} & M_{CI} \\ M_{IC} & M_{CC} \end{pmatrix}$$

is positive definite. ∎

Note that Lemma 5.1 ensures the unique solvability of the linear system (4.6) and therefore of the perturbed variational problem (4.3). Since the approximate operator $\widetilde{T}_\varrho$ is $S_h^1(\Gamma)$–elliptic, an error estimate for the approximate solution $\widetilde{z}_h$ of the perturbed Galerkin variational problem (4.3) follows from the Strang lemma, see, e.g., [21, Theorem 8.2, Theorem 8.3].

Theorem 5.2 *Let z be the unique solution of the operator equation* (2.21) *minimizing the reduced cost functional* (2.13). *Let $\widetilde{z}_h$ be the unique solution of the perturbed variational problem* (4.3). *Then there holds the error estimate*

$$\|z - \widetilde{z}_h\|_{H^{1/2}(\Gamma)} \leq c_1 \inf_{w_h \in S_h^1(\Gamma)} \|z - w_h\|_{H^{1/2}(\Gamma)} + c_2 \|(T_\varrho - \widetilde{T}_\varrho)z\|_{H^{-1/2}(\Gamma)} + c_3 \|g - \widetilde{g}\|_{H^{-1/2}(\Gamma)}. \quad (5.1)$$

When combining the error estimate (5.1) with the approximation property of the ansatz space $S_h^1(\Gamma)$, and with the error estimates (3.22) and (3.24), we finally obtain the error estimate

$$\begin{aligned}
\|z - \widetilde{z}_h\|_{H^{1/2}(\Gamma)} \;\leq\;& c_1 h^{3/2} |z|_{H^2(\Gamma)} + c_2 h \|z\|_{H^{1/2}(\Gamma)} + c_3 h^2 \left[|u_0|_{H^2(\Omega)} + \|z\|_{H^{3/2}(\Gamma)} \right] \\
& + c_4 h \left[\|\bar{u}\|_{L_2(\Omega)} + \|u_f\|_{L_2(\Omega)} \right] + c_5 h^2 |u_f|_{H^2(\Omega)} \qquad (5.2)
\end{aligned}$$

In particular, we can expect linear convergence when considering the error in the energy norm, i.e.,

$$\|z - \widetilde{z}_h\|_{H^{1/2}(\Gamma)} \;\leq\; c(z, \overline{u}, f)\, h\,. \quad (5.3)$$

Moreover, applying the Aubin–Nitsche trick we are also able to derive an error estimate in $L_2(\Gamma)$, i.e.,

$$\|z - \widetilde{z}_h\|_{L_2(\Gamma)} \;\leq\; c(z, \overline{u}, f)\, h^{3/2}\,. \quad (5.4)$$

If the approximate control $\widetilde{z}_h$ is known, i.e., $\underline{\widetilde{z}}$, the related state can be obtained from the second equation in (4.7),

$$K_{II}\underline{u}_I = \underline{f}_I - K_{CI}\underline{\widetilde{z}}.$$

In particular, $\underline{u}_I \in \mathbb{R}^{M_\Omega} \leftrightarrow \widetilde{u}_{0,h} \in S_h^1(\Omega)$ is the unique solution of the variational problem (4.4), i.e.,

$$\int_\Omega \nabla \widetilde{u}_{0,h}(x) \cdot \nabla v_h(x) dx = -\int_\Omega \nabla Q_h \mathcal{E} \widetilde{z}_h(x) \cdot \nabla v_h(x) dx \quad \text{for all } v_h \in S_{h,0}^1(\Omega).$$

Theorem 5.3 *Let* $u_0 \in H^2(\Omega)$ *be the unique solution of the variational problem* (3.1), *and let* $\widetilde{u}_{0,h}$ *be the unique solution of the perturbed variational problem* (4.4). *Then there holds the error estimate*

$$\|u_0 - \widetilde{u}_{0,h}\|_{H^1(\Omega)} \le c_1 h |u_0|_{H^2(\Omega)} + c_2(z, \overline{u}, f) h + c_3 h \|z\|_{H^{3/2}(\Gamma)}. \tag{5.5}$$

Proof. When subtracting (4.4) from the variational formulation (3.2), we obtain the perturbed Galerkin orthogonality

$$\int_\Omega \nabla [u_{0,h} - \widetilde{u}_{0,h}](x) \cdot \nabla v_h(x) dx = \int_\Omega \nabla [Q_h \mathcal{E} \widetilde{z}_h - \mathcal{E} z(x)] \cdot \nabla v_h(x) dx \quad \text{for all } v_h \in S_{h,0}^1(\Omega).$$

For $v_h = u_{0,h} - \widetilde{u}_{0,h}$ we therefore obtain by using the boundedness (3.7), the inverse trace theorem (2.9), and the error estimate (3.8)

$$\begin{aligned} \|u_{0,h} - \widetilde{u}_{0,h}\|_{H^1(\Omega)} &\le \frac{c_2^A}{c_1^A} \|Q_h \mathcal{E} \widetilde{z}_h - \mathcal{E} z\|_{H^1(\Omega)} \\ &\le \frac{c_2^A}{c_1^A} \left[\|Q_h \mathcal{E}(\widetilde{z}_h - z)\|_{H^1(\Omega)} + \|(I - Q_h)\mathcal{E} z\|_{H^1(\Omega)} \right] \\ &\le c_1 \|z - \widetilde{z}_h\|_{H^{1/2}(\Gamma)} + c_2 h |\mathcal{E} z|_{H^2(\Omega)}. \end{aligned}$$

The assertion now follows from the triangle inequality, and from the error estimates (3.3) and (5.3). ■

In particular we can expect linear convergence when considering the error for the approximate state $\widetilde{u}_h := \widetilde{u}_{0,h} + Q_h \mathcal{E} \widetilde{z}_h$ in the energy norm,

$$\|u - \widetilde{u}_h\|_{H^1(\Omega)} \le c(z, \bar{u}, f) h. \tag{5.6}$$

Moreover, applying the Aubin–Nitsche trick we are also able to derive an error estimate in $L_2(\Omega)$, i.e.,

$$\|u - \widetilde{u}_h\|_{L_2(\Omega)} \le c(z, \bar{u}, f) h^2. \tag{5.7}$$

6 Numerical results

As numerical example we consider the Dirichlet boundary control problem (2.1) and (2.2) for the domain $\Omega = (0, \frac{1}{2})^2 \subset \mathbb{R}^2$ where

$$\overline{u}(x) = -\left(4 + \frac{1}{\alpha}\right)[x_1(1-2x_1) + x_2(1-2x_2)], \quad f(x) = -\frac{8}{\alpha}, \quad \alpha = 0.01\,.$$

We introduce a uniform triangulation of Ω on several levels L where the mesh size is $h_L = 2^{-(L+1)}$ for the finite element discretization. Since the minimizer of (2.1) is not known in this case, we use the finite element solutions (z_{h_9}, u_{h_9}) on the 9th level as reference solutions.

L	$\|u_{h_L} - u_{h_9}\|_{L_2(\Omega)}$	eoc	$\|z_{h_L} - z_{h_9}\|_{L_2(\Gamma)}$	eoc
2	9.98 –2		3.89 –1	
3	2.78 –2	1.84	1.07 –1	1.86
4	7.46 –3	1.90	2.81 –2	1.93
5	1.93 –3	1.95	7.28 –3	1.95
6	4.89 –4	1.98	1.87 –3	1.96
7	1.20 –4	2.03	4.69 –4	2.00
8	2.69 –5	2.16	1.06 –4	2.15

Table 1: Finite element errors for Dirichlet optimal control problem.

In Table 1 we give the finite element errors for the primal variable u in the $L_2(\Omega)$ norm, and for the control variable z in the $L_2(\Gamma)$ norm. While for the primal variable u we observe a quadratic convergence behavior as predicted in the error estimate (5.7), we also observe a quadratic convergence for the control variable z. Note that we may only expect a convergence rate of 1.5 due to the error estimate (5.4) which may also result in a reduced convergence

This higher order convergence behavior may be explained by the error estimate (5.2) which includes several terms of different orders; it is not obvious, which term is dominant. Note that the reduced convergence rate is due to the approximation of the normal derivative τ which was based on the use of the finite element solution p_h, see also Remark 3.1.

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