Control of PDE models involving memory terms

Enrique Zuazua[1] [2]

Ikerbasque – BCAM
Bilbao, Basque Country, Spain
zuazua@bcamath.org
http://enzuazua.net

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[2]Joint work with F. Chaves, Q. Lü, L. Rosier and X. Zhang

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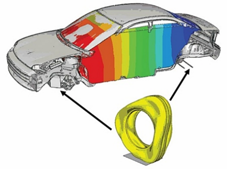
Why viscoelastic materials?[a]

[a] See H. T. Banks, S. Hu and Z. R. Kenz, A Brief Review of Elasticity and Viscoelasticity for Solids, Adv. Appl. Math. Mech., 3 (1), (2011), 1-51.

Viscoelastic materials are those for which the behavior combines liquid-like and solid-like characteristics.

Viscoelasticity is important in areas such as **biomechanics**, **power industry** or **heavy construction**:

- Synthetic polymers;
- Wood;
- Human tissue, cartilage;
- Metals at high temperature;
- Concrete, bitumen;
- ...



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Viscoelasticity

A wave equation with both viscous Kelvin-Voigt damping:

$$\begin{aligned} y_{tt} - \Delta y - \Delta y_t &= 1_\omega h, \qquad x \in \Omega,\ t \in (0, T), && (1)\\ y &= 0, \qquad x \in \partial\Omega,\ t \in (0, T), && (2)\\ y(x,0) = y_0(x),\ y_t(x,0) &= y_1(x) \qquad x \in \Omega. && (3) \end{aligned}$$

Here, Ω is a smooth, bounded open set in $\mathbb{R}^N$ and $h = h(x,t)$ is a control located in a open subset ω of Ω.

We want to study the following problem:

Given (y_0, y_1), to find a control h such that the associated solution to (1)-(3) satisfies

$$y(T) = y_t(T) = 0.$$

A geometric obstruction

Standard results on unique continuation do not apply. The principal part of the operator is

$$\partial_t \Delta.$$

Then characteristic hyperplanes are of the form

$$t = t_0 \quad \text{and} \quad x \cdot e = 1.$$

And the zero sets do not propagate by standard unique continuation arguments.

This phenomenon was previously observed by S. Micu in the context of the Benjamin-Bona-Mahoni equation [3] [4]

In that context the underlying operator is

$$\partial_t - \partial^3_{xxt}$$

but its principal part is the same

$$\partial^3_{xxt}.$$

[3] S. Micu, SIAM J. Control Optim., 39(2001), 1677–1696.

[4] X. Zhang and E. Z. Matematische Annalen, 325 (2003), 543-582.

Viscoelasticity = Waves + Heat

$$y_{tt} - \Delta y - \Delta y_t = 0$$

$$=$$

$$y_{tt} - \Delta y = 0$$

$$+$$

$$\partial_t [y_t] - \Delta y_t = 0$$

Both equations are controllable. Should then the superposition be controllable as well?

Interesting open question: The role of splitting and alternating directions in the controllability of PDE.

Viscoelasticity = Heat + ODE

$$
\begin{aligned}
y_t - \Delta y &= z, && (4)\\
z_t + z &= 1_\omega h, && (5)\\
y(x,t) = v(x,t) &= 0, \qquad (x,t) \in \partial\Omega \times (0,T), && (6)\\
z(x,0) &= z_0(x), \qquad x \in \Omega, && (7)\\
y(x,0) &= y_0(x), \qquad x \in \Omega. && (8)
\end{aligned}
$$

The question now becomes:
Given (y_0, z_0), to find a control h such that the associated solution to (9)-(13) satisfies

$$y(T) = z(T) = 0.$$

In this form the controllability of the system is less clear. We are acting on the ODE variable z. But the control action does not allow to control the whole z. We are effectively acting on y through z. What is the overall impact of the control?

Viscoelasticity = Heat + ODE. Second version

Note that

$$y_{tt} - \Delta y - \Delta y_t + y_t = (\partial_t - \Delta)(\partial_t + I).$$

Then

$$\begin{aligned}
y_t + y &= v, & (9)\\
v_t - \Delta v &= 1_\omega h, & (10)\\
v(x,t) = y(x,t) &= 0, \qquad (x,t) \in \partial\Omega \times (0,T), & (11)\\
v(x,0) &= y_1(x) + y_0(x), \qquad x \in \Omega, & (12)\\
y(x,0) &= y_0(x), \qquad x \in \Omega. & (13)
\end{aligned}$$

The question now becomes:
Given (y_0, z_0) to find a control h such that the associated solution to (9)-(13) satisfies

$$y(T) = v(T) = 0.$$

Viscoelasticity = Heat + Memory

Note that

$$y_{tt} - \Delta y - \Delta y_t = \partial_t [y_t - \Delta y - \Delta \int_0^t y].$$

The later, heat with memory, was addressed by Guerrero and Imanuvilov[5], showing that the system is not null controllable.

[5] S. Guerrero, O. Yu. Imanuvilov, Remarks on non controllability of the heat equation with memory, ESAIM: COCV, 19 (1)(2013), 288–300.

The controllability of the system is unclear:

$$\begin{aligned} v_t - \Delta v &= 1_\omega h, \\ y_t + y &= v. \end{aligned} \tag{14}$$

But we can consider the system with an added ficticious control:

$$\begin{aligned} v_t - \Delta v &= 1_\omega h, \\ y_t + y &= v + 1_\omega k. \end{aligned} \tag{15}$$

Control in two steps:

- Use the control h to control v to zero in time $T/2$.
- Then use the control k to control the ODE dynamics in the time-interval $[T/2, T]$.

Warning. The second step cannot be fulfilled since the ODE does not propagate the action of the controller which is confined in ω.

Possible solution: Make the control in the second equation move or, equivalently, replace the ODE by a transport equation.

This strategy was introduced and found to be successful in

P. Martin, L. Rosier, P. Rouchon, Null Controllability of the Structurally Damped Wave Equation with Moving Control, SIAM J. Control Optim., 51 (1)(2013), 660–684.
L. Rosier, B.-Y. Zhang, Unique continuation property and control for the Benjamin-Bona-Mahony equation on a periodic domain, J. Differential Equations 254 (2013), 141-178.

by using Fourier series decomposition.

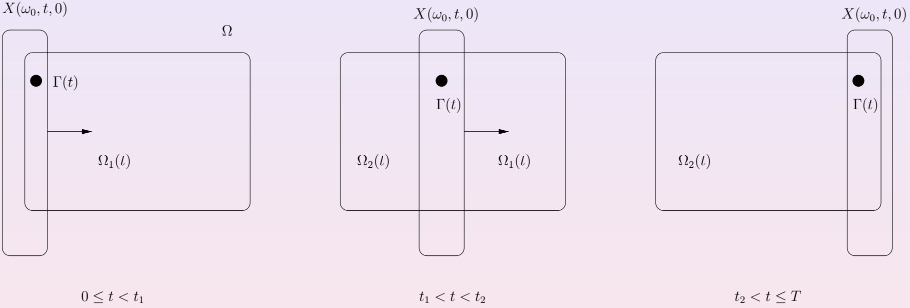
In the context of the example under consideration, if we make the control set ω move to $\omega(t)$ with a velocity field $a(t)$, then the ODE becomes:

$$y_t + a(t) \cdot \nabla y = 1_\omega k.$$

And it is sufficient that all characteristic lines pass by ω to ensure controllability or, in other words, that the set $\omega(t)$ covers the whole domain Ω in its motion.

Question: How to prove this kind of result in a more general setting so that the system does not decouple?

An example of moving support of the control



Other related systems

This issue of moving control is closely related to the works by J. M. Coron, S. Guerrero and G. Lebeau[6][7] on the vanishing viscosity limit for the control of convection-diffusion equations. It is also linked to the recent work by S. Ervedoza, O. Glass, S. Guerrero & J.-P. Puel [8] on the control of $1-d$ compressible Navier-Stokes equations.

[6] J.-M. Coron and S. Guerrero, A singular optimal control: A linear 1-D parabolic hyperbolic example, Asymp. Analisys, 44 (2005), pp. 237-257.

[7] S. Guerrero and G. Lebeau, Singular Optimal Control for a transport-diffusion equation, Comm. Partial Differential Equations, 32 (2007), 1813-1836.

[8] S. Ervedoza, O. Glass, S. Guerrero, J.-P. Puel, Local exact controllability for the 1-D compressible Navier- Stokes equation, Archive for Rational Mechanics and Analysis, 206 (1)(2012), 189-238.

Observability

We consider the dual problem of (16)-(20):

$$\begin{aligned} -p_t - \Delta p &= 0, & (x,t) \in \Omega \times (0,T), && (16)\\ -q_t + q &= p, & (x,t) \in \Omega \times (0,T), && (17)\\ p(x,t) &= 0, & (x,t) \in \partial\Omega \times (0,T), && (18)\\ p(x,T) &= p_0(x), & x \in \Omega, && (19)\\ q(x,T) &= q_0(x), & x \in \Omega. && (20) \end{aligned}$$

The null controllability property i equivalent to the following observability one

$$||p(0)||^2 + ||q(0)||^2 \leq C \int_0^T \int_\omega |q|^2 dxdt, \qquad (21)$$

for all solutions of (16)-(20).

But the structure of the underlying PDE operator and, in particular, the existence of time-like characteristic hyperplanes, makes impossible the propagation of information in the space-like directions, thus making the observability inequality (21) also impossible.

Lack of observability

$$
\begin{aligned}
-p_t - \Delta p &= 0 \quad , \qquad (x,t) \in \Omega \times (0,T), && (22)\\
-q_t + q &= \quad p, \qquad (x,t) \in \Omega \times (0,T), && (23)
\end{aligned}
$$

It is impossible that

$$
||p(0)||^2 + ||q(0)||^2 \leq C \int_0^T \int_\omega |q|^2 dxdt, \tag{24}
$$

Those negative results are well-known in a number of other models:

- Benjamin-Bona-Mahoni (S. Micu, X. Zhang & E. Z.);
- Heat equations with memory (closely related to the coupled systems under consideration "heat + ODE") (S. Guerrero & O. Yu. Imanuvilov)

In both cases the controllability fails because of the presence of accumulation points in the spectrum. A similar situation can be encountered in:

F. Ammar Khodja, K. Mauffrey and A. Münch, Exact boundary controllability of a system of mixed order with essential spectrum, , SIAM J. Cont. Optim. 49 (4) (2011), 1857Ð1879.

In the context of the system of viscoelasticity under consideration the accumulation point in the spectrum is due to the ODE component of the system. In the BBM case is due to the compactness of the generator of the dynamics.

Remedy: Moving control

Let us assume that $\omega \equiv \omega(t)$.
The controllable system under consideration then reads:

$$
\begin{aligned}
y_t - \Delta y &= z, && && (25)\\
z_t + z &= 1_{\omega(t)} h, && && (26)\\
y(x,t) &= 0, && (x,t) \in \partial\Omega \times (0,T), && (27)\\
z(x,0) &= z_0(x), && x \in \Omega, && (28)\\
y(x,0) &= y_0(x), && x \in \Omega. && (29)
\end{aligned}
$$

Motion of the support of the control

In practice, the trajectory of the control can be taken to be determined by the flow $X(x, t, t_0)$ generated by some vector field $f \in C([0, T]; W^{2,\infty}(\mathbb{R}^N; \mathbb{R}^N))$, i.e. X solves

$$\begin{cases} \dfrac{\partial X}{\partial t}(x, t, t_0) = f(X(x, t, t_0), t), \\ X(x, t_0, t_0) = x. \end{cases} \tag{30}$$

Admissible trajectories: There exist a bounded, smooth, open set $\omega_0 \subset \mathbb{R}^N$, a curve $\Gamma \in C^\infty([0, T]; \mathbb{R}^N)$, and two times t_1, t_2 with $0 \le t_1 < t_2 \le T$ such that:

$$\Gamma(t) \in X(\omega_0, t, 0) \cap \Omega, \quad \forall t \in [0, T]; \tag{31}$$

$$\overline{\Omega} \subset \cup_{t\in[0,T]} X(\omega_0, t, 0) = \{X(x, t, 0); \;\; x \in \omega_0, \;\; t \in [0, T]\}; \tag{32}$$

$$\Omega \setminus \overline{X(\omega_0, t, 0)} \text{ is nonempty and connected for } t \in [0, t_1] \cup [t_2, T]; \tag{33}$$

$$\Omega \setminus \overline{X(\omega_0, t, 0)} \text{ has two connected components for } t \in (t_1, t_2); \tag{34}$$

A failing moving support

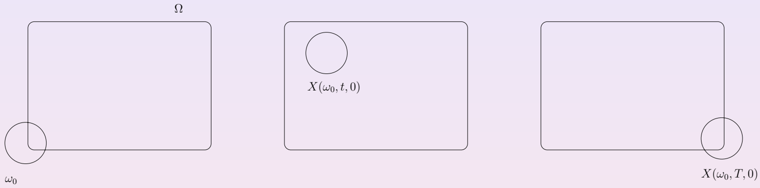
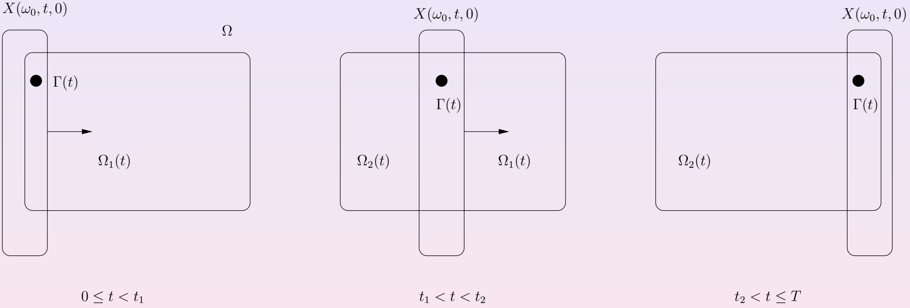


Figure: Example for which condition (34) fails.

A successful motion



Observability inequality

The system is null controllable under the assumptions above on the moving support. But the proof needs to employ Carleman inequalities to prove the observability one.

Two main difficulties appear:

1. Carleman inequalities for heat and ODE equations with a moving control region;
2. We must have the same weight functions in the Carleman for both equations.

Fortunately, we can handle both difficulties. Note that similar strategies were implemented successfully for the system of thermoelasticity in

P. Albano, D. Tataru, Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system, Electron. J. Differential Equations, 22 (2000), 1–15.

G. Lebeau, E. Zuazua, Null controllability of a system of linear thermoelasticity. ARMA, 141 (4)(1998), 297-329.

As a consequence, we have the null controllability of (1)-(3):

Theorem

Let $T > 0$, $X(x, t, t_0)$ *and* ω_0 *be as in* (31)-(35), *and let* ω *be any open set in* Ω *such that* $\overline{\omega}_0 \subset \omega$. *Then for all* $(y_0, y_1) \in L^2(\Omega)^2$ *with* $y_1 - \Delta y_0 \in L^2(\Omega)$, *there exists a function* $h \in L^2(0, T; L^2(\Omega))$ *for which the solution of*

$$y_{tt} - \Delta y - \Delta y_t + b(x) y_t = \mathbf{1}_{\omega(t)}(x) h, \qquad (x, t) \in \Omega \times (0, T), \tag{36}$$

$$y(x, t) = 0, \qquad (x, t) \in \partial\Omega \times (0, T), \tag{37}$$

$$y(., 0) = y_0, \ y_t(., 0) = y_1, \tag{38}$$

fulfills $y(., T) = y_t(., T) = 0$.

Comments

- Can the technical geometric assumptions on the moving control be removed?
- Can one derive similar results by simply assuming that the support of the control covers the whole domain?
- To which extent this methodology can be applied in problems where there are vertical characteristic hyperplanes (BBM, heat with memory,...)?
- Nonlinear versions.

Heat processes with memory terms

A simple system of heat process with memory:

$$\begin{cases} y_t - \Delta y + \int_0^t y(s)ds = u\chi_\omega(x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{39}$$

Setting $z(t) = \int_0^t y(s)ds$, this system can be rewritten as

$$\begin{cases} y_t - \Delta y + z = u\chi_\omega(x) & \text{in } Q, \\ z_t = y & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ y(0) = y_0,\, z(0) = 0 & \text{in } \Omega. \end{cases} \tag{40}$$

And the previous results apply.

More general exponential/polynomial memory kernels

$$\begin{cases} y_t - \Delta y + \int_0^t M(t-s)y(s)ds = u\chi_\omega(x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{41}$$

with

$$M(t) = e^{at}\sum_{k=0}^{K} a_k t^k \tag{42}$$

where $K \in \mathbb{N}$, and $a, a_0, \cdots, a_K, b_0, \cdots, b_K$ are real constants.
Writing

$$Z = \int_0^t M(s-t)y(s)ds \tag{43}$$

we get

$$\begin{cases} y_t + \Delta y = Z & \text{in } Q, \\ \partial_t^{K+1} Z = \sum_{k=0}^{K} k! a_k \partial_t^{K-k} y & \text{in } Q. \end{cases} \tag{44}$$

What about more general memory kernels?
Note, for instance, that for general analytic kernels we get a coupled PDE+ODE system involving an infinite number of ODEs.
Can a strategy in the spirit of Cauchy-Kovalewski be applied?

Waves with memory

Similar techniques can be applied to reduce the following wave equation with memory

$$\begin{cases} y_{tt} - \Delta y + \int_0^t y(s)ds = \chi_O u & \text{in } Q, \\ z_t = y & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ y(0) = y_0, \; y_t(0) = y_1, \; z(0) = 0 & \text{in } \Omega, \end{cases} \tag{45}$$

into

$$\begin{cases} y_{tt} - \Delta y + z = \chi_O u & \text{in } Q, \\ z_t = y & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \end{cases} \tag{46}$$

by setting

$$z(t) = \int_0^t y(s)ds.$$

In view of this structure it is natural to introduce the following Moving geometric Control Condition (MGCC):
We say that an open set $U \subset (0, T) \times \Omega$ satisfies the MGCC, if

1. all rays of geometric optics of the wave equation enter into U before time T;
2. the projection of U onto the x variable covers the whole domain Ω.

This geometric condition turns out to be sufficient for moving control.

Perspectives

What about?

- Uniformity on the vanishing viscosity or velocity of propagation on the ODE?
  In other words, in
  $$\begin{cases} y_{tt} - \Delta y + z = 0 & \text{in } Q, \\ z_t = y & \text{in } Q, \end{cases} \tag{47}$$
  we could replace
  $$z_t = y$$
  by
  $$z_t = \epsilon \Delta y, \quad z_t = \epsilon V \cdot \nabla y \quad z_{tt} = \epsilon \Delta y.$$
- Delay systems?
- More general memory terms (in the principal part of the PDE operator for instance)
- Nonlinear models
- PDE-ODE models appear systematically in other contexts such as population dynamics.