Optimization of nonlocal distributed feedback controllers with time delay

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Mathematics for key technologies

SFB 910

Joint work with

Peter Nestler and Eckehard Schöll

1D Schlögl model (Nagumo equation)

1D Schlögl model

$$\begin{aligned} \partial_t y - \partial_{xx} y + R(y) &= u && (x,t) \in Q := (a,b) \times (0,T). \\ y(x,0) &= y_0(x), && x \in (a,b) \\ \partial_x y(a,t) = \partial_x y(b,t) &= 0, && t \in (0,T). \end{aligned}$$

with control function ("forcing") $u = u(x,t)$ and cubic reaction term

$$R(y) = \rho\,(y - y_1)(y - y_2)(y - y_3),$$

$\rho > 0,\ y_1 \le y_2 \le y_3.$

Optimal (open loop) control

By optimal control, the state y is controlled in a desired way. One might be interested in approximating a desired state $\widehat{y}$ by an optimal control u:

Optimal control problem

$$\min_{u} J(y_u) := \frac{1}{2} \iint_Q (y_u(x,t) - \widehat{y}(x,t))^2 \, dxdt$$

where y_u is the unique solution of

$$\partial_t y - \partial_{xx} y + R(y) = u$$

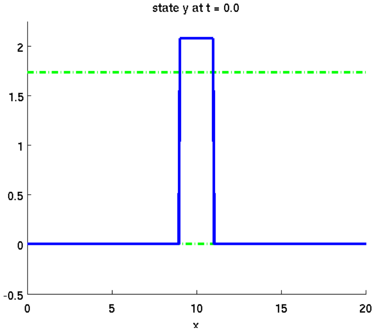
subject to given initial - and boundary conditions and certain constraints on u.

This is a problem of open loop control that some theoretical physicists call "optimal forcing".

Uncontrolled "natural" wave front, $u = 0$

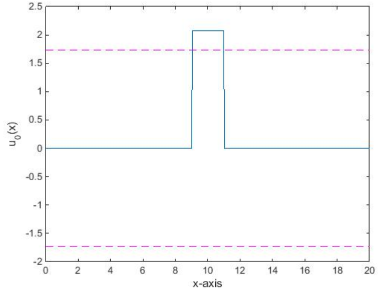
$$R = \frac{1}{3}y^3 - y = \frac{1}{3}(y + \sqrt{3})y(y - \sqrt{3}), \quad (a,b) = (0,L) = (0,20)$$

$$y_0(x) = \begin{cases} 1.2\sqrt{3}, & x \in [9,11] \\ 0, & \text{else.} \end{cases}$$

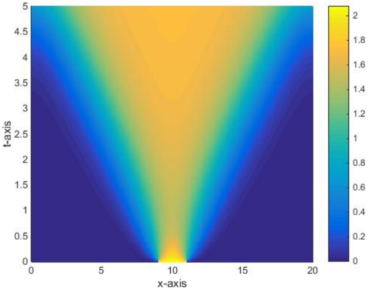


Two propagating fronts

Different visualization



Initial state



Uncontrolled wave fronts

Time delayed feedback control

In theoretical physics, in particular related to lasers, a *nonlocal coupling* of the control u with the state y by time-delayed feedback is popular.

Two particular options:

$$
\begin{aligned}
u(x,t) &= \kappa\,(y(x,t-\tau) - y(x,t)) && \text{"Pyragas type"} \\
u(x,t) &= \kappa \left(\int_0^\infty g(\tau) y(x,t-\tau)\, d\tau - y(x,t) \right) && \text{"Nonlocal time-delayed".}
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We concentrate on a *finite interval of time* and the *nonlocal version*:

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u(x,t) = \kappa\left(\int_0^T g(\tau)y(x,t-\tau)d\tau - y(x,t)\right).
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$$u(x,t) = \kappa \left(\int_0^T g(\tau) y(x,t-\tau)\,d\tau - y(x,t) \right).$$

We will often suppress the dependence on x. Notice, however, that $g = g(t)$ does not depend on the spatial variable.

Feedback system

Related feedback system

$$\begin{aligned}
\partial_t y(x,t) - \partial_{xx} y(x,t) + R(y(x,t)) &= \kappa \left(\int_0^T g(\tau) y(x, t-\tau) d\tau - y(x,t) \right) \\
y(x,s) &= y_0(x,s), \qquad s \le 0,\ x \in \Omega, \\
\partial_x y(a,t) &= \partial_x y(b,t) = 0, \quad t \in (0,T).
\end{aligned}$$

Some references

J. Löber, R. Coles, J. Siebert, H. Engel, E. Schöll,
Control of chemical wave propagation in Engineering of Chemical Complexity II.
A. S. Mikhailov, G. Ertl (Eds.), World Scientific, Singapore, 2014.

J. Siebert, S. Alonso, M. Bär, E. Schöll,
Dynamics of reaction-diffusion patterns controlled by asymmetric nonlocal coupling as limiting case of differential advection.
Phys. Rev. E 89, 052909 (2014).

J. Siebert, E. Schöll,
Front and Turing patterns induced by Mexican-hat-like nonlocal feedback.
arXiv 1411.6561 (2014).

Forward problem: $g \mapsto y$

Depending on the chosen feedback kernel g, different solutions y are generated. We numerically confirmed some results by Löber et al. (2014).

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"Weak gamma delay kernel" $g(t) = e^{-t}$, $y_1 = 0,\ y_3 = 1$

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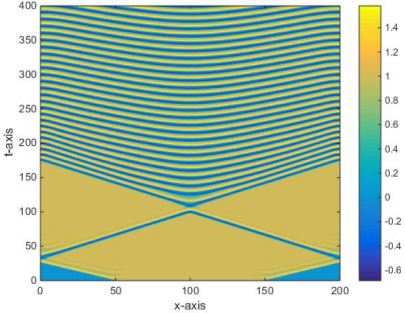
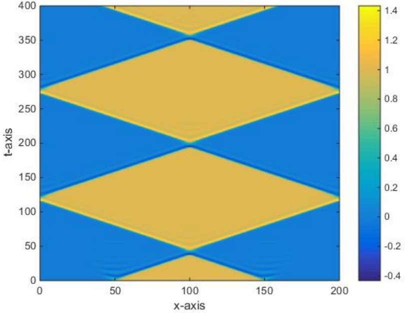


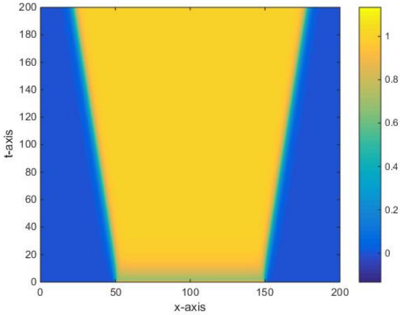
$y_2 = 0.25$, $\kappa = -1.65$



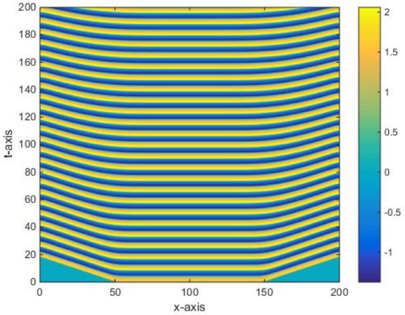
$y_2 = 0.5$, $\kappa = -1.4$

$\Omega = (0, 200)$, $T = 200$, y_0: Natural wave starting from a step function.

"Strong gamma delay kernel" $g(t) = t\, e^{-t}$



$y_2 = 0,\ \kappa = 2$



$y_2 = 0,\ \kappa = -2$

Design problem: $\widehat{y} \mapsto g$

In the *forward problem*, we computed the function y associated with a given kernel g. In the *design problem*, this is reversed:

Find a kernel g such that the solution y_g associated with g is as close as possible to a given desired function $\widehat{y}$.

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Related optimal control problem: "Optimize the Controller"

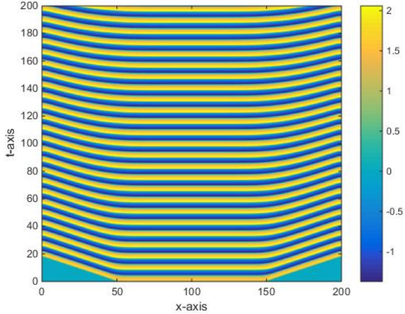
$$\min_{g \in C} \frac{1}{2} \iint_Q (y_g - \widehat{y})^2 \, dxdt$$

$$\left\{ \begin{array}{rcl} \partial_t y(t) - \partial_{xx} y(t) + R(y(t)) & = & \kappa \int_0^T g(\tau) y(t-\tau) d\tau - \kappa\, y(t) \\ y(x,s) & = & y_0(x,s), \; s \le 0, \; x \in \Omega \\ \partial_x y(a,t) & = & \partial_x y(b,t) = 0, \end{array} \right.$$

where $C = \{g \in L^\infty(0,T) : 0 \le g(\tau) \le \beta, \quad \int_0^T g(\tau)\, d\tau = 1\}.$

Design problem: $\widehat{y} \mapsto g$

We must select realistic patterns $\widehat{y}$ . . .



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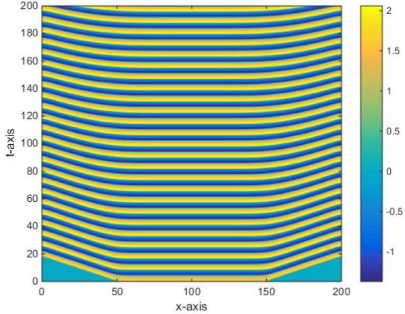

Realistic

Unrealistic

Well-posedness of the problem

Theorem (Control-to-state mapping)

For each $g \in L^\infty(0,T)$ and each $y_0 \in C(\bar\Omega \times [-T,0])$, the feedback equation has a unique weak solution $y_g \in C(\overline{Q})$. The mapping $g \mapsto y_g$ is of class C^∞.

Idea of the proof:

$$
\begin{aligned}
\partial_t y + \partial_{xx} u + R(y) + \kappa\, y &= \kappa \int_0^T g(\tau) y(x, t-\tau)\, d\tau \\
&= \kappa \int_0^t g(\tau) y(x, \underbrace{t-\tau}_{s})\, d\tau + \underbrace{\kappa \int_t^T g(\tau) y_0(x, t-\tau)\, d\tau}_{Y_g(x,t)} \\
&= \kappa \int_0^t g(t-s) y(x,s)\, ds + Y_g(x,t) \\
&= K(g)\, y + Y_g.
\end{aligned}
$$

Control-to-state mapping

Next, we substitute $y = e^{\lambda t} v$ and get

$$\partial_t v + \partial_{xx} v + e^{-\lambda t} R(e^{\lambda t} v) + (\lambda + \kappa)\, v - e^{-\lambda t} K(g)(e^{\lambda \cdot} v) = e^{-\lambda t} Y_g.$$

If λ is sufficiently large, the mapping in blue behaves like a monotone mapping. Now we proceed as in

E. Casas, C. Ryll, F. Tröltzsch,
Sparse optimal control of the Schlögl and FitzHugh-Nagumo systems,
CMAM, 2013.

We get a unique v_g and the differentiability of the mapping $g \mapsto v_g$. $\square$

Existence of an optimal kernel

Corollary (Existence)

The problem of optimal feedback design is solvable, i.e. there exists at least one optimal kernel $\bar{g} \in C$.

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The problem of optimal feedback design is solvable, i.e. there exists at least one optimal kernel $\bar{g} \in C$.

We have associated necessary conditions. However, from now on we concentrate on a special choice of the kernel g. We optimize with respect to a particular class of step functions g.

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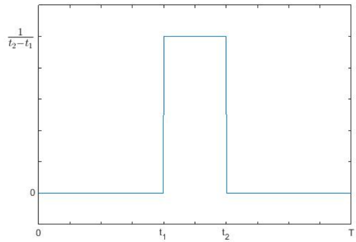
A. S. Mikhailov, G. Ertl (Eds.), World Scientific, Singapore, 2014.

Class of step functions

Class of kernels g: Select $0 \leq t_1 < t_2 \leq T$;

$$g(\tau) = \begin{cases} \dfrac{1}{t_2 - t_1}, & t_1 \leq \tau \leq t_2, \\ 0, & \text{else.} \end{cases}$$



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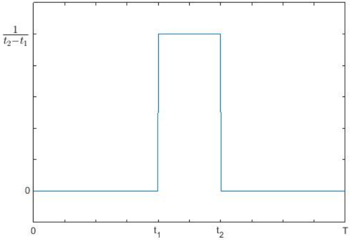

Nonlocal feedback

$$\partial_t y(x,t) - \partial_{xx} y(x,t) + R(y(x,t)) = \frac{\kappa}{t_2 - t_1} \int_{t_1}^{t_2} y(x, t-\tau)\, d\tau - \kappa\, y(x,t)$$

Here, κ, t_1, t_2 are our control parameters to be optimized.

Structure of the optimal "control" problem

$$\partial_t y(x,t) - \partial_{xx} y(x,t) + R(y(x,t)) = \frac{\kappa}{t_2 - t_1}\int_{t_1}^{t_2} y(x,t-\tau)\,d\tau - \kappa\, y(x,t)$$

- The state y is uniquely determined by the parameter vector (κ, t_1, t_2) (and by the initial data y_0, but we keep them fixed).
- We indicate this by $y = y_{(\kappa,t_1,t_2)}$.
- Altogether, we obtain the

Objective function

$$F(\kappa, t_1, t_2) := \frac{1}{2}\iint_Q (y_{(\kappa,t_1,t_2)} - \widehat{y})^2\, dxdt + \frac{\nu}{2}(\kappa^2 + t_1^2 + t_2^2)$$

with regularization parameter $\nu \geq 0$.

Optimization problem 1

Nonlinear optimization problem (P_1)

$$\min_{(\kappa, t_1, t_2) \in C_\delta} F(\kappa, t_1, t_2) \tag{P1}$$

where

$$C_\delta = \{(\kappa, t_1, t_2) \in \mathbb{R}^3 : \; 0 \le t_1 \le t_2 \le T, \quad t_2 - t_1 \ge \delta\}$$

with a (small) distance$\delta > 0$.

Linearized equations

Let $z := \partial_{t_1} y_{(\kappa,t_1,t_2)}$, $y := y_{(\kappa,t_1,t_2)}$. Then z is the unique solution of the linearized equation

$$\partial_t z - \Delta z + R'(y)z + \kappa\, z = \frac{\partial}{\partial t_1}\left(\frac{\kappa}{t_2 - t_1}\int_{t_1}^{t_2} y_{(\kappa,t_1,t_2)}(x, t-\tau)\, d\tau\right)$$

$$= \frac{\kappa}{t_2 - t_1}\left(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} y(x, t-\tau)\, d\tau - y(x, t - t_1) + \int_{t_1}^{t_2} z(x, t-\tau)\, d\tau\right),$$

$$z(x, s) = 0, \quad s \in [-T, 0], \; x \in \Omega,$$

subject to homogeneous Neumann boundary conditions.

Similar equations are obtained for $\partial_{t_2} y_{(\kappa,t_1,t_2)}$ and $\partial_{\kappa} y_{(\kappa,t_1,t_2)}$.

Adjoint equation

For necessary optimality conditions we introduce an adjoint equation:

Adjoint equation

$$
\begin{aligned}
-\partial_t\varphi(x,t) - \partial_{xx}\varphi(x,t) + R'(y_{(\kappa,t_1,t_2)}(x,t))\,\varphi(x,t)& \\
= \frac{\kappa}{t_2 - t_1}\int_0^T \varphi(x,t+\tau)\,d\tau - \kappa\,\varphi(x,t) + y_{(\kappa,t_1,t_2)}(x,t) - \widehat{y}(x,t),&
\end{aligned}
$$

$$\varphi(x,s) = 0, \quad s \in [T, 2T],$$

$$\partial_x\varphi(a,t) = \partial_x\varphi(b,t) = 0,$$

$$x \in \Omega, t \in (0,T).$$

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$$x \in \Omega, t \in (0,T).$$

This is again an equation with time delay and a nonlocal term.

Main steps in adjoining

Let z be the solution of the linearized state equation. Then $z(x,t) = 0,\ t \leq 0$.

$$(K(\bar g)z\,,\,\varphi)_{L^2(Q)} = \iint_Q \int_0^T \bar g(\tau)\,\underbrace{z(x,t-\tau)}_{=0,\text{ if } t\leq\tau}\,\bar\varphi(x,t)\,d\tau dt\,dx$$

$$= \iint_Q \int_0^t \bar g(\tau)\,z(x,\underbrace{t-\tau}_{\eta})\,\bar\varphi(x,t)\,d\tau dt\,dx = \iint_Q \int_0^t \bar g(t-\eta)\,z(x,\eta)\,\bar\varphi(x,t)\,d\eta dt\,dx$$

$$= \iint_Q \int_\eta^T \bar g(\underbrace{t-\eta}_{\sigma})\,\bar\varphi(x,t)\,dt\,z(x,\eta)\,d\eta\,dx = \iint_Q \int_0^{T-\eta} \bar g(\sigma)\,\underbrace{\bar\varphi(x,\eta+\sigma)}_{\varphi(x,t):=0,\ t\geq T}\,d\sigma z(x,\eta)d\eta dx$$

$$= \iint_Q \int_0^T \bar g(\sigma)\,\bar\varphi(x,\eta+\sigma)\,d\sigma\,z(x,\eta)\,d\eta\,dx = \iint_Q \int_0^T \bar g(\tau)\,\bar\varphi(x,t+\tau)\,z(x,t)\,d\tau\,dxdt$$

$$= (K^*(\bar g)\varphi), z)_{L^2(Q)}$$

Gradient of F

Theorem (Partial derivatives of F)

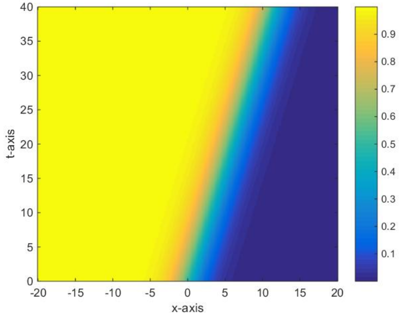
Let (κ, t_1, t_2) be given, $y := y_{(\kappa,t_1,t_2)}$ be the associated state, and let $\varphi := \varphi_{(\kappa,t_1,t_2)}$ be the associated adjoint state. Then the partial derivatives $\partial_{t_i} F(\kappa, t_1, t_2)$, $i = 1, 2$, and $\partial_\kappa F(\kappa, t_1, t_2)$ can be determined as follows:

$$
\begin{aligned}
\partial_{t_1} F &= \nu t_1 + \frac{\kappa}{t_2 - t_1} \iint_Q \varphi(x,t) \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} y(x, t-\tau)\,d\tau - y(x, t - t_1) \right] dxdt \\
\partial_{t_2} F &= \nu t_2 - \frac{\kappa}{t_2 - t_1} \iint_Q \varphi(x,t) \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} y(x, t-\tau)\,d\tau - y(x, t - t_2) \right] dxdt \\
\partial_{\kappa} F &= \nu \kappa + \iint_Q \varphi(x,t) \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} y(x, t-\tau)\,d\tau - y(x, t) \right] dxdt.
\end{aligned}
$$

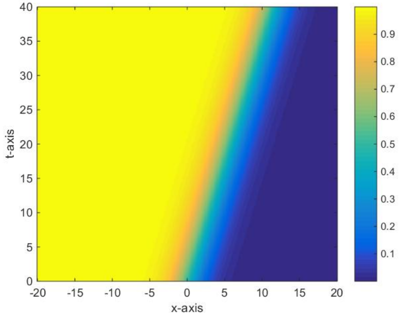
Example 1 (Test)

Spatial interval: $\Omega = (-20, 20)$, Time interval: $[0, 40]$

$$y_0(x,t) := \frac{1}{2}\left(1 - \tanh\left(\frac{2(x - vt)}{4\sqrt{2}}\right)\right), \; v = \frac{1 - 2y_2}{\sqrt{2}}, \; x \in \Omega, \; t \le 0.$$



Desired $\widehat{y}$ (pre-computed)



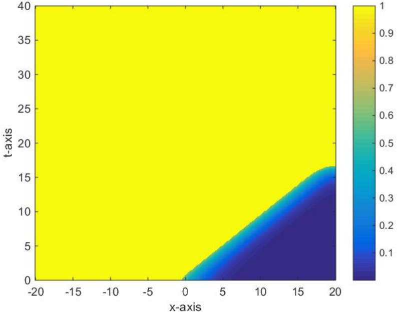
Optimal pattern ($\widehat{y}$ recovered)

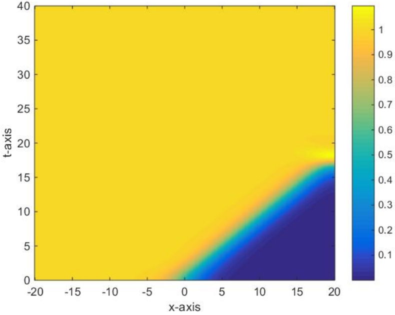
Initial vector for the optimization process: $\kappa = 0.5, \; t_1 = 0, t_2 = 2$

Computed optimal vector: $[\kappa = 0.5], \; t_1 = 0.456, t_2 = 0.541$ κ was fixed

Example 2

Spatial interval: $(-20, 20)$, Time interval: $[0, 40]$





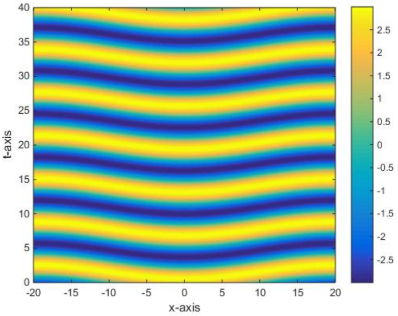
Artificial desired pattern $\widehat{y}$ *Optimal pattern*

Initial vector: $\kappa = -1.5,\ t_1 = 0, t_2 = 1$
Computed optimal vector: $[\kappa = -1.5],\ t_1 = 0.05, t_2 = 0.94,$ κ fixed

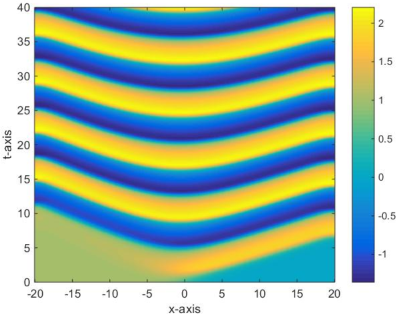
(Discouraging) Example 3

Spatial interval: $(-20, 20)$, Time interval: $[20, 40]$

$\widehat{y}(x,t) = 3\,\sin(t - \cos(\frac{\pi}{20}(x+20)))$, Init: $[t_1 = 0]$, $t_2 = 2$, $[\kappa = -2]$



Desired pattern $\widehat{y}$



Optimal pattern

Computed optimal value: $F = 1.32 \cdot 10^3$, $\|\nabla F\| = 0.045$, $t_2 = 3.71$

Why this is so desastrous?

Re-definition of the objective function

Our objective function does not really express our needs, if we are just interested in a periodic pattern.

- Do not compare u and $\widehat{y}$ at the beginning. Consider

$$F := \frac{1}{2}\int_{T/2}^{T}\int_a^b (y - \widehat{y})^2 \, dxdt.$$

- From now on,

$$Q := (a, b) \times (T/2, T).$$

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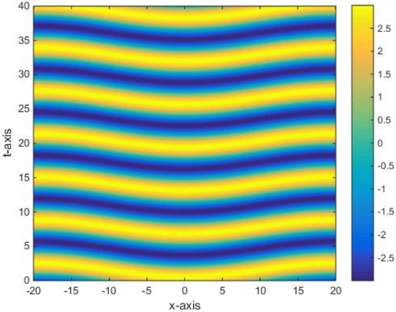
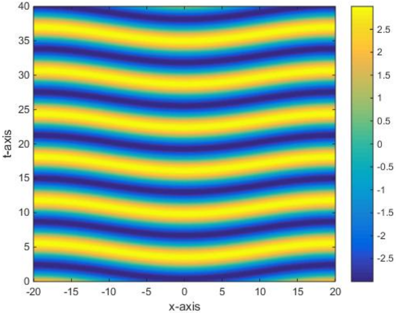
- Moreover, two patterns should be equivalent, if they differ only by a time shift.

Two equivalent patterns





$$\widehat{y} = 3\sin\left(t - \cos\left(\frac{\pi}{20}(x+20)\right)\right) \qquad \widetilde{y} = 3\sin\left(t - \cos\left(\frac{\pi}{20}(x+20)\right) + 3\right)$$

The right pattern is a simple time shift of the left, but their L^2-difference is large,

$$\frac{1}{2}\iint_Q (\widehat{y} - \widetilde{y})^2 dxdt \approx 1.4374 \cdot 10^4$$

Adapting the objective function

Measure the difference of y to the "best time shift" of $\widehat{y} : \Omega \times \mathbb{R} \to \mathbb{R}$:

Shifted objective function

$$f(\kappa, t_1, t_2) := \min_{s} \iint_Q (y_{(\kappa,t_1,t_2)}(x,t) - \widehat{y}(x, t - s))^2 \, dxdt.$$

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This is the core of the idea. Skipping x and the differential *dxdt* for short,

$$\iint_Q (y(t) - \widehat{y}(t-s))^2 = \underbrace{\iint_Q y^2(t)}_{\text{independent of } s} - 2 \iint_Q y(t)\, \widehat{y}(t-s) + \underbrace{\iint_Q \widehat{y}^2(t-s)}_{\substack{\text{independent of s} \\ \text{for periodic } \widehat{y}}}$$

$$= \iint_Q y^2(t) - 2 \iint_Q y(t)\, \widehat{y}(t-s) + \iint_Q \widehat{y}^2(t).$$

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Instead of minimizing the left-hand side, we maximize the red term that is known as cross correlation.

A second optimization problem

The maximum of the red term is achieved, if y and $\widehat{y}(\cdot - s)$ are collinear, i.e.

$$\frac{\iint_Q y(t)\,\widehat{y}(t-s)dxdt}{\sqrt{\iint_Q y^2(t)dxdt}\,\sqrt{\iint_Q \widehat{y}^2(t-s)dxdt}} = 1.$$

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Therefore, for time-periodic $\widehat{y}$ with period $p > 0$, we consider the following

Optimization problem 2

$$\min_{(\kappa,t_1,t_2)\in C_\delta} F_{\text{corr}}(\kappa, t_1, t_2) := \min_{s\in[0,p]} \left(1 - \frac{\left(y_{(\kappa,t_1,t_2)}\,,\,\widehat{y}(\cdot - s)\right)_{L^2(Q)}}{\|y_{(\kappa,t_1,t_2)}\|_{L^2(Q)}\,\|\widehat{y}(\cdot - s)\|_{L^2(Q)}}\right) \qquad \text{(P2)}$$

Remark

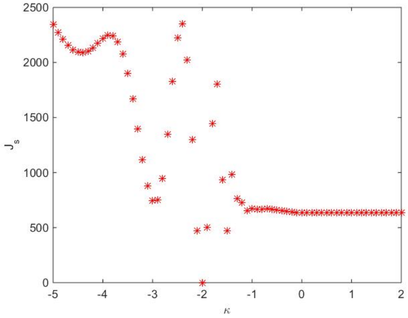
Under natural assumptions, in particular an SSC-condition, we have an adjoint calculus (with a slightly changed adjoint equation).

Advantage of the shifted function

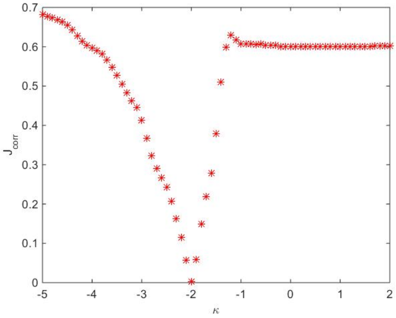
We see the functions $\kappa \mapsto F(\kappa, t_1, t_2)$ and $\kappa \mapsto F_{\text{corr}}(\kappa, t_1, t_2)$ for

$$\widehat{y} := y_{(\kappa, t_1, t_2)} = y_{(-2, 0, 2.5)}.$$

Here, $\kappa = -2$ is the global minimum of $\kappa \mapsto F(\kappa, 0, 2.5)$.

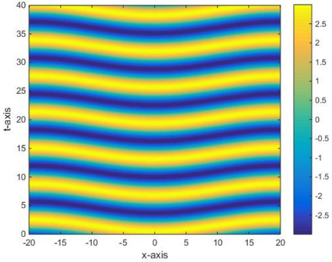


Standard function F

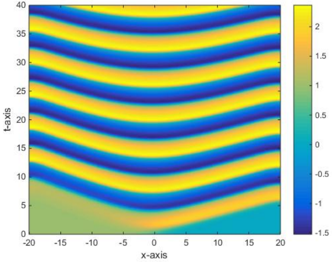


Shifted function F_{corr}

Example 3 with the shifted function



Desired pattern $\widehat{y}$



Optimal pattern by `fmincon`

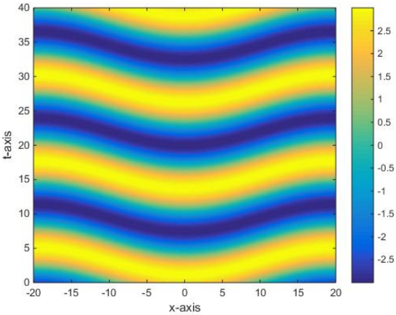
Computed optimal values:

$$F_{\text{corr}} = 0.1229,\ [t_1 = 0],\ t_2 = 3.0031,\ \kappa = -2.4318$$

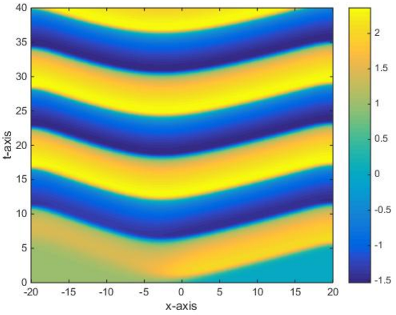
Example 4

Spatial interval: $(-20, 20)$, Time interval of observation: $[20, 40]$

$$\widehat{y}(x,t) = 3\,\sin\left(\frac{t}{2} - \cos\left(\frac{\pi}{20}(x+20)\right)\right)$$, Init: $t_1 = 0, t_2 = 2, K = -2$



Desired pattern $\widehat{y}$



Optimal pattern

Computed optimal value: $F_{\text{corr}} = 0.12,\ [t_1 = 0], t_2 = 6.94191,\ \kappa = -2.28$

Conclusions

- We considered a design problem for nonlocal feedback controllers that generate desired patterns.
- The problem was formulated as an optimal control problem with kernel g as "control".
- An associated adjoint calculus was developed.
- To approximate desired time-periodic patterns, a shifted objective function turned out to be useful.

Thank you