Approximation of the exact controls for the beam equation

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Controlled hinged beam equation

Given any time $T > 0$ and initial data

$$(u^0, u^1) \in \mathcal{H} := H_0^1(0,\pi) \times H^{-1}(0,\pi),$$

the exact controllability in time T of the linear beam equation with hinged (simply-supported) ends,

$$\begin{cases} u''(t,x) + u_{xxxx}(t,x) = 0, \quad x \in (0,\pi),\ t > 0 \\ u(t,0) = u(t,\pi) = u_{xx}(t,0) = 0, \quad t > 0 \\ u_{xx}(t,\pi) = v(t), \quad t > 0 \\ u(0,x) = u^0(x),\ u'(0,x) = u^1(x), \quad x \in (0,\pi) \end{cases} \tag{1}$$

consists of finding a scalar function $v \in L^2(0,T)$, called control, such that the corresponding solution (u, u') of (1) verifies

$$u(T,\cdot\,) = u'(T,\cdot\,) = 0. \tag{2}$$

(Many) methods to study the controllability

Several approaches are available for the study of a controllability problem:

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  - Microlocal Analysis

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- **Optimization methods (Hilbert Uniqueness Method)**
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Fattorini H. O. and Russell D. L., *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rat. Mech. Anal., 4 (1971), 272-292.

J.-L. Lions, *Controlabilité exacte, stabilisation et perturbations des systèmes distribués*, Vol. 1, Masson, Paris, 1988.

Optimization method

Lemma

Let $T > 0$ and $(u^0, u^1) \in \mathcal{H}$. The function $v \in L^2(0,T)$ is a control which drives to zero the solution of (1) *in time T if and only if, for any $(\varphi^0, \varphi^1) \in \mathcal{H}$,*

$$\int_0^T v(t)\overline{\varphi}_x(t,1)\,dt = -\left\langle u^1(x), \varphi(0,x)\right\rangle_{-1,1} + \left\langle u^0(x), \varphi'(0,x)\right\rangle_{1,-1},$$

where $(\varphi, \varphi') \in \mathcal{H}$ is the solution of the backward equation

$$\begin{cases} \varphi''(t,x) + \varphi_{xxxx}(t,x) = 0 & (t,x) \in (0,T) \times (0,1) \\ \varphi(t,0) = \varphi(t,1) = 0 & t \in (0,T) \\ \varphi_{xx}(t,0) = \varphi_{xx}(t,1) = 0 & t \in (0,T) \\ \varphi(T,x) = \varphi^0(x) & x \in (0,1) \\ \varphi'(T,x) = \varphi^1(x) & x \in (0,1). \end{cases} \tag{3}$$

Optimization method

For each $(u^0, u^1) \in \mathcal{H}$, define the functional $J : \mathcal{H} \to \mathbb{R}$,

$$J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(t,1)|^2 \, dt + \left\langle u^1(x), \varphi(0,x) \right\rangle_{-1,1} - \left\langle u^0(x), \varphi'(0,x) \right\rangle_{1,-1},$$

where (φ, φ') is the solution of (3) with initial data (φ^0, φ^1).

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- If J has a minimum at $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in \mathcal{H}$ then $\widehat{v}(t) = \widehat{\varphi}_x(1,t)$ is a control for (1).

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- If J has a minimum at $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in \mathcal{H}$ then $\widehat{v}(t) = \widehat{\varphi}_x(1,t)$ is a control for (1).
- J has a minimum if it is coercive and it is coercive if the following observability inequality holds for any $(\varphi^0, \varphi^1) \in \mathcal{H}$:

$$\|(\varphi(0), \varphi'(0))\|^2_{\mathcal{H}} \le C \int_0^T |\varphi_x(t,\pi)|^2 dt. \tag{4}$$

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- J has a minimum if it is coercive and it is coercive if the following observability inequality holds for any $(\varphi^0, \varphi^1) \in \mathcal{H}$:

$$\|(\varphi(0), \varphi'(0))\|_{\mathcal{H}}^2 \le C\int_0^T |\varphi_x(t,\pi)|^2 dt. \tag{4}$$

- Hence, if (4) holds, for any initial data $(u^0, u^1) \in \mathcal{H}$, there exists a control $v \in L^2(0,T)$ with the property

$$\|v\|_{L^2} \le \sqrt{C}\|(u^0, u^1)\|_{\mathcal{H}}. \tag{5}$$

Ingham's inequality

Observability inequality (4) is equivalent to inequality of the form

$$\sum_{n\in\mathbb{Z}^*} |\alpha_n|^2 \leq C(T) \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{n\in\mathbb{Z}^*} \alpha_n e^{\nu_n t} \right|^2 dt, \ \ (\alpha_n)_{n\in\mathbb{Z}^*} \in \ell^2. \quad (6)$$

Ingham's inequality

For any $T > \frac{2\pi}{\gamma_\infty}$, $\gamma_\infty = \liminf_{n\to\infty} |\nu_{n+1} - \nu_n|$, inequality (6) holds.

A. E. Ingham, Some trigonometric inequalities with applications to the theory of series, Math. Zeits., 41 (1936), 367-379.

J. Ball and M. Slemrod, Nonharmonic Fourier series and the stabilization of distributed semilinear control systems, Comm. Pure Appl. Math., 32 (1979), 555-587.

J. P. Kahane: Pseudo-Périodicité et Séries de Fourier Lacunaires, Ann. Sci. Ecole Norm. Super. 37, 93-95 (1962).

Observability inequality

In our particular case

$$\nu_n = i\,\mathsf{sgn}(n)\,n^2, \qquad \gamma_\infty = \liminf_{n\to\infty} |\nu_{n+1} - \nu_n| = \infty.$$

Ingham's inequality implies that the observability inequality (4) is verified for any $T > 0$.

Consequently, given any $T > 0$, there exists a control $v \in L^2(0, T)$ for each $(u^0, u^1) \in \mathcal{H}$.

The control function v is not unique.

Moment problem for the beam equation

The null-controllability of the beam equation is equivalent to solve a moment problem.

Lemma

Let $T > 0$ *and* $(u^0, u^1) = \left(\sum_{n=1}^{\infty} a_n^0 \sin(nx), \sum_{n=1}^{\infty} a_n^1 \sin(nx)\right) \in \mathcal{H}$. *The function* $v \in L^2(0,T)$ *is a control which drives to zero the solution of* (1) *in time* T *if and only if*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} v\left(t + \frac{T}{2}\right) e^{t\overline{\nu}_n} dt = \frac{(-1)^n e^{-\frac{T}{2}\overline{\nu}_n}}{\sqrt{2}n\pi}\left(\overline{\nu}_n a_n^0 - a_n^1\right) \quad (n \in \mathbb{Z}^*), \tag{7}$$

where $\nu_n = i\,\text{sgn}(n)\, n^2$ *are the eigenvalues of the unbounded skew-adjoint differential operator corresponding to (1).*

A solution v of the moment problem may be constructed by means of a biorthogonal family to the sequence $(e^{\nu_n t})_{n \in \mathbb{Z}^*}$.

Moment problem for the beam equation

Definition

A family of functions $(\phi_m)_{m\in\mathbb{Z}^*} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ *with the property*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_m(t) e^{\overline{\nu}_n t} dt = \delta_{mn} \qquad \forall\, m, n \in \mathbb{Z}^*, \tag{8}$$

is called a biorthogonal sequence to $(e^{\nu_n t})_{n\in\mathbb{Z}^*}$ *in* $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$.

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Once we have a biorthogonal sequence to $(e^{\nu_n t})_{n\in\mathbb{Z}^*}$, a "formal" solution of the moment problem is given by

$$v(t) = \sum_{n\in\mathbb{Z}^*} \frac{(-1)^n e^{-\frac{T}{2}\overline{\nu}_n}}{\sqrt{2} n\pi} \left(\overline{\nu}_n a_n^0 - a_n^1\right) \phi_n \left(t - \frac{T}{2}\right). \tag{9}$$

Ingham's inequality and the existence of a biorthogonal

Consider a Hilbert space H and a family $(f_n)_{n\in\mathbb{Z}^*} \subset H$ such that

$$\sum_{n\in\mathbb{Z}^*} |a_n|^2 \leq C_1 \left\| \sum_{n\in\mathbb{Z}^*} a_n f_n \right\|^2, \quad (a_n)_{n\in\mathbb{Z}^*} \in \ell^2. \tag{10}$$

Then there exists a biorthogonal sequence to the family $(f_n)_{n\in\mathbb{Z}^*}$.

- $(f_n)_{n\in\mathbb{Z}^*}$ is minimal i. e.

$$f_m \notin \overline{\text{Span}\left\{(f_n)_{n\in\mathbb{Z}^*\setminus\{m\}}\right\}} \qquad (m \in \mathbb{Z}^*).$$

- Apply Hahn-Banach Theorem to $\{f_m\}$ and $\overline{\text{Span}\left\{(f_n)_{n\in\mathbb{Z}^*\setminus\{m\}}\right\}}$. There exists $\phi_m \in H$ such that

$$(\phi_m, f_m) = 1 \text{ and } (\phi_m, f_n) = 0 \text{ for any } n \neq m.$$

- The biorthogonal sequence which is bounded:

$$\left\| \sum_{n\in\mathbb{Z}^*} b_n \phi_n \right\|^2 \leq \frac{1}{C_1} \sum_{n\in\mathbb{Z}^*} |b_n|^2.$$

No Ingham?

If we are in a context in which no Ingham's type inequality is available? We can take the inverse way:

- Construction of the biorthogonal

  Paley-Wiener Theorem: Let $F : \mathbb{C} \to \mathbb{C}$ be an entire function of exponential type ($|F(z)| \leq Me^{T|z|}$) which belongs to $L^2(\mathbb{R})$ on the real axis. Then $\int_{\mathbb{R}} F(t)e^{ixt}dt$ is a function from $L^2(-T,T)$.

  R. E. A. C. Paley and N. Wiener, Fourier Transforms in Complex Domains, AMS Colloq. Publ., Vol. 19, Amer. Math. Soc., 1934.

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} F(t)e^{ixt}dt \quad \Rightarrow \quad \begin{cases} F(t) = \displaystyle\int_{-T}^{T} f(x)e^{-ixt}dx; \\ \\ \|f\|_{L^2} = \sqrt{2\pi}\|F\|_{L^2(\mathbb{R})}. \end{cases}$$

- Evaluation of its norm
- Construction of the control

Finite differences for the beam equation

$N \in \mathbb{N}^*$, $h = \frac{\pi}{N+1}$, $x_j = jh$, $0 \leq j \leq N+1$,
$x_{-1} = -h$, $x_{N+2} = \pi + h$.

$$\begin{cases} u_j''(t) = -\frac{u_{j+2}(t) - 4u_{j+1} + 6u_j(t) - 4u_{j-1}(t) + u_{j-2}(t)}{h^4}, \ t > 0 \\ u_0(t) = u_{N+1}(t) = 0, \ u_{-1}(t) = -u_1(t), \ t > 0 \\ u_{N+2} = -u_N + h^2 v_h(t), \ t > 0 \\ u_j(0) = u_j^0, \ u_j'(0) = u_j^1, \ \ 1 \leq j \leq N. \end{cases} \tag{11}$$

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Discrete controllability problem: given $T > 0$ and $(U_h^0, U_h^1) = (u_j^0, u_j^1)_{1 \leq j \leq N} \in \mathbb{C}^{2N}$, there exists a control function $v_h \in L^2(0, T)$ such that the solution u of (11) satisfies

$$u_j(T) = u_j'(T) = 0, \; \forall j = 1, 2, ..., N. \tag{12}$$

System (11) consists of N linear differential equations with N unknowns $u_1, u_2, ..., u_N$.
$u_j(t) \approx u(t, x_j)$ if $(U_h^0, U_h^1) \approx (u^0, u^1)$.

Discrete controls

- Existence of the discrete control v_h.
- Boundedness of the sequence $(v_h)_{h>0}$ in $L^2(0,T)$.
- Convergence of the sequence $(v_h)_{h>0}$ to a control v of the beam equation (1).

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L. LEON and E. ZUAZUA: Boundary controllability of the finite-difference space semi-discretizations of the beam equation. ESAIM:COCV, A Tribute to Jacques- Louis Lions, Tome 2, 2002, pp. 827-862.

Equivalent vectorial form

System (11) is equivalent to

$$\begin{cases} U_h''(t) + (A_h)^2 U_h(t) = F_h(t) & t \in (0,T) \\ U_h(0) = U_h^0 \\ U_h'(0) = U_h^1, \end{cases} \tag{13}$$

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}, \quad U_h(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{pmatrix}$$

$$F_h(t) = \frac{1}{h^2}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -v_h(t) \end{pmatrix}, \quad U_h^0 = \begin{pmatrix} u_1^0 \\ u_2^0 \\ \vdots \\ u_N^0 \end{pmatrix}, \quad U_h^1 = \begin{pmatrix} u_1^1 \\ u_2^1 \\ \vdots \\ u_N^1 \end{pmatrix}.$$

Discrete observability inequality

$$\begin{cases} W_h''(t) + A_h^2 W_h(t) = 0 \quad t \in (0,T) \\ W_h(T) = W_h^0 \in \mathbb{C}^N \\ W_h'(T) = W_h^1 \in \mathbb{C}^N. \end{cases} \tag{14}$$

The energy of (14) is defined by

$$E_h(t) = \frac{1}{2}\left(\langle A_h W_h(t), W_h(t)\rangle + \langle A_h^{-1} W_h'(t), W_h'(t)\rangle\right), \tag{15}$$

and the following relation holds:

$$\frac{d}{dt} E_h(t) = 0. \tag{16}$$

The exact controllability in time T of (11) holds if the following discrete observability inequality is true

$$E_h(t) \leq C(T,h) \int_0^T \left| \frac{W_{hN}(t)}{h} \right|^2 dt, \quad (W_h^0, W_h^1) \in \mathbb{C}^{2N}. \tag{17}$$

One or two problems

Eigenvalues:
$\nu_n = i\,\mathsf{sgn}\,(n)\,\mu_n,\ \mu_n = \frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right), \quad 1 \leq |n| \leq N.$
Eigenvectors form an orthogonal basis in $\mathbb{C}^{2N}$:

$$\phi^n = \frac{1}{\sqrt{2\mu_n}}\begin{pmatrix} \varphi^n \\ \\ -\nu_n\,\varphi^n \end{pmatrix}, \quad \varphi^n = \sqrt{2}\begin{pmatrix} \sin(nh\pi) \\ \sin(2nh\pi) \\ \vdots \\ \sin(Nnh\pi) \end{pmatrix}, 1 \leq |n| \leq N.$$

The observability constant is not uniform in h:

$$(W_h^0, W_h^1) = \phi^N \Rightarrow C(T,h) = \frac{1}{T\cos^2\left(\frac{N\pi h}{2}\right)} \approx \frac{1}{Th^2}.$$

There are initial data $(u^0, u^1) \in \mathcal{H}$ such that the sequence of discrete minimal L^2-norm controls $(\widehat{v}_h)_{h>0}$ diverges!!!

Cures (L. Leon and E. Zuazua, COCV 2002)

Problems from the bad numerical approximation of high eigenmodes (spurious numerical eigenmodes).

- Control the projection of the solution over the space Span$\{\phi^n : 1 \leq |n| \leq \gamma N\}$, with $\gamma \in (0,1)$.

$$\sum_{1\leq|n|\leq\gamma N} |\alpha_n|^2 \leq C \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{1\leq|n|\leq\gamma N} \alpha_n e^{\nu_n t} \right|^2 dt. \qquad (18)$$

- Introduce a new control which vanishes in the limit

$$E_h(t) \leq C \left[\int_0^T \left| \frac{W_{hN}(t)}{h} \right|^2 dt + h^2 \int_0^T \left| \frac{W'_{hN}(t)}{h} \right|^2 dt \right]. \qquad (19)$$

$C = C(T) \Rightarrow$ uniform controllability $\Rightarrow$

convergence of the discrete controls.

Regularity and filtration of the initial data

We consider the controlled system

$$\begin{cases} U_h''(t) + (A_h)^2 U_h(t) = F_h(t) \quad t \in (0,T) \\ U_h(0) = U_h^0 \\ U_h'(0) = U_h^1, \end{cases} \tag{20}$$

We suppose that one of the following properties holds:

- Initial data (u^0, u^1) are sufficiently smooth (for instance, in $H^3(0,1) \times H_0^1(0,1)$) and discretized by points

$$U^0 = (u^0(jh))_{1 \le j \le N}, \quad U^1 = (u^1(jh))_{1 \le j \le N};$$

- Initial data (u^0, u^1) are in the energy space $\mathcal{H}$ and the high frequencies of their discretization are filtered out,

$$(U^0, U^1) = \sum_{1 \le |n| \le \delta N} a_{nh} \Phi^n \qquad (\delta \in (0,1));$$

Can we obtain the uniform controllability in any $T > 0$?

Discrete moments problem

Lemma

Let $T > 0$ and $\varepsilon > 0$. System (20) is null-controllable in time T if and only if, for any initial datum $(U_h^0, U_h^1) \in \mathbb{C}^{2N}$ of form

$$(U_h^0, U_h^1) = \left(\sum_{j=1}^{N} a_{jh}^0 \varphi^j, \sum_{j=1}^{N} a_{jh}^1 \varphi^j \right), \tag{21}$$

there exists a control $v_h \in L^2(0,T)$ such that

$$\int_0^T v_h(t) e^{\overline{\nu}_n t} dt = \frac{(-1)^n h}{\sqrt{2}\sin(|n|\pi h)} \left(-a_{|n|h}^1 + \overline{\nu}_n a_{|n|h}^0 \right), \tag{22}$$

for any $n \in \mathbb{Z}^*$ such that $|n| \leq N$.

Biorthogonal family

If $(\theta_m)_{1\leq|m|\leq N} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ is a *biorthogonal sequence to the family of exponential functions* $\left(e^{\nu_n t}\right)_{1\leq|n|\leq N}$ *in* $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ then a control of (13) will be given by

$$v_h(t) = \sum_{1\leq|n|\leq N} \frac{(-1)^n h e^{-\nu_n \frac{T}{2}}}{\sqrt{2}\sin(|n|\pi h)} \left(-a^1_{|n|h} + \overline{\nu}_n a^0_{|n|h}\right) \theta_n\left(t - \frac{T}{2}\right).$$

We look for a biorthogonal sequence $(\theta_m)_{1\leq|m|\leq N}$ to $\left(e^{i\nu_n t}\right)_{1\leq|n|\leq N}$ and we try to estimate the right hand side sum. The exponents are real:

$$\nu_n = \mathrm{sgn}(n)\frac{4}{h^2}\sin\left(\frac{n\pi h}{2}\right) \qquad (1 \leq |n| \leq N).$$

Biorthogonal sequence

Taking into account that

$$\nu_{n+1}-\nu_n = \frac{4}{h^2}\sin\left(\frac{n\pi h}{2}\right)\sin\left(\frac{(2n+1)\pi h}{2}\right) > \begin{cases} n & \text{if } \delta < |n| < \delta N \\ 4 & \text{otherwise,} \end{cases}$$

we can use Ingham's inequality and a Kahane's argument to show that, for any $T > 0$, there exists a biorthogonal $(\theta_m)_{1\le|m|\le N}$ to the family $\left(e^{i\nu_n t}\right)_{1\le|n|\le N}$ with the property that

$$\left\|\sum_{1\le|n|\le N} b_n\theta_n\right\|^2 \le C\exp\left(\frac{C}{T}\right)\sum_{1\le|n|\le N}|b_n|^2.$$

It follows that

$$\|v_h(t)\|^2 = \left\|\sum_{1\le|n|\le N}\frac{(-1)^n h e^{-\nu_n\frac{T}{2}}}{\sqrt{2}\sin(|n|\pi h)}\left(-a^1_{|n|h}+\overline{\nu}_n a^0_{|n|h}\right)\theta_n\left(t-\frac{T}{2}\right)\right\|^2$$

$$\le C\exp\left(\frac{C}{T}\right)\sum_{1\le|n|\le N}\frac{h^2}{\sin^2(n\pi h)}\left(|a^1_{|n|h}|^2+|\nu_n|^2|a^0_{|n|h}|^2\right).$$

Regularity or filtration

$$\|v_h(t)\|^2 \leq C \exp\left(\frac{C}{T}\right) \sum_{1\leq|n|\leq N} \frac{h^2}{\sin^2(n\pi h)} \left(|a^1_{|n|h}|^2 + |\nu_n|^2 |a^0_{|n|h}|^2\right).$$

- The initial data to be controlled are in $H^3(0,1) \times H^1_0(0,1)$

$$\sum_{1\leq|n|\leq N} n^2 \left(|a^1_{|n|h}|^2 + |\nu_n|^2 |a^0_{|n|h}|^2\right) \leq C\|(u^0,u^1)\|^2_{3,1}$$

$$\Rightarrow \|v_h\|^2 \leq C \exp\left(\frac{C}{T}\right) \|(u^0,u^1)\|^2_{3,1}.$$

- The high frequencies of the discrete initial data are filtered out

$$\|v_h\|^2 \leq C(\delta) \exp\left(\frac{C}{T}\right) \sum_{1\leq|n|\leq \delta N} \frac{1}{n^2} \left(|a^1_{|n|h}|^2 + |\nu_n|^2 |a^0_{|n|h}|^2\right)$$

$$\leq C'(\delta) \exp\left(\frac{C}{T}\right) \|(u^0,u^1)\|^2_{1,-1}.$$

Numerical results

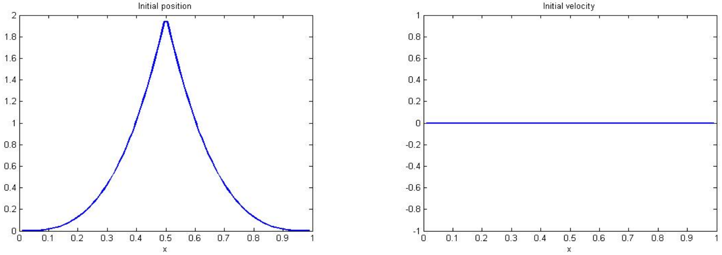

Figure: Initial data to be controlled.

$$N = 100; T = .3;$$

A conjugate gradient method for the corresponding discrete optimization approach.

Numerical results

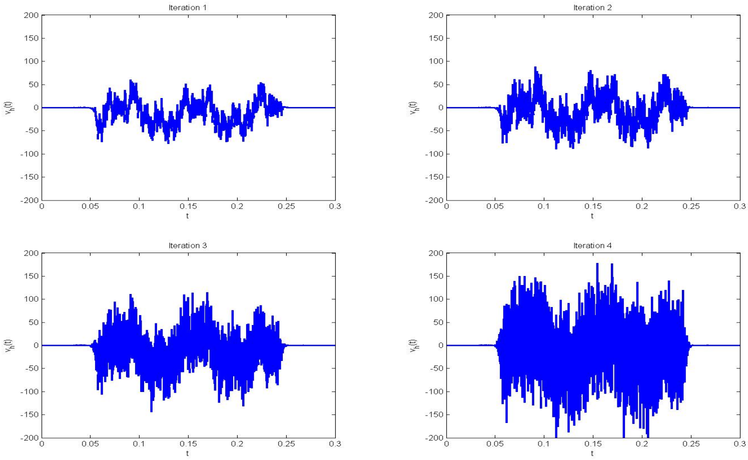

Figure: Example 2 - The first four iterations of the conjugate gradient method for the approximation of v_h with $N = 100$ without filtration.

Numerical results

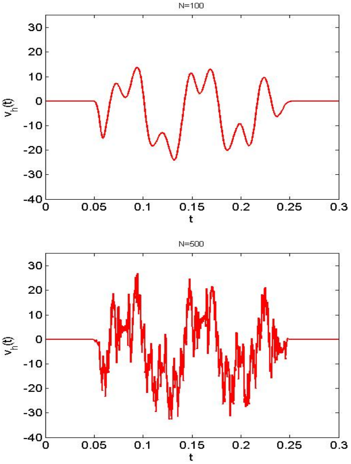

Figure: The approximation of the control v_h with $N = 100$, 200, 500 and 1000 by using filtration of the initial data with $\delta = \frac{1}{40}$.

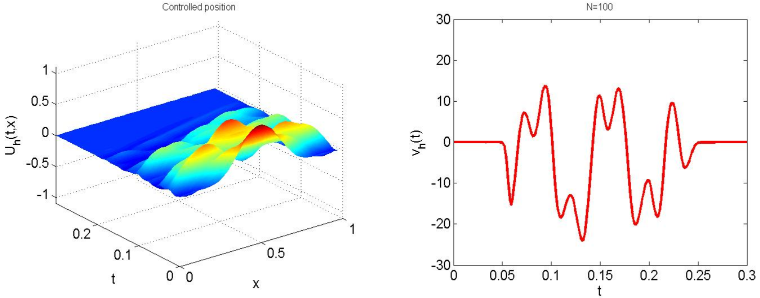

Figure: Controlled solution and the approximation of the control with $N = 100$ by using filtration of the initial data $\delta = \frac{1}{40}$.

Numerical vanishing viscosity

Instead of (13) we consider the system

$$\begin{cases} U_h''(t) + (A_h)^2 U_h(t) + \varepsilon A_h U_h'(t) = F_h(t) & t \in (0,T) \\ U_h(0) = U_h^0 \\ U_h'(0) = U_h^1, \end{cases} \tag{23}$$

- $\varepsilon = \varepsilon(h), \quad \lim_{h\to 0} \varepsilon = 0$
- If $F_h = 0$, $\dfrac{dE_h}{dt}(t) = -\varepsilon \langle A_h U_h'(t), U_h'(t) \rangle \leq 0$
- The term $\varepsilon A_h U_h'(t)$ represents a numerical vanishing viscosity.

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Can we obtain the uniform controllability in any $T > 0$ (without projection or additional controls) using this new discrete scheme?

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Spectral analysis. Good news but no Ingham.

Eigenvalues: $\lambda_n = \frac{1}{2}\left(\varepsilon + i\,\mathsf{sgn}\,(n)\sqrt{4-\varepsilon^2}\right)\mu_{|n|}$, $1 \leq |n| \leq N$.
Eigenvectors:

$$\phi^n = \frac{1}{\sqrt{2\mu_n}}\begin{pmatrix} \varphi^n \\ \\ -\lambda_n\,\varphi^n \end{pmatrix}, \quad \varphi^n = \sqrt{2}\begin{pmatrix} \sin(nh\pi) \\ \sin(2nh\pi) \\ \vdots \\ \sin(Nnh\pi) \end{pmatrix}, 1 \leq |n| \leq N.$$

If $(W_h^0, W_h^1) = \phi^N$ we obtain that

$$C(T,h) = \frac{\int_0^T \left|\frac{W_{hN}(t)}{h}\right|^2 dt}{\|(W_h(0), W_h'(0))\|^2} \approx \frac{1}{\cos^2\left(\frac{N\pi h}{2}\right)}\frac{\Re(\lambda_N)}{e^{2T\Re(\lambda_N)}-1}.$$

To ensure the uniform observability of these initial data we need
$\varepsilon > C\ln\left(\frac{1}{h}\right)h^2$

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To ensure the uniform observability of these initial data we need

$\varepsilon > C\ln\left(\frac{1}{h}\right)h^2 \Rightarrow \Re(\lambda_N) > C\ln\left(\frac{1}{h}\right)$.

Discrete moments problem

Lemma

Let $T > 0$ and $\varepsilon > 0$. System (13) is null-controllable in time T if and only if, for any initial datum $(U_h^0, U_h^1) \in \mathbb{C}^{2N}$ of form

$$(U_h^0, U_h^1) = \left(\sum_{j=1}^{N} a_{jh}^0 \varphi^j, \sum_{j=1}^{N} a_{jh}^1 \varphi^j \right), \tag{24}$$

the exists a control $v_h \in L^2(0,T)$ such that

$$\int_0^T v_h(t) e^{\overline{\lambda}_n t} dt = \frac{(-1)^n h}{\sqrt{2}\sin(|n|\pi h)} \left(-a_{|n|h}^1 + (\overline{\lambda}_n - \varepsilon \mu_{|n|}) a_{|n|h}^0 \right), \tag{25}$$

for any $n \in \mathbb{Z}^*$ such that $|n| \leq N$.

Biorthogonal family

If $(\theta_m)_{1\leq|m|\leq N} \subset L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ is a *biorthogonal sequence to the family of exponential functions* $\left(e^{\lambda_n t}\right)_{1\leq|n|\leq N}$ *in* $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ then a control of (13) will be given by

$$v_h(t) = \sum_{1\leq|n|\leq N} \frac{(-1)^n h e^{-\lambda_n \frac{T}{2}}}{\sqrt{2}\sin(|n|\pi h)} \left(-a^1_{|n|h} + (\overline{\lambda}_n - \varepsilon\mu_{|n|})a^0_{|n|h}\right)\theta_n\left(t-\frac{T}{2}\right).$$

Now the main task in to show that there exists a biorthogonal sequence $(\theta_m)_{1\leq|m|\leq N}$ and to evaluate its L^2-norm in order to estimate the right hand side sum.

S.M., *Uniform boundary controllability of a semi–discrete 1–D wave equation with vanishing viscosity*, SIAM J. Cont. Optim., 47 (2008), 2857-2885.

Main differences:

- We have the optimal value of the viscosity parameter ε:

$$\varepsilon \geq Ch^2 \ln\left(\frac{1}{h}\right).$$

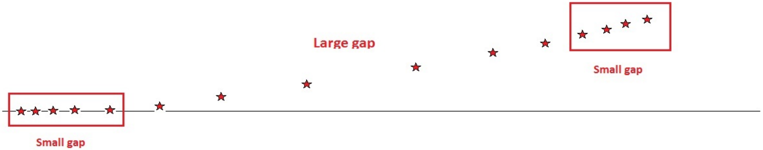
S.M., *Uniform boundary controllability of a semi–discrete 1–D wave equation with vanishing viscosity*, SIAM J. Cont. Optim., 47 (2008), 2857-2885.

Main differences:

- We have the optimal value of the viscosity parameter ε:

$$\varepsilon \geq Ch^2 \ln\left(\frac{1}{h}\right).$$

- The controllability time T should be arbitrarily small.



Construction of a biorthogonal (I) - The big picture

Suppose that $(\theta_m)_{1\leq|m|\leq N}$ is a biorthogonal sequence to the family of exponential functions $\left(e^{\lambda_n t}\right)_{1\leq|n|\leq N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ and define

$$\Psi_m(z)=\int_{-\frac{T}{2}}^{\frac{T}{2}}\overline{\theta_m(t)}e^{-i\,tz}dt.$$

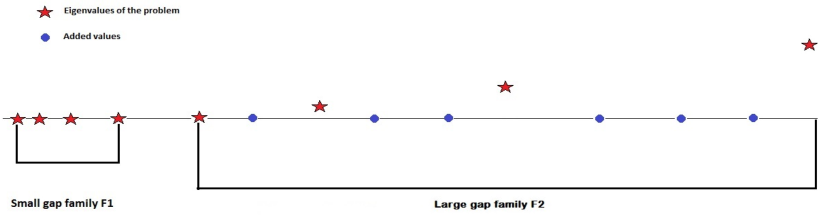
- $\Psi_m(i\lambda_n)=\delta_{nm}$
- Ψ_m is an entire function of exponential type $\frac{T}{2}$
- $\Psi_n\in L^2(\mathbb{R})$

Paley-Wiener Theorem ensures that the reciprocal is true and gives a constructive way to obtain a biorthogonal sequence.

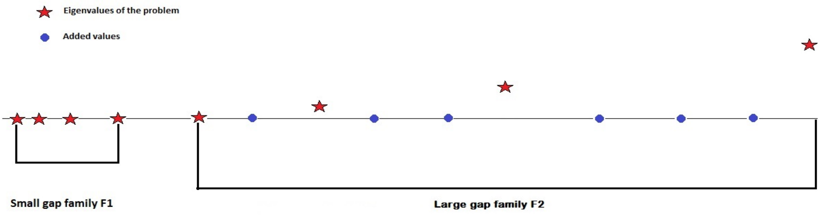
$$\Psi_m(z)=P_m(z)\times M_m(z)=\prod_{n\neq m}\frac{i\lambda_n-z}{i\lambda_n-i\lambda_m}\times M_m(z).$$

P_m (the product) and M_m (the multiplier) should have small exponential type and good behavior on the real axis.

Construction of a biorthogonal (II) - A small picture



Construction of a biorthogonal (II) - A small picture



- $(\xi_l^1)_l$ is a biorthogonal to family F_1 which is finite.
- $(\xi_k^2)_k$ is a biorthogonal to family F_2 with good gap properties.
- A biorthogonal $(\theta_m)_m$ to full family $F_1 \cup F_2$ can be constructed by using the Fourier transforms $\widehat{\theta_k^1}$ and $\widehat{\theta_l^2}$.

Construction of a biorthogonal (III): The main result

Theorem

Let $T > 0$. There exist two positive constants h_0 and ε_0 such that for any $h \in (0, h_0)$ and $\varepsilon \in \left(c_0 h^2 \ln\left(\frac{1}{h}\right), c_0 h\right)$ there exists a biorthogonal $(\theta_m)_m$ to $(e^{\lambda_n t})_n$ and two constants $\alpha < T$ and $C = C(T) > 0$ (independent of ε and h) such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_m \alpha_m \theta_m(t) \right|^2 dt \leq C(T) \sum_m |\alpha_m|^2 e^{\alpha |\Re(\lambda_m)|}, \qquad (26)$$

for any finite sequence $(\alpha_m)_m$.

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for any finite sequence $(\alpha_m)_m$.

Since

$$v_h(t) = \sum_{1 \leq |n| \leq N} \frac{(-1)^n h e^{-\frac{T\lambda_n}{2}}}{\sqrt{2}\sin(|n|\pi h)} \left(-a^1_{|n|h} + (\overline{\lambda}_n - \varepsilon\mu_{|n|}) a^0_{|n|h}\right) \theta_n\left(t - \frac{T}{2}\right).$$

we obtain immediately from (26) the uniform boundedness (in h) of the family of controls $(v_h)_{h>0}$.

Numerical results

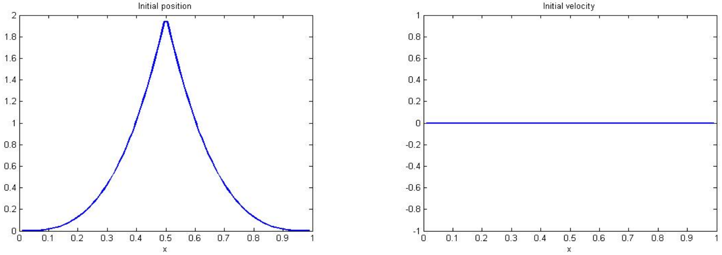


Figure: Initial data to be controlled.

$$N = 100; T = 2.3; \varepsilon = h$$

A conjugate gradient method for the corresponding discrete optimization approach.

Numerical results

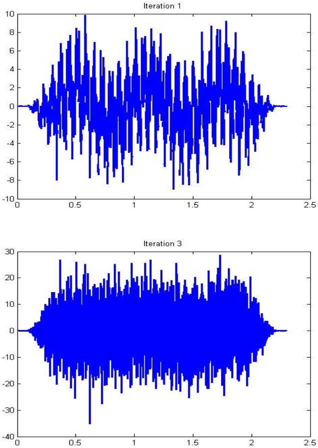
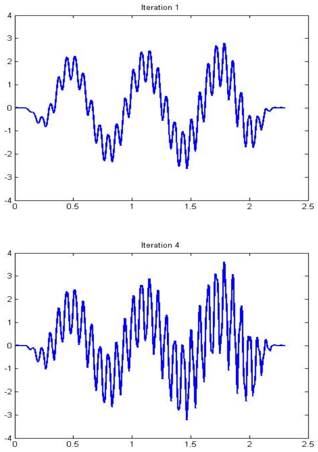


Figure: The first four iterations with $\varepsilon = 0$.

Numerical results



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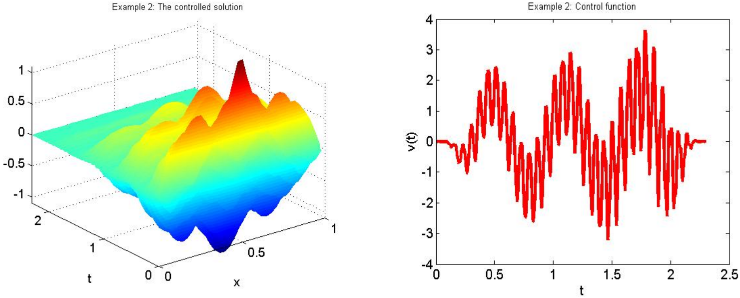

Figure: Controlled solution and the control.

Controlled clamped beam equation

Given any time $T > 0$ and initial data

$$(u^0, u^1) \in \mathcal{H} := L^2(0,\pi) \times H^{-2}(0,\pi),$$

the exact controllability in time T of the linear clamped beam equation,

$$\begin{cases} u''(t,x) + u_{xxxx}(t,x) = 0, & x \in (0,\pi),\ t > 0 \\ u(t,0) = u(t,\pi) = u_x(t,0) = 0, & t > 0 \\ u_x(t,\pi) = v(t), & t > 0 \\ u(0,x) = u^0(x),\ u'(0,x) = u^1(x), & x \in (0,\pi) \end{cases} \tag{27}$$

consists of finding a scalar function $v \in L^2(0,T)$, called control, such that the corresponding solution (u, u') of (27) verifies

$$u(T,\cdot\,) = u'(T,\cdot\,) = 0. \tag{28}$$

Finite differences for the clamped beam equation

$N \in \mathbb{N}^*$, $h = \frac{\pi}{N+1}$, $x_j = jh$, $0 \leq j \leq N+1$,
$x_{-1} = -h$, $x_{N+2} = \pi + h$.

$$\begin{cases} u_j''(t) = -\frac{u_{j+2}(t) - 4u_{j+1} + 6u_j(t) - 4u_{j-1}(t) + u_{j-2}(t)}{h^4}, \ t > 0 \\ u_0(t) = u_{N+1}(t) = 0, \ u_{-1}(t) = u_1(t), \ t > 0 \\ u_{N+2} = u_N + 2hv_h(t), \ t > 0 \\ u_j(0) = u_j^0, \ u_j'(0) = u_j^1, \ \ 1 \leq j \leq N. \end{cases} \tag{29}$$

Discrete controllability problem: given $T > 0$ and $(U_h^0, U_h^1) = (u_j^0, u_j^1)_{1 \leq j \leq N} \in \mathbb{C}^{2N}$, there exists a control function $v_h \in L^2(0,T)$ such that the solution u of (11) satisfies

$$u_j(T) = u_j'(T) = 0, \ \forall j = 1, 2, ..., N. \tag{30}$$

Discrete observability inequality

$$\begin{cases} W_h''(t) + \widetilde{B_h} W_h(t) = 0 \quad t \in (0,T) \\ W_h(T) = W_h^0 \in \mathbb{C}^N \\ W_h'(T) = W_h^1 \in \mathbb{C}^N. \end{cases} \tag{31}$$

The energy of (31) is defined by

$$E_h(t) = \frac{1}{2}\left(\langle \widetilde{B_h} W_h(t), W_h(t)\rangle + \langle W_h'(t), W_h'(t)\rangle\right), \tag{32}$$

and the following relation holds:

$$\frac{d}{dt}E_h(t) = 0. \tag{33}$$

The exact controllability in time T of (29) holds if the following discrete observability inequality is true

$$E_h(t) \leq C(T,h)\int_0^T \left|\frac{2W_{hN}(t)}{h^2}\right|^2 dt, \quad (W_h^0, W_h^1) \in \mathbb{C}^{2N}. \tag{34}$$

Spectral analysis

- **Continuous spectrum:** The eigenvalues of the corresponding differential operator are given by the positive roots of the equation $\cos(z) - \cosh^{-1}(z) = 0$, which are asymptotically exponentially close to the zeros of the $\cos(z)$ function.
- **Discrete spectrum:** The eigenvalues of the corresponding discrete operator are given by the positive roots of the equation $f(z) = 0$, where

$$f(z) = \cos z \pm \sin^2\left(\frac{hz}{2}\right) + \frac{2\left(1 - \sin^4\left(\frac{hz}{2}\right)\right) r^{N+1}(z)}{r^{2(N+1)}(z) - 2\sin^2\left(\frac{hz}{2}\right) r^{N+1}(z) + 1},$$

$$r(z) = 1 + 2\sin^2\left(\frac{zh}{2}\right) + \sqrt{\sin^2\left(\frac{zh}{2}\right)\left(1 + \sin^2\left(\frac{zh}{2}\right)\right)}.$$

Function f has a sequence of well separated roots $(z_n)_{1 \leq n \leq N} \subset (0, (N+1)\pi)$. We obtain that our problem has a sequence of eigenvalues $\lambda_n = \frac{1}{h^4}\cos^4\left(\frac{z_n h}{2}\right)$ and a complete set of eigenfunctions Φ^n, $1 \leq n \leq N$.

Observability inequality for discrete clamped beam

The observability inequality is equivalent to

$$\sum_{1\le|n|\le N} |a_n|^2 \le C \int_0^T \left| \sum_{1\le|n|\le N} a_n e^{i\,\mathrm{sgn}(n)\sqrt{\lambda_{|n|}}t} \frac{\Phi_N^{|n|}}{\sqrt{\lambda_{|n|}}} \right|^2 dt. \tag{35}$$

Inequality (35) follows with $C = C(T) = \mathcal{O}\left(\frac{\kappa}{T}\right)$ since

1. For any $T > 0$ there exists $n_T = \mathcal{O}(1/T) \in \mathbb{N}$, independent of h, such that the following inequality holds

$$\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \ge \frac{2\pi}{T} \qquad (n_T \le n \le N - n_T)\,. \tag{36}$$

2. There exists a constant $C > 0$, independent of h, such that

$$\Phi_N^n \ge C\sqrt{\lambda_n} \qquad (1 \le n \le N)\,. \tag{37}$$

We obtain that the discrete clamped beam equation is uniformly controllable in any time. As in the continuous case, the observability constant explodes as $\exp(\kappa/T)$ as T tends to zero.

Thank you very much for your attention!