Approximation of the exact controls for the beam equation

Sorin Micu

University of Craiova (Romania)

Graz, June 25, 2015

Joint works with Florin Bugariu, Nicolae Cîndea, Ionel Rovenţa and Laurenţiu Temereancă
Given any time $T > 0$ and initial data

$$(u^0, u^1) \in \mathcal{H} := H^1_0(0, \pi) \times H^{-1}(0, \pi),$$

the exact controllability in time $T$ of the linear beam equation with hinged (simply-supported) ends,

\[ \begin{cases} u''(t, x) + u_{xxxx}(t, x) = 0, & x \in (0, \pi), \ t > 0 \\ u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = 0, & t > 0 \\ u_{xx}(t, \pi) = v(t), & t > 0 \\ u(0, x) = u^0(x), \ u'(0, x) = u^1(x), & x \in (0, \pi) \end{cases} \] (1)

consists of finding a scalar function $v \in L^2(0, T)$, called control, such that the corresponding solution $(u, u')$ of (1) verifies

$$u(T, \cdot) = u'(T, \cdot) = 0.$$ (2)
Several approaches are available for the study of a controllability problem:

- Moment theory
Several approaches are available for the study of a controllability problem:

- Moment theory
- Direct methods
Several approaches are available for the study of a controllability problem:

- Moment theory
- Direct methods
- Transmutation methods
Several approaches are available for the study of a controllability problem:

- Moment theory
- Direct methods
- Transmutation methods
- Uniform stabilization
Several approaches are available for the study of a controllability problem:

- Moment theory
- Direct methods
- Transmutation methods
- Uniform stabilization
- Optimization methods (Hilbert Uniqueness Method)
Several approaches are available for the study of a controllability problem:

- Moment theory
- Direct methods
- Transmutation methods
- Uniform stabilization
- Optimization methods (Hilbert Uniqueness Method)
  - Multipliers
Several approaches are available for the study of a controllability problem:

- Moment theory
- Direct methods
- Transmutation methods
- Uniform stabilization
- Optimization methods (Hilbert Uniqueness Method)
  - Multipliers
  - Carleman estimates
Several approaches are available for the study of a controllability problem:

- Moment theory
- Direct methods
- Transmutation methods
- Uniform stabilization
- Optimization methods (Hilbert Uniqueness Method)
  - Multipliers
  - Carleman estimates
  - Microlocal Analysis
Several approaches are available for the study of a controllability problem:

- **Moment theory**
- Direct methods
- Transmutation methods
- Uniform stabilization
- **Optimization methods (Hilbert Uniqueness Method)**
  - Multipliers
  - Carleman estimates
  - Microlocal Analysis


Lemma

Let $T > 0$ and $(u^0, u^1) \in \mathcal{H}$. The function $v \in L^2(0,T)$ is a control which drives to zero the solution of (1) in time $T$ if and only if, for any $(\varphi^0, \varphi^1) \in \mathcal{H}$,

$$
\int_{0}^{T} v(t) \varphi_x(t, 1) \, dt = - \langle u^1(x), \varphi(0, x) \rangle_{-1,1} + \langle u^0(x), \varphi'(0, x) \rangle_{1,-1},
$$

where $(\varphi, \varphi') \in \mathcal{H}$ is the solution of the backward equation

$$
\begin{cases}
\varphi''(t, x) + \varphi_{xxxx}(t, x) = 0 & (t, x) \in (0, T) \times (0, 1) \\
\varphi(t, 0) = \varphi(t, 1) = 0 & t \in (0, T) \\
\varphi_{xx}(t, 0) = \varphi_{xx}(t, 1) = 0 & t \in (0, T) \\
\varphi(T, x) = \varphi^0(x) & x \in (0, 1) \\
\varphi'(T, x) = \varphi^1(x) & x \in (0, 1).
\end{cases}
$$

(3)
For each \((u^0, u^1) \in \mathcal{H}\), define the functional \(J : \mathcal{H} \rightarrow \mathbb{R}\),

\[
J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(t, 1)|^2 \, dt + \langle u^1(x), \varphi(0, x) \rangle_{1,1} - \langle u^0(x), \varphi'(0, x) \rangle_{1,-1},
\]

where \((\varphi, \varphi')\) is the solution of (3) with initial data \((\varphi^0, \varphi^1)\).
For each \((u^0, u^1) \in \mathcal{H}\), define the functional \(J : \mathcal{H} \to \mathbb{R}\),

\[
J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(t, 1)|^2 \, dt + \langle u^1(x), \varphi(0, x) \rangle_{-1,1} - \langle u^0(x), \varphi'(0, x) \rangle_{1,-1},
\]

where \((\varphi, \varphi')\) is the solution of (3) with initial data \((\varphi^0, \varphi^1)\).

- If \(J\) has a minimum at \((\hat{\varphi}^0, \hat{\varphi}^1) \in \mathcal{H}\) then \(\hat{v}(t) = \hat{\varphi}_x(1, t)\) is a control for (1).
Optimization method

For each \((u^0, u^1) \in \mathcal{H}\), define the functional \(J : \mathcal{H} \rightarrow \mathbb{R}\),

\[
J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(t, 1)|^2 dt + \langle u^1(x), \varphi(0, x) \rangle_{-1,1} - \langle u^0(x), \varphi'(0, x) \rangle_{1,-1},
\]

where \((\varphi, \varphi')\) is the solution of (3) with initial data \((\varphi^0, \varphi^1)\).

- If \(J\) has a minimum at \((\hat{\varphi}^0, \hat{\varphi}^1) \in \mathcal{H}\) then \(\hat{v}(t) = \hat{\varphi}_x(1, t)\) is a control for (1).

- \(J\) has a minimum if it is coercive and it is coercive if the following observability inequality holds for any \((\varphi^0, \varphi^1) \in \mathcal{H}\):

\[
\| (\varphi(0), \varphi'(0)) \|^2_{\mathcal{H}} \leq C \int_0^T |\varphi_x(t, \pi)|^2 dt. \tag{4}
\]
Optimization method

For each \((u^0, u^1) \in \mathcal{H}\), define the functional \(J : \mathcal{H} \to \mathbb{R}\),

\[
J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(t, 1)|^2 \, dt + \langle u^1(x), \varphi(0, x) \rangle_{-1,1} - \langle u^0(x), \varphi'(0, x) \rangle_{1,-1},
\]

where \((\varphi, \varphi')\) is the solution of (3) with initial data \((\varphi^0, \varphi^1)\).

- If \(J\) has a minimum at \((\hat{\varphi}^0, \hat{\varphi}^1) \in \mathcal{H}\) then \(\hat{v}(t) = \hat{\varphi}_x(1, t)\) is a control for (1).
- \(J\) has a minimum if it is coercive and it is coercive if the following observability inequality holds for any \((\varphi^0, \varphi^1) \in \mathcal{H}\):

\[
\|(\varphi(0), \varphi'(0))\|_{\mathcal{H}}^2 \leq C \int_0^T |\varphi_x(t, \pi)|^2 dt. \tag{4}
\]

- Hence, if (4) holds, for any initial data \((u^0, u^1) \in \mathcal{H}\), there exists a control \(v \in L^2(0, T)\) with the property

\[
\|v\|_{L^2} \leq \sqrt{C} \|(u^0, u^1)\|_{\mathcal{H}}. \tag{5}
\]
Ingham’s inequality

Observability inequality (4) is equivalent to inequality of the form

\[ \sum_{n \in \mathbb{Z}^*} |\alpha_n|^2 \leq C(T) \int_{-T/2}^{T/2} \left| \sum_{n \in \mathbb{Z}^*} \alpha_n e^{\nu_n t} \right|^2 dt, \quad (\alpha_n)_{n \in \mathbb{Z}^*} \in \ell^2. \tag{6} \]

Ingham’s inequality

For any \( T > \frac{2\pi}{\gamma_\infty} \), \( \gamma_\infty = \liminf_{n \to \infty} |\nu_{n+1} - \nu_n| \), inequality (6) holds.


In our particular case

\[ \nu_n = i \, \text{sgn}(n) \, n^2, \quad \gamma_\infty = \lim_{n \to \infty} \inf |\nu_{n+1} - \nu_n| = \infty. \]

Ingham’s inequality implies that the observability inequality (4) is verified for any \( T > 0 \).

Consequently, given any \( T > 0 \), there exists a control \( v \in L^2(0, T) \) for each \( (u^0, u^1) \in \mathcal{H} \).

The control function \( v \) is not unique.
The null-controllability of the beam equation is equivalent to solve a moment problem.

**Lemma**

Let $T > 0$ and

$$(u^0, u^1) = \left(\sum_{n=1}^{\infty} a^0_n \sin(nx), \sum_{n=1}^{\infty} a^1_n \sin(nx)\right) \in \mathcal{H}.$$

The function $v \in L^2(0, T)$ is a control which drives to zero the solution of $(1)$ in time $T$ if and only if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} v \left(t + \frac{T}{2}\right) e^{t\nu_n} dt = \frac{(-1)^n e^{-\frac{T}{2} \nu_n}}{\sqrt{2n\pi}} \left(\nu_n a^0_n - a^1_n\right) \quad (n \in \mathbb{Z}^*),$$

(7)

where $\nu_n = i \text{sgn}(n) n^2$ are the eigenvalues of the unbounded skew-adjoint differential operator corresponding to $(1)$.

A solution $v$ of the moment problem may be constructed by means of a biorthogonal family to the sequence $(e^{\nu_n t})_{n\in\mathbb{Z}^*}$. 
## Moment problem for the beam equation

### Definition

A family of functions $(\phi_m)_{m \in \mathbb{Z}^*} \subset L^2 \left(-\frac{T}{2}, \frac{T}{2}\right)$ with the property

\[
\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_m(t)e^{\nu n t} \, dt = \delta_{mn} \quad \forall \, m, n \in \mathbb{Z}^*,
\] (8)

is called a biorthogonal sequence to $(e^{\nu n t})_{n \in \mathbb{Z}^*}$ in $L^2 \left(-\frac{T}{2}, \frac{T}{2}\right)$. 

---

**Definition**

A family of functions $(\phi_m)_{m \in \mathbb{Z}^*} \subset L^2 \left(-\frac{T}{2}, \frac{T}{2}\right)$ with the property

\[
\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_m(t)e^{\nu n t} \, dt = \delta_{mn} \quad \forall \, m, n \in \mathbb{Z}^*,
\] (8)

is called a biorthogonal sequence to $(e^{\nu n t})_{n \in \mathbb{Z}^*}$ in $L^2 \left(-\frac{T}{2}, \frac{T}{2}\right)$. 

---
Moment problem for the beam equation

Definition

A family of functions \((\phi_m)_{m \in \mathbb{Z}^*} \subset L^2\left(\frac{-T}{2}, \frac{T}{2}\right)\) with the property

\[
\int_{\frac{-T}{2}}^{\frac{T}{2}} \phi_m(t)e^{\nu n t} dt = \delta_{mn} \quad \forall m, n \in \mathbb{Z}^*,
\]  

(8)

is called a biorthogonal sequence to \((e^{\nu n t})_{n \in \mathbb{Z}^*}\) in \(L^2\left(\frac{-T}{2}, \frac{T}{2}\right)\).

Once we have a biorthogonal sequence to \((e^{\nu n t})_{n \in \mathbb{Z}^*}\), a “formal” solution of the moment problem is given by

\[
v(t) = \sum_{n \in \mathbb{Z}^*} \frac{(-1)^n e^{-\frac{T}{2} \nu n}}{\sqrt{2n\pi}} \left(\nu_n a_0^n - a_1^n\right) \phi_n \left(t - \frac{T}{2}\right). \quad (9)
\]
Ingham’s inequality and the existence of a biorthogonal sequence

Consider a Hilbert space $H$ and a family $(f_n)_{n \in \mathbb{Z}^*} \subset H$ such that

$$
\sum_{n \in \mathbb{Z}^*} |a_n|^2 \leq C_1 \left\| \sum_{n \in \mathbb{Z}^*} a_n f_n \right\|^2, \quad (a_n)_{n \in \mathbb{Z}^*} \in \ell^2.
$$

(10)

Then there exists a biorthogonal sequence to the family $(f_n)_{n \in \mathbb{Z}^*}$.

- $(f_n)_{n \in \mathbb{Z}^*}$ is minimal i.e.

  $$
f_m \notin \text{Span} \left\{ (f_n)_{n \in \mathbb{Z}^* \setminus \{m\}} \right\} \quad (m \in \mathbb{Z}^*).$$

- Apply Hahn-Banach Theorem to $\{f_m\}$ and $\text{Span} \left\{ (f_n)_{n \in \mathbb{Z}^* \setminus \{m\}} \right\}$. There exists $\phi_m \in H$ such that

  $$
  (\phi_m, f_m) = 1 \quad \text{and} \quad (\phi_m, f_n) = 0 \quad \text{for any} \quad n \neq m.
  $$

- The biorthogonal sequence which is bounded:

  $$
  \left\| \sum_{n \in \mathbb{Z}^*} b_n \phi_n \right\|^2 \leq \frac{1}{C_1} \sum_{n \in \mathbb{Z}^*} |b_n|^2.
  $$
If we are in a context in which no Ingham’s type inequality is available? We can take the inverse way:

Construction of the biorthogonal

**Paley-Wiener Theorem:** Let $F : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function of exponential type ($|F(z)| \leq M e^{T|z|}$) which belongs to $L^2(\mathbb{R})$ on the real axis. Then $\int_{\mathbb{R}} F(t) e^{ixt} dt$ is a function from $L^2(-T, T)$.


$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} F(t) e^{ixt} dt \quad \Rightarrow \quad \left\{ \begin{array}{l} F(t) = \int_{-T}^{T} f(x) e^{-ixt} dx; \\ \|f\|_{L^2} = \sqrt{2\pi}\|F\|_{L^2(\mathbb{R})}. \end{array} \right.$$
Finite differences for the beam equation

\( N \in \mathbb{N}^*, h = \frac{\pi}{N+1}, x_j = jh, 0 \leq j \leq N + 1, \)
\( x_{-1} = -h, x_{N+2} = \pi + h. \)

\[
\begin{align*}
\begin{cases}
  u_j''(t) = -\frac{u_{j+2}(t)-4u_{j+1}+6u_j(t)-4u_{j-1}(t)+u_{j-2}(t)}{h^4}, & t > 0 \\
  u_0(t) = u_{N+1}(t) = 0, & u_{-1}(t) = -u_1(t), & t > 0 \\
  u_{N+2} = -u_N + h^2v_h(t), & t > 0 \\
  u_j(0) = u_j^0, & u_j'(0) = u_j^1, ~ 1 \leq j \leq N.
\end{cases}
\end{align*}
\]
(11)
Finite differences for the beam equation

\[ N \in \mathbb{N}^*, \ h = \frac{\pi}{N+1}, \ x_j = jh, \ 0 \leq j \leq N + 1, \]
\[ x_{-1} = -h, \ x_{N+2} = \pi + h. \]

\[
\begin{cases}
  u_j''(t) = -\frac{u_{j+2}(t) - 4u_{j+1} + 6u_j(t) - 4u_{j-1}(t) + u_{j-2}(t)}{h^4}, \ t > 0 \\
  u_0(t) = u_{N+1}(t) = 0, \ u_{-1}(t) = -u_1(t), \ t > 0 \\
  u_{N+2} = -u_N + h^2v_h(t), \ t > 0 \\
  u_j(0) = u_0^j, \ u_j'(0) = u_1^j, \ 1 \leq j \leq N.
\end{cases}
\]  \hspace{1cm} \text{(11)}

**Discrete controllability problem:** given \( T > 0 \) and \((U_h^0, U_h^1) = (u_j^0, u_j^1)_{1 \leq j \leq N} \in \mathbb{C}^{2N}\), there exists a control function \( v_h \in L^2(0, T) \) such that the solution \( u \) of (11) satisfies

\[ u_j(T) = u_j'(T) = 0, \ \forall j = 1, 2, ..., N. \]  \hspace{1cm} \text{(12)}

System (11) consists of \( N \) linear differential equations with \( N \) unknowns \( u_1, u_2, ..., u_N \).

\[ u_j(t) \approx u(t, x_j) \text{ if } (U_h^0, U_h^1) \approx (u^0, u^1). \]
Discrete controls

- Existence of the discrete control $v_h$.
- Boundedness of the sequence $(v_h)_{h>0}$ in $L^2(0, T)$.
- Convergence of the sequence $(v_h)_{h>0}$ to a control $v$ of the beam equation (1).
Discrete controls

- Existence of the discrete control $v_h$.
- Boundedness of the sequence $(v_h)_{h>0}$ in $L^2(0, T)$.
- Convergence of the sequence $(v_h)_{h>0}$ to a control $v$ of the beam equation (1).

Equivalent vectorial form

System (11) is equivalent to

\[
\begin{aligned}
U_h''(t) + (A_h)^2 U_h(t) &= F_h(t) \quad t \in (0, T) \\
U_h(0) &= U_h^0 \\
U'_h(0) &= U_h^1,
\end{aligned}
\]  

(13)

\[A_h = \frac{1}{h^2} \begin{pmatrix}
2 & -1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 2 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 2 & -1 \\
0 & 0 & 0 & 0 & \ldots & -1 & 2
\end{pmatrix}, \quad \begin{pmatrix}
U_0(t) \\
U_1(t) \\
\vdots \\
U_N(t)
\end{pmatrix}, \quad \begin{pmatrix}
u_1(t) \\
u_2(t) \\
\vdots \\
u_N(t)
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
\vdots \\
-v_h(t)
\end{pmatrix}, \quad U_h^0 = \begin{pmatrix}
u_1^0 \\
u_2^0 \\
\vdots \\
u_N^0
\end{pmatrix}, \quad U_h^1 = \begin{pmatrix}
u_1^1 \\
u_2^1 \\
\vdots \\
u_N^1
\end{pmatrix}.
\]
Discrete observability inequality

\[
\begin{align*}
W''_h(t) + A^2_h W_h(t) &= 0 \quad t \in (0, T) \\
W_h(T) &= W^0_h \in \mathbb{C}^N \\
W'_h(T) &= W^1_h \in \mathbb{C}^N.
\end{align*}
\] (14)

The energy of (14) is defined by

\[
E_h(t) = \frac{1}{2} \left( \langle A_h W_h(t), W_h(t) \rangle + \langle A^{-1}_h W'_h(t), W'_h(t) \rangle \right),
\] (15)

and the following relation holds:

\[
\frac{d}{dt} E_h(t) = 0.
\] (16)

The exact controllability in time \( T \) of (11) holds if the following discrete observability inequality is true

\[
E_h(t) \leq C(T, h) \int_0^T \left| \frac{W_{h,N}(t)}{h} \right|^2 dt, \quad (W^0_h, W^1_h) \in \mathbb{C}^{2N}.
\] (17)
One or two problems

Eigenvalues:
\[ \nu_n = i \text{sgn}(n) \mu_n, \quad \mu_n = \frac{4}{h^2} \sin^2 \left( \frac{n\pi h}{2} \right), \quad 1 \leq |n| \leq N. \]

Eigenvectors form an orthogonal basis in \( \mathbb{C}^{2N} \):

\[ \phi^n = \frac{1}{\sqrt{2\mu_n}} \begin{pmatrix} \varphi^n \\ -\nu_n \varphi^n \end{pmatrix}, \quad \varphi^n = \sqrt{2} \begin{pmatrix} \sin(nh\pi) \\ \sin(2nh\pi) \\ \vdots \\ \sin(Nnh\pi) \end{pmatrix}, \quad 1 \leq |n| \leq N. \]

The observability constant is not uniform in \( h \):

\[ (W_h^0, W_h^1) = \phi^N \Rightarrow C(T, h) = \frac{1}{T \cos^2 \left( \frac{N\pi h}{2} \right)} \approx \frac{1}{Th^2}. \]

There are initial data \( (u^0, u^1) \in \mathcal{H} \) such that the sequence of discrete minimal \( L^2 \)-norm controls \( (\hat{\upsilon}_h)_h \) diverges!!!
Problems from the bad numerical approximation of high eigenmodes (spurious numerical eigenmodes).

- Control the projection of the solution over the space $\text{Span}\{\phi^n : 1 \leq |n| \leq \gamma N\}$, with $\gamma \in (0, 1)$.

\[
\sum_{1 \leq |n| \leq \gamma N} |\alpha_n|^2 \leq C \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{1 \leq |n| \leq \gamma N} \alpha_n e^{\nu_n t} \right|^2 dt. \tag{18}
\]

- Introduce a new control which vanishes in the limit

\[
E_h(t) \leq C \left[ \int_0^T \left| \frac{W_{hN}(t)}{h} \right|^2 dt + h^2 \int_0^T \left| \frac{W'_{hN}(t)}{h} \right|^2 dt \right]. \tag{19}
\]

$C = C(T) \Rightarrow$ uniform controllability $\Rightarrow$ convergence of the discrete controls.
Regularity and filtration of the initial data

We consider the controlled system

\[
\begin{align*}
U''_h(t) + (A_h)^2 U_h(t) &= F_h(t) \quad t \in (0, T) \\
U_h(0) &= U^0_h \\
U'_h(0) &= U^1_h,
\end{align*}
\]

We suppose that one of the following properties holds:

- Initial data \((u^0, u^1)\) are sufficiently smooth (for instance, in \(H^3(0, 1) \times H^1_0(0, 1)\)) and discretized by points
  \[U^0 = (u^0(jh))_{1 \leq j \leq N}, \quad U^1 = (u^1(jh))_{1 \leq j \leq N};\]
- Initial data \((u^0, u^1)\) are in the energy space \(\mathcal{H}\) and the high frequencies of their discretization are filtered out,
  \[(U^0, U^1) = \sum_{1 \leq |n| \leq \delta N} a_{nh} \Phi^n \quad (\delta \in (0, 1));\]

Can we obtain the uniform controllability in any \(T > 0\)?
Lemma

Let $T > 0$ and $\varepsilon > 0$. System (20) is null-controllable in time $T$ if and only if, for any initial datum $(U^0_h, U^1_h) \in \mathbb{C}^{2N}$ of form

$$
(U^0_h, U^1_h) = \left( \sum_{j=1}^{N} a^0_{jh} \varphi^j, \sum_{j=1}^{N} a^1_{jh} \varphi^j \right),
$$

(21)

there exists a control $v_h \in L^2(0, T)$ such that

$$
\int_{0}^{T} v_h(t) e^{\bar{n} t} dt = \frac{(-1)^n h}{\sqrt{2} \sin(|n| \pi h)} \left( -a^1_{|n|h} + \bar{n} a^0_{|n|h} \right),
$$

(22)

for any $n \in \mathbb{Z}^*$ such that $|n| \leq N$. 

Biorthogonal family

If \((\theta_m)_{1 \leq |m| \leq N} \subset L^2 \left( -\frac{T}{2}, \frac{T}{2} \right)\) is a biorthogonal sequence to the family of exponential functions \((e^{\nu_n t})_{1 \leq |n| \leq N}\) in \(L^2 \left( -\frac{T}{2}, \frac{T}{2} \right)\) then a control of (13) will be given by

\[
v_h(t) = \sum_{1 \leq |n| \leq N} \frac{(-1)^n h e^{-\nu_n \frac{T}{2}}}{\sqrt{2} \sin(|n|\pi h)} \left( -a^1_{|n|h} + \nu_n a^0_{|n|h} \right) \theta_n \left( t - \frac{T}{2} \right).
\]

We look for a biorthogonal sequence \((\theta_m)_{1 \leq |m| \leq N}\) to \((e^{i\nu_n t})_{1 \leq |n| \leq N}\) and we try to estimate the right hand side sum. The exponents are real:

\[
\nu_n = \text{sgn}(n) \frac{4}{h^2} \sin \left( \frac{n\pi h}{2} \right) \quad (1 \leq |n| \leq N).
\]
Taking into account that
\[ \nu_{n+1} - \nu_n = \frac{4}{h^2} \sin \left( \frac{n\pi h}{2} \right) \sin \left( \frac{(2n+1)\pi h}{2} \right) > \begin{cases} n & \text{if } \delta < |n| < \delta N \\ 4 & \text{otherwise}, \end{cases} \]

we can use Ingham’s inequality and a Kahane’s argument to show that, for any \( T > 0 \), there exists a biorthogonal \((\theta_m)_{1 \leq |m| \leq N}\) to the family \((e^{i\nu_n t})_{1 \leq |n| \leq N}\) with the property that

\[
\left\| \sum_{1 \leq |n| \leq N} b_n \theta_n \right\|^2 \leq C \exp \left( \frac{C}{T} \right) \sum_{1 \leq |n| \leq N} |b_n|^2.
\]

It follows that
\[
\|v_h(t)\|^2 = \left\| \sum_{1 \leq |n| \leq N} \frac{(-1)^n h e^{-\nu_n \frac{T}{2}}}{\sqrt{2} \sin(|n|\pi h)} \left(-a_{|n|h}^1 + \nu_n a_{|n|h}^0 \right) \theta_n \left(t - \frac{T}{2}\right) \right\|^2
\]
\[
\leq C \exp \left( \frac{C}{T} \right) \sum_{1 \leq |n| \leq N} \frac{h^2}{\sin^2(n\pi h)} \left(|a_{|n|h}^1|^2 + |\nu_n|^2 |a_{|n|h}^0|^2\right).
\]
\[ \|v_h(t)\|^2 \leq C \exp \left( \frac{C}{T} \right) \sum_{1 \leq |n| \leq N} \frac{h^2}{\sin^2(n\pi h)} \left( |a^1_{|n|h}|^2 + |\nu_n|^2 |a^0_{|n|h}|^2 \right). \]

- The initial data to be controlled are in \( H^3(0,1) \times H^1_0(0,1) \)

\[ \sum_{1 \leq |n| \leq N} n^2 \left( |a^1_{|n|h}|^2 + |\nu_n|^2 |a^0_{|n|h}|^2 \right) \leq C \| (u^0, u^1) \|_{3,1}^2 \]

\[ \Rightarrow \|v_h\|^2 \leq C \exp \left( \frac{C}{T} \right) \| (u^0, u^1) \|_{3,1}^2. \]

- The high frequencies of the discrete initial data are filtered out

\[ \|v_h\|^2 \leq C(\delta) \exp \left( \frac{C}{T} \right) \sum_{1 \leq |n| \leq \delta N} \frac{1}{n^2} \left( |a^1_{|n|h}|^2 + |\nu_n|^2 |a^0_{|n|h}|^2 \right) \]

\[ \leq C''(\delta) \exp \left( \frac{C}{T} \right) \| (u^0, u^1) \|_{1,-1}^2. \]
Numerical results

Figure: Initial data to be controlled.

\[ N = 100; \quad T = .3; \]

A conjugate gradient method for the corresponding discrete optimization approach.
Numerical results

Figure: Example 2 - The first four iterations of the conjugate gradient method for the approximation of $v_h$ with $N = 100$ without filtration.
Numerical results

Figure: The approximation of the control $v_h$ with $N = 100, 200, 500$ and $1000$ by using filtration of the initial data with $\delta = \frac{1}{40}$. 
Figure: Controlled solution and the approximation of the control with \( N = 100 \) by using filtration of the initial data \( \delta = \frac{1}{40} \).
Instead of (13) we consider the system

\[
\begin{aligned}
U_{hh}''(t) + (A_h)^2 U_{hh}(t) + \varepsilon A_h U_{hh}'(t) &= F_h(t) \quad t \in (0, T) \\
U_h(0) &= U_{h0} \\
U_h'(0) &= U_{h1},
\end{aligned}
\]

(23)

- \( \varepsilon = \varepsilon(h), \quad \lim_{h \to 0} \varepsilon = 0 \)

- If \( F_h = 0 \), \( \frac{dE_h}{dt}(t) = -\varepsilon \langle A_h U_h'(t), U_h'(t) \rangle \leq 0 \)

- The term \( \varepsilon A_h U_{hh}'(t) \) represents a numerical vanishing viscosity.
Numerical vanishing viscosity

Instead of (13) we consider the system

\[
\begin{aligned}
U''_h(t) + (A_h)^2 U_h(t) + \varepsilon A_h U'_h(t) &= F_h(t) \quad t \in (0, T) \\
U_h(0) &= U^0_h \\
U'_h(0) &= U^1_h,
\end{aligned}
\]  

(23)

- \[ \varepsilon = \varepsilon(h), \quad \lim_{h \to 0} \varepsilon = 0 \]

- If \( F_h = 0 \), \[ \frac{dE_h}{dt}(t) = -\varepsilon \langle A_h U'_h(t), U'_h(t) \rangle \leq 0 \]

- The term \( \varepsilon A_h U'_h(t) \) represents a numerical vanishing viscosity.

Can we obtain the uniform controllability in any \( T > 0 \) (without projection or additional controls) using this new discrete scheme?


At the interface between parabolic and hyperbolic equations: singular limit control problem.


Spectral analysis. Good news but no Ingham.

Eigenvalues: \( \lambda_n = \frac{1}{2} \left( \varepsilon + i \text{sgn} \left( n \right) \sqrt{4 - \varepsilon^2} \right) \mu |n|, \ 1 \leq |n| \leq N. \)

Eigenvectors:

\[
\phi^n = \frac{1}{\sqrt{2\mu_n}} \begin{pmatrix} \varphi^n \\ -\lambda_n \varphi^n \end{pmatrix}, \quad \varphi^n = \sqrt{2} \begin{pmatrix} \sin(nh\pi) \\ \sin(2nh\pi) \\ \vdots \\ \sin(Nnh\pi) \end{pmatrix}, \ 1 \leq |n| \leq N.
\]

If \((W_h^0, W_h^1) = \phi^N\) we obtain that

\[
C(T, h) = \int_0^T \left| \frac{W_{hN}(t)}{h} \right|^2 dt \approx \frac{1}{\cos^2 \left( \frac{N\pi h}{2} \right)} \frac{\Re(\lambda_N)}{e^{2T\Re(\lambda_N)} - 1}.
\]

To ensure the uniform observability of these initial data we need

\[
\varepsilon > C \ln \left( \frac{1}{h} \right) h^2
\]
Spectral analysis. Good news but no Ingham.

Eigenvalues: \( \lambda_n = \frac{1}{2} \left( \varepsilon + i \text{sgn}(n) \sqrt{4 - \varepsilon^2} \right) \mu_{|n|}, \ 1 \leq |n| \leq N. \)

Eigenvectors:

\[
\phi^n = \frac{1}{\sqrt{2\mu_n}} \begin{pmatrix}
\varphi^n \\
-\lambda_n \varphi^n
\end{pmatrix}, \quad \varphi^n = \sqrt{2} \begin{pmatrix}
sin(nh\pi) \\
sin(2nh\pi) \\
\vdots \\
sin(Nnh\pi)
\end{pmatrix}, \ 1 \leq |n| \leq N.
\]

If \((W^0_h, W^1_h) = \phi^N\) we obtain that

\[
C(T, h) = \int_0^T \left| \frac{W_{hN}(t)}{h} \right|^2 dt \approx \frac{1}{\cos^2 \left( \frac{N\pi h}{2} \right)} \frac{\Re(\lambda_N)}{e^{2T\Re(\lambda_N)} - 1}.
\]

To ensure the uniform observability of these initial data we need

\[
\varepsilon > C \ln \left( \frac{1}{h} \right) h^2 \Rightarrow \Re(\lambda_N) > C \ln \left( \frac{1}{h} \right).
\]
Discrete moments problem

Lemma

Let $T > 0$ and $\varepsilon > 0$. System (13) is null-controllable in time $T$ if and only if, for any initial datum $(U_0^h, U_1^h) \in \mathbb{C}^{2N}$ of form

$$(U_0^h, U_1^h) = \left( \sum_{j=1}^{N} a_{j}^0 \varphi^j , \sum_{j=1}^{N} a_{j}^1 \varphi^j \right),$$

(24)

the exists a control $v_h \in L^2(0, T)$ such that

$$\int_0^T v_h(t)e^{\lambda nt}dt = \frac{(-1)^n h}{\sqrt{2 \sin(|n|\pi h)}} \left( -a_{|n|}^1 + (\lambda n - \varepsilon \mu |n|)a_{|n|}^0 \right),$$

(25)

for any $n \in \mathbb{Z}^*$ such that $|n| \leq N$. 
If $(\theta_m)_{1 \leq |m| \leq N} \subset L^2 \left( -\frac{T}{2}, \frac{T}{2} \right)$ is a biorthogonal sequence to the family of exponential functions $(e^{\lambda_n t})_{1 \leq |n| \leq N}$ in $L^2 \left( -\frac{T}{2}, \frac{T}{2} \right)$ then a control of (13) will be given by

$$v_h(t) = \sum_{1 \leq |n| \leq N} \frac{(-1)^n h e^{-\lambda_n T}}{\sqrt{2} \sin(|n|\pi h)} \left( -a_{|n|h}^1 + (\lambda_n - \varepsilon \mu_{|n|})a_{|n|h}^0 \right) \theta_n \left( t - \frac{T}{2} \right).$$

Now the main task is to show that there exists a biorthogonal sequence $(\theta_m)_{1 \leq |m| \leq N}$ and to evaluate its $L^2$-norm in order to estimate the right hand side sum.

Main differences:

- We have the optimal value of the viscosity parameter $\varepsilon$:

\[
\varepsilon \geq C h^2 \ln \left( \frac{1}{h} \right).
\]

Main differences:

- We have the optimal value of the viscosity parameter $\varepsilon$:
  $$\varepsilon \geq C h^2 \ln \left( \frac{1}{h} \right).$$

- The controllability time $T$ should be arbitrarily small.
Suppose that \((\theta_m)^{1 \leq |m| \leq N}\) is a biorthogonal sequence to the family of exponential functions \((e^{\lambda_n t})^{1 \leq |n| \leq N}\) in \(L^2\left(\frac{-T}{2}, \frac{T}{2}\right)\) and define

\[
\Psi_m(z) = \int_{\frac{-T}{2}}^{\frac{T}{2}} \theta_m(t)e^{-itz} \, dt.
\]

- \(\Psi_m(i\lambda_n) = \delta_{nm}\)
- \(\Psi_m\) is an entire function of exponential type \(\frac{T}{2}\)
- \(\Psi_n \in L^2(\mathbb{R})\)

Paley-Wiener Theorem ensures that the reciprocal is true and gives a constructive way to obtain a biorthogonal sequence.

\[
\Psi_m(z) = P_m(z) \times M_m(z) = \prod_{n \neq m} \frac{i\lambda_n - z}{i\lambda_n - i\lambda_m} \times M_m(z).
\]

\(P_m\) (the product) and \(M_m\) (the multiplier) should have small exponential type and good behavior on the real axis.
Construction of a biorthogonal (II) - A small picture

\((\xi_1 l)\) is a biorthogonal to family \(F_1\) which is finite.

\((\xi_2 k)\) is a biorthogonal to family \(F_2\) with good gap properties.

A biorthogonal \((\theta_m)\) to full family \(F_1 \cup F_2\) can be constructed by using the Fourier transforms \(\hat{\theta}_1 k\) and \(\hat{\theta}_2 l\).
Construction of a biorthogonal (II) - A small picture

- $(\xi_1^1)_l$ is a biorthogonal to family $F_1$ which is finite.
- $(\xi_2^2)_k$ is a biorthogonal to family $F_2$ with good gap properties.
- A biorthogonal $(\theta_m)_m$ to full family $F_1 \cup F_2$ can be constructed by using the Fourier transforms $\hat{\theta}_k^1$ and $\hat{\theta}_l^2$. 

Diagram:
- Eigenvalues of the problem
- Added values

Small gap family $F_1$

Large gap family $F_2$
Construction of a biorthogonal (III): The main result

Theorem

Let \( T > 0 \). There exist two positive constants \( h_0 \) and \( \varepsilon_0 \) such that for any \( h \in (0, h_0) \) and \( \varepsilon \in (c_0 h^2 \ln \left( \frac{1}{h} \right), c_0 h) \) there exists a biorthogonal \((\theta_m)_m\) to \((e^{\lambda_n t})_n\) and two constants \( \alpha < T \) and \( C = C(T) > 0 \) (independent of \( \varepsilon \) and \( h \)) such that

\[
\int_{-T/2}^{T/2} \left| \sum_m \alpha_m \theta_m(t) \right|^2 dt \leq C(T) \sum_m |\alpha_m|^2 e^{\alpha |\Re(\lambda_m)|},
\]

(26)

for any finite sequence \((\alpha_m)_m\).
Construction of a biorthogonal (III): The main result

Theorem

Let $T > 0$. There exist two positive constants $h_0$ and $\varepsilon_0$ such that for any $h \in (0, h_0)$ and $\varepsilon \in \left( c_0 h^2 \ln \left( \frac{1}{h} \right), c_0 h \right)$ there exists a biorthogonal $(\theta_m)_m$ to $(e^{\lambda_n t})_n$ and two constants $\alpha < T$ and $C = C(T) > 0$ (independent of $\varepsilon$ and $h$) such that

$$
\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_m \alpha_m \theta_m(t) \right|^2 dt \leq C(T) \sum_m |\alpha_m|^2 e^{\alpha |\Re(\lambda_m)|}, \quad (26)
$$

for any finite sequence $(\alpha_m)_m$.

Since

$$
v_h(t) = \sum_{1 \leq |n| \leq N} \frac{(-1)^n h e^{-\frac{T\lambda_n}{2}}}{\sqrt{2} \sin(|n| \pi h)} \left( -a^1_{|n|h} + (\lambda_n - \varepsilon \mu_{|n|}) a^0_{|n|h} \right) \theta_n \left( t - \frac{T}{2} \right).
$$

we obtain immediately from (26) the uniform boundedness (in $h$) of the family of controls $(v_h)_{h>0}$. 
Numerical results

Figure: Initial data to be controlled.

$$N = 100; T = 2.3; \varepsilon = h$$

A conjugate gradient method for the corresponding discrete optimization approach.
Numerical results

Figure: The first four iterations with $\varepsilon = 0$. 
Figure: The first four iterations with $\varepsilon = h$. 
Figure: Controlled solution and the control.
Given any time $T > 0$ and initial data

$$(u^0, u^1) \in \mathcal{H} := L^2(0, \pi) \times H^{-2}(0, \pi),$$

the exact controllability in time $T$ of the linear clamped beam equation,

$$\begin{cases}
  u''(t, x) + u_{xxxx}(t, x) = 0, & x \in (0, \pi), \ t > 0 \\
  u(t, 0) = u(t, \pi) = u_x(t, 0) = 0, & t > 0 \\
  u_x(t, \pi) = v(t), & t > 0 \\
  u(0, x) = u^0(x), \ u'(0, x) = u^1(x), & x \in (0, \pi)
\end{cases} \quad (27)$$

consists of finding a scalar function $v \in L^2(0, T)$, called control, such that the corresponding solution $(u, u')$ of (27) verifies

$$u(T, \cdot) = u'(T, \cdot) = 0. \quad (28)$$
Finite differences for the clamped beam equation

\[ N \in \mathbb{N}^*, \ h = \frac{\pi}{N+1}, \ x_j = jh, \ 0 \leq j \leq N + 1, \]
\[ x_{-1} = -h, \ x_{N+2} = \pi + h. \]

\[
\begin{cases}
  u''_j(t) = -\frac{u_{j+2}(t) - 4u_{j+1}(t) + 6u_j(t) - 4u_{j-1}(t) + u_{j-2}(t)}{h^4}, \ t > 0 \\
  u_0(t) = u_{N+1}(t) = 0, \ u_{-1}(t) = u_1(t), \ t > 0 \\
  u_{N+2} = u_N + 2hv_h(t), \ t > 0 \\
  u_j(0) = u^0_j, \ u'_j(0) = u^1_j, \ 1 \leq j \leq N.
\end{cases}
\] (29)

Discrete controllability problem: given \( T > 0 \) and \((U^0_h, U^1_h) = (u^0_j, u^1_j)_{1 \leq j \leq N} \in \mathbb{C}^{2N}\), there exists a control function \( v_h \in L^2(0, T) \) such that the solution \( u \) of (11) satisfies

\[ u_j(T) = u'_j(T) = 0, \ \forall j = 1, 2, ..., N. \] (30)
Discrete observability inequality

\[
\begin{cases}
  W_h''(t) + \tilde{B}_h W_h(t) = 0 \quad t \in (0, T) \\
  W_h(T) = W_h^0 \in \mathbb{C}^N \\
  W_h'(T) = W_h^1 \in \mathbb{C}^N.
\end{cases}
\]  

(31)

The energy of (31) is defined by

\[
E_h(t) = \frac{1}{2} \left( \langle \tilde{B}_h W_h(t), W_h(t) \rangle + \langle W_h'(t), W_h'(t) \rangle \right),
\]

and the following relation holds:

\[
\frac{d}{dt} E_h(t) = 0.
\]  

(33)

The exact controllability in time \( T \) of (29) holds if the following discrete observability inequality is true

\[
E_h(t) \leq C(T, h) \int_0^T \left| \frac{2W_{hN}(t)}{h^2} \right|^2 dt, \quad (W_h^0, W_h^1) \in \mathbb{C}^{2N}.
\]  

(34)
Spectral analysis

- **Continuous spectrum:** The eigenvalues of the corresponding differential operator are given by the positive roots of the equation \( \cos(z) - \cosh^{-1}(z) = 0 \), which are asymptotically exponentially close to the zeros of the \( \cos(z) \) function.

- **Discrete spectrum:** The eigenvalues of the corresponding discrete operator are given by the positive roots of the equation \( f(z) = 0 \), where

\[
f(z) = \cos z \pm \sin^2 \left( \frac{hz}{2} \right) + \frac{2 \left( 1 - \sin^4 \left( \frac{hz}{2} \right) \right) r^{N+1}(z)}{r^{2(N+1)}(z) - 2 \sin^2 \left( \frac{hz}{2} \right) r^{N+1}(z) + 1},
\]

\[
r(z) = 1 + 2 \sin^2 \left( \frac{zh}{2} \right) + \sqrt{\sin^2 \left( \frac{zh}{2} \right) \left( 1 + \sin^2 \left( \frac{zh}{2} \right) \right)}.
\]

Function \( f \) has a sequence of well separated roots \((z_n)_{1 \leq n \leq N} \subset (0, (N + 1)\pi)\). We obtain that our problem has a sequence of eigenvalues \( \lambda_n = \frac{1}{h^4} \cos^4 \left( \frac{z_n h}{2} \right) \) and a complete set of eigenfunctions \( \Phi^n, 1 \leq n \leq N \).
Observability inequality for discrete clamped beam

The observability inequality is equivalent to

\[
\sum_{1\leq |n|\leq N} |a_n|^2 \leq C \int_0^T \left| \sum_{1\leq |n|\leq N} a_ne^{i\text{sgn}(n)\sqrt{\lambda_{|n|}}t} \Phi_{|n|} \right|^2 dt. \quad (35)
\]

Inequality (35) follows with \( C = C(T) = \Theta \left( \frac{\kappa}{T} \right) \) since

1. For any \( T > 0 \) there exists \( n_T = \Theta(1/T) \in \mathbb{N} \), independent of \( h \), such that the following inequality holds

\[
\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \geq \frac{2\pi}{T} \quad (n_T \leq n \leq N - n_T). \quad (36)
\]

2. There exists a constant \( C > 0 \), independent of \( h \), such that

\[
\Phi^n_N \geq C \sqrt{\lambda_n} \quad (1 \leq n \leq N). \quad (37)
\]

We obtain that the discrete clamped beam equation is uniformly controllable in any time. As in the continuous case, the observability constant explodes as \( \exp(\kappa/T) \) as \( T \) tends to zero.
Thank you very much for your attention!