

Open loop synthesis for closed loop control

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Closed loop synthesis

- ▶ Receding horizon synthesis
- ▶ Lagrange manifold method
- ▶ Time-splitting method for Hamilton Jacobi equation
- ▶ Coordinate decomposition (state dependent) synthesis
- ▶ Applications — Delay system and diffusion system
- ▶ Open loop synthesis by Sequential Programming method

Receding horizon (closed loop) synthesis Given $x \in X$, Hilbert space space on $[t, t + T]$ we solve for control $u \in U$ (a constraint set):

$$\min \int_t^{t+T} f^0(x(s), u(s)) ds + V(x(t+T)) \text{ over } u \in L^2(t, t+T; U)$$

subject to

$$\frac{d}{dt}x = f(x, u), \quad u(t) \in U.$$

Remarks

(1) For the short horizon it is much easier to find the feedback map $x \rightarrow u \in L^2(t, t + T; U)$

(2) One can treat the control constraint $u(t) \in U$ much easier.

(3) If we select a good "look-up" cost V we have a good asymptotic ad $t \rightarrow \infty$ [Ito-Kunisch]. For example V is the value function for the infinite horizon problem. Or, if V is a control Liapunov function in the sense that there exists a Lipschitz feedback law $-K$ such that

$$f(x, -K(x)) \cdot V_x + f^0(x, -K(x)) \leq 0$$

(4) For example $u = -B^*x$ (dissipative control, [Ito-Kang]) for $f(x, u) = Ax + f(x) + Bu$ assume that

$$(x, (A - BB^*)x) + (f(x), x) + \ell(x) \leq 0, \quad \text{for all } x \in \text{dom}(A).$$

(5) Test Example, Burgers equation and Navier Stokes equation, damped wave equations.

Lagrange manifold method Consider the optimal control problem

$$\int_0^T (\ell(x(t)) + h(u(t))) dt + G(x(T)) \quad (1)$$

subject to

$$\frac{d}{dt}x(t) = Ax + f(x(t)) + Bu(t), \quad x(0) = x. \quad (2)$$

The necessary optimality condition for (1)–(2) is of the form of Two Point Boundary value problem:

$$\begin{cases} \frac{d}{dt}x^*(t) = Ax^*(t) + f(x^*(t)) + Bu^*(t), & x^*(0) = x \\ -\frac{d}{dt}p(t) = (A + f'(x^*))^*p(t) + \ell'(x^*(t)), & p(T) = G'(x^*(T)) \\ u(t) \in \partial h^*(-B^*p(t)) \end{cases} \quad (3)$$

Given $x \in X$ we define a map $K(x) = p(0) \in X$, which defines the Lagrange manifold for (3).

Thus, for sufficiently large $T > 0$ we define a feedback law

$$u \in \partial h^*(-B^*K(x))$$

Moreover, it can be shown that

$$p(0) = V_x(0, x)$$

where the value function V for (1)–(2) is a viscosity solution to the Hamilton Jacobi equation

$$\frac{\partial}{\partial t} V + (Ax + f(x), V_x) - h^*(B^*V_x) + \ell(x) = 0, \quad V(T, x) = G(x)$$

Remarks We need sampling points $x \in \Sigma$ and interpolation method for $K(x)$ — central difference, sparse sampling and interpolation for high dimensions. Extrapolation by the stationary HJ equation.

Exit (Time optimal) problem

$$\min \int_0^{\tau} (\ell(x(s)) + h(u(s))) ds$$

subject to

$$\frac{d}{dt}x(t) = f(x(t), u(t)), \quad x(0) = x, \quad u(t) \in U$$

$$Cx(\tau) = c.$$

The necessary optimality condition is

$$\left\{ \begin{array}{l} \frac{d}{dt}x^*(t) = f(x^*, u^*), \quad x(0) = x, \quad Cx^*(\tau) = c \\ -\frac{d}{dt}p(t) = f_x(x^*, u^*)^* p(t) + \ell'(x^*), \quad p(\tau) = C^* \mu \\ h(u) - h(u^*(t)) + (f_u(x^*, u^*)(u - u^*(t)), p(t)) \geq 0 \quad \text{for all } u \in U \\ 1 + (f(x^*(\tau), u^*(\tau), p(\tau)) = 0, \end{array} \right.$$

Time-splitting method Consider $\frac{d}{dt}x = Ax + f(x) + Bu$ and define

$$V^+(t, x) = V(t, S_{\Delta t}x), \quad S(t) = e^{At}, \quad C_0\text{-semigroup},$$

Update $V(t - \Delta t, x)$ for a local node $x \in \omega$ by the Hamilton Jacobi equation:

$$\frac{\partial}{\partial t} + f(x) \cdot V_x + h^*(-b^t V_x) + \ell(x) = 0. \quad (4)$$

$$\begin{aligned} V(t - \Delta t, x) - V(t, x) &= V(t - \Delta t, x) - V^+(t, x) + V^+(t, x) - V(t, x) \\ &\sim V_x(S_{\Delta t}x - x) + f(x) \cdot V_x^+ + h^*(-b^t V_x^+) + \ell(x). \end{aligned}$$

Navier Stokes system

$$V^+(t, x) = V(t, T_{\Delta t}S_{\Delta t}x)$$

where $S_{\Delta t}$ is the Stokes solution map and $T_{\Delta t}$ is the transport by convective term. Then, we Update $V(t - \Delta t, x)$ for a local node $x \in \omega$ by the Hamilton Jacobi equation (4).

Delay Control Systems By the time splitting we have

$$\frac{d}{dt}x \sim A_{\Delta t}x(t - \Delta t) + f(x(t)) + Bu(t)$$

where the Yosida approximation $A_{\Delta t}$ is defined by

$$A_{\Delta t} = \frac{1}{\Delta t} (I - (I - \Delta t A)^{-1})$$

The resulting system is a delay differential equation.

In general we consider the control of delay differential equations. Let $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$ be the history function for the state $x(t) \in \mathbb{R}^n$.

$$\frac{d}{dt}x(t) = f(x(t), x_t, x(t - r)) + Bu(t) \tag{5}$$

$$x(0) = x_0, \quad x(\theta) = \phi(\theta), \quad \theta \in (-r, 0)$$

where $f = f(x, \phi, x_1) : R^n \times L^2(-r, 0; R^n) \times R^n \rightarrow R^n$ is locally Lipschitz. The optimality condition is given by

$$\left\{ \begin{array}{l} \frac{d}{dt}x^*(t) = f(x^*(t), x_t^*, x^*(t-r)) + Bu(t) \\ x(0) = x_0, \quad x(\theta) = \phi(\theta), \quad \theta \in (-r, 0) \\ -\frac{d}{dt}\chi(t) = f_x^t \chi(t) + f_\phi^* \chi(t + \cdot) + f_{x_1}^t \chi(t+r) + \ell'(x(t)) \\ \chi(T) = G'(x(T)) \quad \chi(t) = 0, \quad t > T \\ u(t) = -B^t \chi(t). \end{array} \right. \quad (6)$$

Coordinate decomposition

$$\frac{d}{dt}x_1 = f(x_1, x_2) + Bu$$

$$\frac{d}{dt}x_2 = g(x_1, x_2)$$

Assume x_2 is a fixed on horizon $t \in (t_n, t_n + \Delta t)$ for the first (control dynamics) equation and thus

$$x_1(t_n + \Delta t) \sim x_1^+ = S_1(\Delta t, x_2)x_1(t_n) + Bu \Delta t$$

and let

$$x_2^+ = S_2(\Delta t, \frac{x_1 + x_1^+}{2})$$

Then we minimize over $u \in U$

$$\min G_1(x_1^+) + G(x_2^+) + \Delta t |u|^2$$

— We have applied for the Lorenz 3×3 system.

Sequential Programming method

Consider the constrained optimization of the form

$$\min F(x) + H(u) \text{ subject to } E(x, u) = 0, \quad u \in \mathcal{C}$$

We consider a sequence linearized constrained optimization

$$\min F(x) + H(u) \text{ subject to } u \in \mathcal{C}$$

$$E_x(x_n, u_n)(x - x_n) + E_u(x_n, u_n)(u - u_n) + E(x_n, u_n) = 0.$$

The necessary optimality is of the saddle point problem form:

$$\left\{ \begin{array}{l} E_x(x_n, u_n)(\bar{x} - x_n) + E_u(x_n, u_n)(\bar{u} - u_n) + E(x_n, u_n) = 0. \\ E_x(x_n, u_n)^* \lambda + F'(\bar{x}) = 0 \\ H(u) - H(\bar{u}) + \langle E_u(x_n, u_n)(u - \bar{u}), \lambda \rangle \geq \text{for all } u \in \mathcal{C} \end{array} \right.$$

Fixed point formulation of saddle point problem

Assume $E(x, u) = E_0(x) + Bu$.

$$\begin{cases} x^+ = x_n + (E'_0(x_n))^{-1}(Bu + E(x_n)) \\ \lambda = -(E'_0(x_n))^{-*} F'(x^+) \\ u = \Psi(p) = \operatorname{argmin}_{u \in \mathcal{C}} \{H(u) - (u, p)\}, \quad p = -B^* \lambda \end{cases} \quad (7)$$

where $u^+ = \Psi(-B^* \lambda)$ solves the optimality condition.

$$H(u) - H(u^+) + (B^* \lambda, u - u^+) \geq 0 \text{ for all } u \in \mathcal{C}.$$

Fixed point iterate: for $\alpha \in (0, 1]$

- ▶ Given $u_c \in \mathcal{C}$, determine $\lambda = \lambda(u_c)$ by the first two equations of (7).
- ▶ Update $u_{new} = \alpha \Psi(-B^* \lambda) + (1 - \alpha) u$

If $H(u) = \frac{1}{2}(u, Ru)$ we have

$$u^+ = \operatorname{Proj}_{\mathcal{C}}(u^+ - \alpha R^{-1} B^* \lambda(u^+))$$

We use the nonlinear CG method for the fixed point.

The second order variant is given by

$$\min \quad F(x) + H(u) + \langle E(x, u), \lambda_n \rangle \text{ subject to } u \in \mathcal{C}$$

$$E_x(x_n, u_n)(x - x_n) + E_u(x_n, u_n)(u - u_n) + E(x_n, u_n) = 0.$$

$$\left\{ \begin{array}{l} E_x(x_n, u_n)(\bar{x} - x_n) + E_u(x_n, u_n)(\bar{u} - u_n) + E(x_n, u_n) = 0. \\ (E_x(\bar{x}, \bar{u}) - E_x(x_n, u_n)^* \lambda_n + E_x(x_n, u_n)^* \lambda + F'(\bar{x})) = 0 \\ H(u) - H(\bar{u}) + \langle (E_u(\bar{x}, \bar{u}) - E_u(x_n, u_n))(u - \bar{u}), \lambda_n \rangle \\ \quad + \langle E_u(x_n, u_n)(u - \bar{u}), \lambda \rangle \geq \text{ for all } u \in \mathcal{C} \end{array} \right.$$

Remarks (1) Non-smoothness in H , F and E is treated directly.
(2) SP is of the first order due to the term $(y = (x, u))$

$$(E'(y_n) - E'(y^*))^* \lambda^*.$$

(3) The fixed point iterate is preconditioned (projected) gradient method of $u^+ \in \mathcal{C}$ and we solve the saddle point (incomplete).

(4) The second order method incorporates the curvature of the Lagrangian $L(y, \lambda) = F(y) + \langle E(y), \lambda \rangle$ by the (secant) term

$$(E'(\bar{y})^* - E'(y_n)^*) \lambda_n$$

without quadratic model of L (SQP).

(5) For large scale problem the fixed point formulation (decomposition of coordinate) with "hot start" via damped fixed point iterate or nonlinear CG method is effective.

(6) We have tested the optimal control problem and non-smooth elliptic control problem (L^1 cost). It works very well for our numerical tests. We use the SP solver for the closed loop synthesis.