Optimal shape of sensors or actuators for heat and wave equations with random initial data

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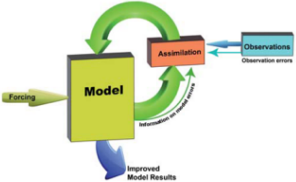
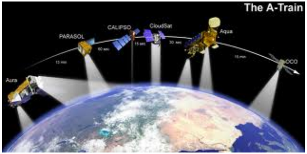
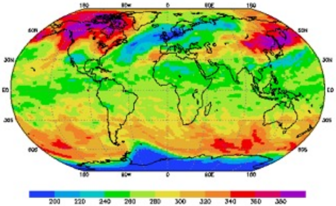
Motivations

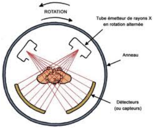


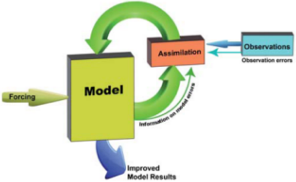


What is the best shape and placement of sensors ?

- Reduce the cost of instruments.
- Maximize the efficiency of reconstruction and estimations.

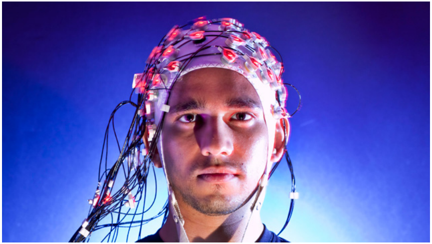






Outlines of this talk

1. Introduction and motivation
2. Modeling of the problem : a randomized criterion
3. Optimal observability for wave and heat equations



N-D wave/heat equation

- ↪ Ω open bounded connected subset of $\mathbb{R}^n$ such that $\partial\Omega \neq \emptyset$
- ↪ $T > 0$ fixed
- ↪ $\omega \subset \Omega$ subset of positive measure

N-D wave equation

$$\begin{array}{ll} \partial_{tt}y - \triangle y = 0 & \text{in } (0,T)\times\Omega \\ y_{|\partial\Omega} = 0 & \\ y(0,\cdot) = y^0,\ \partial_t y(0,\cdot) = y^1 & \text{in } \Omega \end{array}$$

↪ Generalization to many other boundary conditions (Neumann or mixed Dirichlet-Neumann or Robin on $\partial\Omega$)

↪ $\forall (y^0, y^1) \in L^2(\Omega)\times H^{-1}(\Omega)$, there exists a unique $y \in \mathcal{C}^0([0,T], L^2(\Omega)) \cap \mathcal{C}^1([0,T], H^{-1}(\Omega))$.

N-D heat equation

$$\begin{array}{ll} \partial_{t}y - \triangle y = 0 \text{ in } (0,T)\times\Omega & \\ y_{|\partial\Omega} = 0 & \\ y(0,\cdot) = y^0 & \text{in } \Omega \end{array}$$

↪ Assume that $\partial\Omega$ is $\mathcal{C}^2$ (for simplicity, but can be easily weakened, e.g. when Ω is a convex polygon).

↪ $\forall y^0 \in H^1_0 \cap H^2(\Omega)$, there exists a unique $y \in \mathcal{C}^0([0,T], H^1_0 \cap H^2(\Omega)) \cap \mathcal{C}^1([0,T], L^2(\Omega))$

Observable variable ($\omega \subset \Omega$ of positive measure)

$$z(t,x) = \chi_\omega(x) y(t,x) = \begin{cases} y(t,x) & \text{if } x\in\omega \\ 0 & \text{else.} \end{cases}$$

Observability of the N-D wave/heat equation

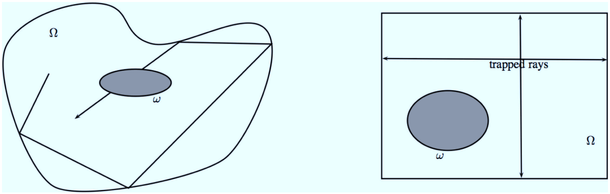
Observability inequality (wave equation)

The time T being chosen large enough, how to choose $\omega \subset \Omega$ to ensure that

$$\exists C > 0 \mid \forall (y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega), \quad C\|(y^0, y^1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \int_0^T \int_\Omega z(t,x)^2 dxdt \ ?$$

- Microlocal Analysis. Bardos, Lebeau and Rauch proved that, roughly in the class of $\mathcal{C}^\infty$ domains, the observability inequality holds iff (ω, T) satisfies the GCC.
- Observability constant :

$$C_T^{\text{wave}} = \inf_{\substack{y \text{ solution of the wave eq.} \\ (y^0,y^1)\in L^2(\Omega)\times H^{-1}(\Omega)}} \frac{\int_0^T \int_\omega y(t,x)^2 dxdt}{\|(y^0, y^1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}}.$$



Observability of the N-D wave/heat equation

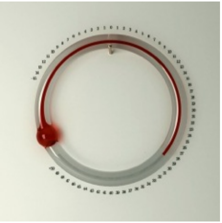
Observability inequality (heat equation)

The time T being fixed, how to choose $\omega \subset \Omega$ to ensure that

$$\exists C > 0 \mid \quad C\|y(T,\cdot)\|^2_{L^2(\Omega)} \leq \int_0^T \int_\omega y(t,x)^2 \, dxdt,$$

for every solution of the heat equation such that $y(0,\cdot) \in H^1_0 \cap H^2(\Omega)$?

- $\hookrightarrow$ this ineq. holds for every open subset ω of Ω ;
- $\hookrightarrow$ related to the inverse problem of recovering its final data from the L^2-observation of its solution on the set ω during a time T.



$$C_T^{\text{heat}}(\chi_\omega) = \inf_{\substack{y \text{ solution of the heat eq.} \\ y(0,\cdot)\in H^1_0\cap H^2(\Omega)}} \frac{\int_0^T \int_\omega |y(t,x)|^2 \, dx\, dt}{\|y(T,\cdot)\|^2_{L^2(\Omega)}}.$$

Toward a new shape optimization problem

The randomization procedure

A first natural idea of modeling

We investigate the problem of maximizing the quantity $C_T(\chi_\omega)$ over all possible subsets $\omega \subset \Omega$ of Lebesgue measure $L|\Omega|$.

BUT, two difficulties

1. Theoretical difficulties
2. The model is not relevant w.r.t. practical expectation

   The usual observability constant is deterministic and gives an account for the worst case. It is pessimistic.

$\hookrightarrow$ In practice : many experiments, many measures.

$\rightarrow$ Objective : optimize the sensor shape and location **in average**.

$\rightarrow$ randomized observability constant.

Toward a new shape optimization problem

The randomized procedure

Introduce $(-\lambda_j^2, \phi_j)$, the j-th eigenpair of the Laplace-Dirichlet operator on Ω.

$\hookrightarrow$ Spectral expansion of the solution y of the wave equation

$$\forall t \in (0,T),\ y(t,\cdot) = \sum_{j=1}^{+\infty} (a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t))\phi_j,$$

where a_j, b_j are determined by the initial conditions.

$\hookrightarrow$ Spectral expansion of the solution y of the heat equation

$$\forall t \in (0,T),\ y(t,\cdot) = \sum_{j=1}^{+\infty} a_j e^{-\lambda_j^2 t}\phi_j,$$

where a_j is determined by the initial condition.

Toward a new shape optimization problem

The randomization procedure

↪ Random selection of the initial data (see Burq - Tzvetkov, Invent. Math. 2008) :

$$y^\nu(t,x) = \sum_{j=1}^{+\infty} \left(\beta_{1,j}^\nu a_j e^{i\lambda_j t} + \beta_{2,j}^\nu b_j e^{-i\lambda_j t} \right) \phi_j(x),$$

where $(\beta_{1,j}^\nu)_{j\in\mathbb{N}^*}$ and $(\beta_{2,j}^\nu)_{j\in\mathbb{N}^*}$ are two sequences of i.i.d random variables (Bernoulli/Gaussian) on a probability space $(X, \mathcal{A}, \mathbb{P})$ of mean 0.

↪ Effects of the randomization

- Gaussian randomization : the map $\nu \in X \mapsto (y^\nu(0,\cdot), y_t^\nu(0,\cdot)$ “generates” a full measure set in $H^s \times H^{s-1}(\Omega)$ for almost every initial data $(y(0,\cdot), y_t(0,\cdot)) \in H^s \times H^{s-1}(\Omega)$.
- Bernoulli randomization : keeps the $H^s \times H^{s-1}(\Omega)$-norm of the original function.
- No regularization effect : the map $\nu \mapsto y^\nu$ is measurable and

$$[\nu \mapsto (y^\nu, \partial_t y^\nu)] \in L^2(X, L^2 \times H^{-1}(\Omega)).$$

If $(y(0,\cdot), y_t(0,\cdot) \notin H^{s+\varepsilon} \times H^{s-1+\varepsilon}(\Omega)$, then $(y^\nu, \partial_t y^\nu) \notin H^{s+\varepsilon} \times H^{s-1+\varepsilon}(\Omega)$ almost surely.

A randomized observability constant

↪ Wave eq. : we consider the randomized observability inequality

$$C_{T,\text{rand}}(\chi_\omega)\|(y^0,y^1)\|^2_{H^1_0\times L^2} \leq \mathbb{E}\left(\int_0^T\int_\omega y^\nu(t,x)^2\,dxdt\right),$$

for all $y^0(\cdot)\in L^2(\Omega)$ and $y^1(\cdot)\in H^{-1}(\Omega)$, where y^ν denotes the solution of the wave eq. with random initial data $y^{0,\nu}$ and $y^{1,\nu}$.

↪ Heat eq. : we consider the randomized observability inequality

$$C_{T,\text{rand}}(\chi_\omega)\|y(T,\cdot)\|^2_{L^2} \leq \mathbb{E}\left(\int_0^T\int_\omega y^\nu(t,x)^2\,dx\,dt\right),$$

for all $y(0,\cdot)\in H^1_0\cap H^2(\Omega)$, where y^ν denotes the solution of the heat equation with the random intial data $y^{0,\nu}$.

Proposition

For every measurable set $\omega\subset\Omega$,

$$C_{T,\text{rand}}(\chi_\omega) = T\inf_{j\in\mathbb{N}^*}\gamma_j\int_\omega \phi_j(x)^2\,dx \quad \text{where} \quad \gamma_j = \begin{cases} \frac{1}{2} & \text{for the wave eq,} \\ \frac{e^{2\lambda_j^2 T}-1}{2\lambda_j^2} & \text{for the heat eq.}\end{cases}$$

There holds $C_{T,\text{rand}}(\chi_\omega)\geq C_T(\chi_\omega)$. There are examples where the inequality is strict.

Optimal observability with respect to the domain

Question

What is the "best possible" observation domain ω of given measure ?

Optimal design problem (energy concentration criterion)

We investigate the problem of maximizing

$$\frac{C_{T,\text{rand}}(\chi_\omega)}{T} = \inf_{j\in\mathbb{N}^*} \gamma_j \int_\omega \phi_j(x)^2\, dx.$$

over all possible subset $\omega \subset \Omega$ of Lebesgue measure $L|\Omega|$.

Related problems

Optimal design for control/stabilization problems

1. What is the "best domain" for achieving HUM optimal control ?

$$y_{tt} - \Delta y = \chi_\omega u$$

2. What is the "best domain" domain for stabilization (with localized damping) ?

$$y_{tt} - \Delta y = -k\chi_\omega y_t$$

See works by
- P. Hébrard, A. Henrot : theoretical and numerical results in 1D for optimal stabilization (for all initial data).
- A. Münch, P. Pedregal, F. Periago : numerical investigations of the optimal domain (for one fixed initial data). Study of the relaxed problem.
- S. Cox, P. Freitas, F. Fahroo, K. Ito, ... : variational formulations and numerics.
- M.I. Frecker, C.S. Kubrusly, H. Malebranche, S. Kumar, J.H. Seinfeld, ... : numerical investigations (among a finite number of possible initial data).
- K. Morris, S.L. Padula, O. Sigmund, M. Van de Wal, ... : numerical investigations for actuator placements (predefined set of possible candidates), Riccati approaches.

- and many others. . .

Additional remark

Let $A > 0$ fixed. If we restrict the search to

$$\{\omega \subset \Omega \quad | \quad |\omega| = L|\Omega| \quad \text{and} \quad P_\Omega(\omega) \leq A\} \qquad \text{(perimeter)}$$

or

$$\{\omega \subset \Omega \quad | \quad |\omega| = L|\Omega| \quad \text{and} \quad \|\chi_\omega\|_{BV(\Omega)} \leq A\} \qquad \text{(total variation)}$$

or

$$\{\omega \subset \Omega \quad | \quad |\omega| = L|\Omega| \quad \text{and} \quad \omega \text{ satisfies the } 1/A\text{-cone property}\}$$

or

ω ranges over some finite-dimensional (or "compact") prescribed set...

then *there always exists (at least) one optimal set ω.*

$\rightarrow$ but then...

- the complexity of ω may increase with A
- we want to know if there is a "very best" set (over all possible measurable)

Solving of the optimal design problem

Generalities

General optimal design problem

$$\sup_{\substack{\omega\subset\Omega\\ |\omega|=L|\Omega|}} J(\chi_\omega) := \sup_{\substack{\omega\subset\Omega\\ |\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \gamma_j \int_\Omega \chi_\omega(x)\phi_j(x)^2 dx$$

- Admissible set for this problem :

$$\mathcal{U}_L = \{\chi_\omega \mid \omega \text{ is a measurable subset of } \Omega \text{ of measure } L|\Omega|\}.$$

- Convex closure of this set for the weak-star topology of L^∞ :

$$\overline{\mathcal{U}}_L = \left\{a \in L^\infty(\Omega;[0,1]) \mid \int_\Omega a(x)dx = L|\Omega|\right\}.$$

Relaxed optimal design problem

$$\sup_{a\in\overline{\mathcal{U}}_L} J(a) := \sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \gamma_j \int_\Omega a(x)\phi_j(x)^2 dx$$

To solve the problem, we distinguish between :

wave or Schrödinger equations	$\neq$	parabolic equations (e.g., heat, Stokes)

Solving of the optimal design problem (wave equation)

Solving the relaxed problem

Geometrical assumption on Ω

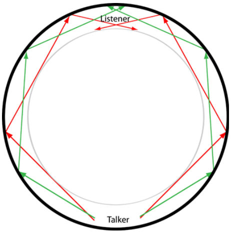
- There exists $p > 1$ such that the sequence $(\phi_j^2)_{j\in\mathbb{N}^*}$ is uniformly bounded in L^p norm
- The whole sequence $\phi_j^2 \rightharpoonup \frac{1}{|\Omega|}$ vaguely as $j \to +\infty$. (QUE conjecture)

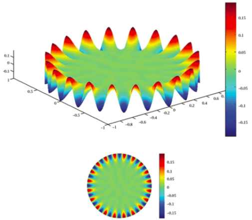
We have

$$\sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2 dx = L \quad \text{(reached with } a = L\text{)}$$

Remarks

- Verified in any flat torus ;
- Not verified in the Euclidean disk (whispering galleries phenomenon).





Solving the optimal design problem (wave equation)

Gap or no-gap ?

A priori,

$$\sup_{\substack{\omega\subset\Omega\\ |\omega|=L|\Omega|}} J(\chi_\omega) \leq \sup_{a\in\overline{\mathcal{U}}_L} J(a).$$

Remarks in 1D :

- Note that, for every ω, $\displaystyle\int_\omega \sin^2(jx)dx \xrightarrow[j\to+\infty]{} \frac{L\pi}{2}$ as $j\to+\infty$.
- No lower semi-continuity (but upper semi-continuity) of the criterion.
- With $\omega_N = \displaystyle\bigcup_{k=1}^{N}\left[\frac{k\pi}{N+1}-\frac{L\pi}{2N},\frac{k\pi}{N+1}+\frac{L\pi}{2N}\right]$, one has $\chi_{\omega_N} \rightharpoonup L$ but

$$\lim_{N\to+\infty} J(\chi_{\omega_N}) < L.$$

Solving of the optimal design problem (wave equation)

Theorem (No-gap)

- Under the L^p boundedness assumption of the sequence $(\phi_j)_{j\in\mathbb{N}^*}$ $(p>1)$ and QUE, there is no gap, that is :

$$\sup_{\chi_\omega\in\mathcal{U}_L}\ \inf_{j\in\mathbb{N}^*}\int_\Omega \chi_\omega(x)\phi_j(x)^2\,dx = \max_{a\in\overline{\mathcal{U}}_L}\ \inf_{j\in\mathbb{N}^*}\int_\Omega a(x)\phi_j(x)^2\,dx = L.$$

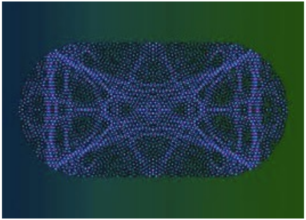
- the result also holds also true in the Euclidean disk.

On the QUE assumption

Quantum Unique Ergodicity property (QUE) in multi-D

- true in 1D, since $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$ on $\Omega = [0, \pi]$
- Gérard-Leichtnam (Duke Math. 1993), Burq-Zworski (SIAM Rev. 2005) : if Ω is a convex ergodic billiard with $W^{2,\infty}$ boundary then $\phi_j^2 \rightharpoonup \frac{1}{|\Omega|}$vaguely for a subset of indices of density 1.
- There exist some convex sets Ω (stadium shaped) that satisfy QE but not QUE (Hassell, Ann. Math. 2010)
- QUE conjecture (Rudnick-Sarnak 1994) : every compact manifold having negative sectional curvature satisfies QUE.

If the QUE assumption fails, we may have scars : energy concentration phenomena (there can be exceptional subsequences converging to other invariant measures, like, for instance, measures carried by closed geodesics : scars)



See Snirelman, Sarnak, Bourgain-Lindenstrauss, Colin de Verdière, Anantharaman, Nonnenmacher, . . .

In summary...(wave equation)

- Under "quantum ergodic assumtions" :

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega \chi_\omega(x)\phi_j(x)^2\,dx = \sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2\,dx = L.$$

- Maximizing sequence : $\chi_{\omega_N} \rightharpoonup L$ is not enough !
  A constructive homogenization procedure of maximizing sequences is known.

- Supremum of J over $\mathcal{U}_L$: reached or not ? ? ?

  Particular cases :
  - in 1D : the supremum is reached if and only if $L = 1/2$ (and there is an infinite number of optimal sets).
  - in the 2D square, if we restrict the search of optimal sets to Cartesian products of 1D subsets, then the supremum is reached if and only if $L \in \{1/4, 1/2, 3/4\}$.

Conjecture

For generic domains Ω and generic values of L, the supremum is not reached and hence there does not exist any optimal set.

Truncated criterion (wave equation)

Truncated shape optimization problem

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{1 \leq j \leq N} \int_\Omega \chi_\omega(x) \phi_j(x)^2 \, dx$$

Theorem

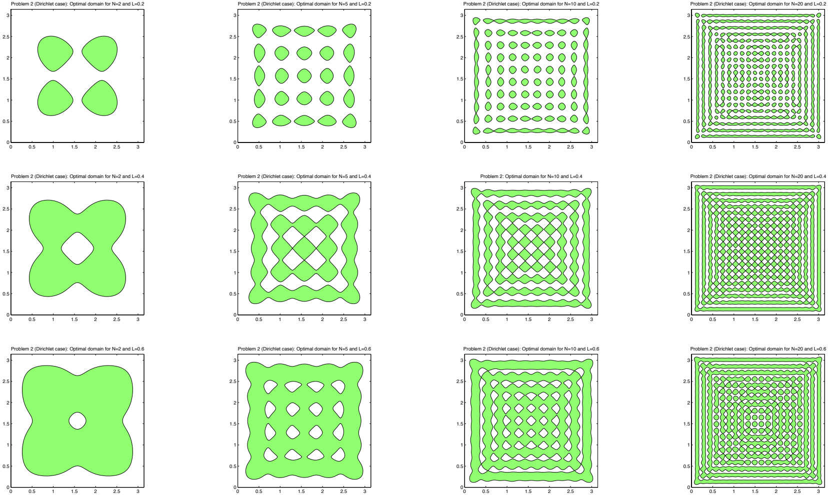
Let $L \in (0,1)$. The shape optimization problem above has a unique solution ω_N^*.

- $\hookrightarrow$ Γ-convergence result : $\lim_{N \to +\infty} \sup_{\chi_\omega \in \mathcal{U}_L} J_N(\chi_\omega) =$ optimal value for the relaxed pb.
- $\hookrightarrow$ If No-gap, L^∞ weak-* convergence of $(\chi_{\omega_N^*})_{N \in \mathbb{N}^*}$ to a minimizer of the optimal design problem.
- $\hookrightarrow$ Spillover phenomenon : the best domain ω^N for the first N modes is the worst possible for $N+1$ modes.

Proved in 1D by Hébrard-Henrot (SICON, 2003) and Privat-Trélat-Zuazua (J. Fourier Anal. Appl., 2013)

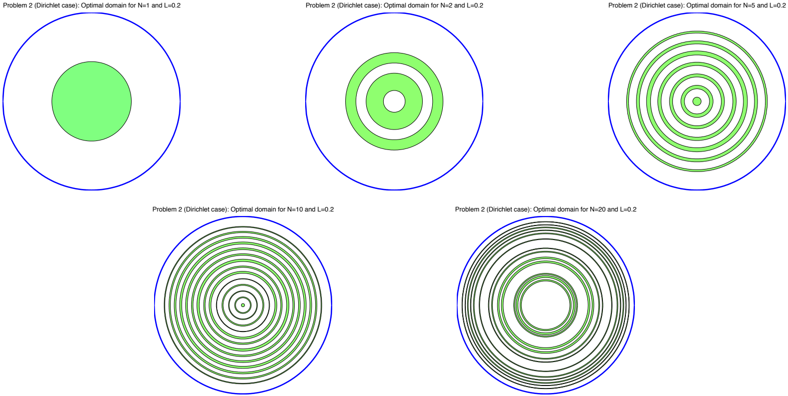
Several numerical simulations : $\Omega = [0, \pi]^2$

For 4, 25, 100 and 500 eigenmodes and $L \in \{0.2, 0.4, 0.6\}$



Several numerical simulations : Ω = unit disk

$L = 0.2$, for 1, 4, 25, 100 and 400 eigenmodes



Solving of the optimal design problem (N-D heat equation)

An existence result

Theorem

Assume that Ω is a bounded connected subset of $\mathbb{R}^n$ such that $\partial\Omega$ is piecewise $\mathcal{C}^1$. There exists $N_0 \in \mathbb{N}^*$ such that

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{1 \leq j} \gamma_j \int_\Omega \chi_\omega(x)\phi_j(x)^2 dx = \max_{\chi_\omega \in \mathcal{U}_L} \min_{1 \leq j \leq N_0} \gamma_j \int_\Omega \chi_\omega(x)\phi_j(x)^2 dx.$$

- $\hookrightarrow$ Stationarity of the optimal domain in the truncation procedure...
- $\hookrightarrow$ The proof requires fine recent results in

  J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, Observability inequalities and measurable sets, J. Europ. Math. Soc. (2013).

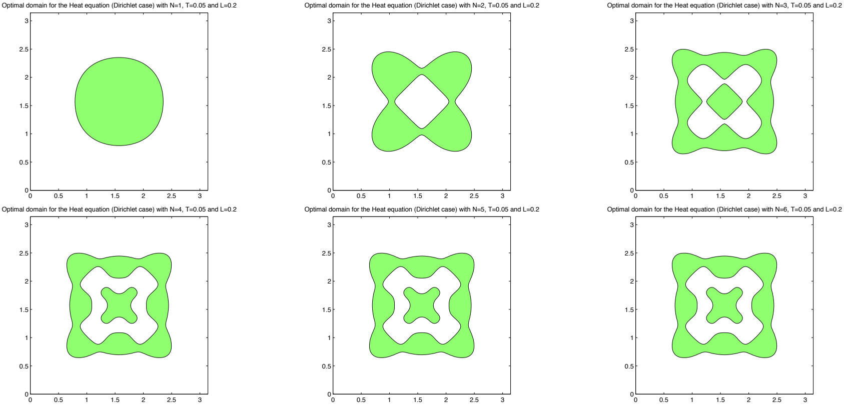
- $\hookrightarrow$ Same kinds of results for optimal design null controllability issues for the N-dimensional heat equation

$$\partial_t y(t,x) = \Delta y(t,x) + \chi_\omega(x) u(t,x) \quad \text{on } (0,T)\times\Omega$$

Control function supported by ω, use of the moment method, maximization of the operator norm of the control w.r.t. ω...

- $\hookrightarrow$ Generalization to parabolic systems (and even Stokes with Dirichlet boundary conditions, ...)

Several numerical simulations : $\Omega = [0, \pi]^2$, $T = 0.05$ and $L = 0.2$ for $N \in \{1, 2, 3, 4, 5, 6\}$



Stationarity of the maximizers from $N = 4$ (i.e. 16 eigenmodes)

Application to anomalous diffusion

Anomalous diffusion equations, Dirichlet : $\partial_t y + (-\triangle)^\alpha y = 0$ ($\alpha > 0$ arbitrary)

↪ protein diffusion within cells, or diffusion through porous media.

↪ Associated optimal design problem :

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \frac{e^{2\lambda_j^\alpha} - 1}{2\lambda_j^\alpha} \int_\Omega \chi_\omega(x) \phi_j(x)^2 \, dx$$

In the square $\Omega = (0,\pi)^2$, with the usual basis (products of sine) : the optimal domain ω^* has a finite number of connected components, $\forall \alpha > 0$.

In the disk $\Omega = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$, with the usual basis (Bessel functions), the optimal domain ω^* is radial, and

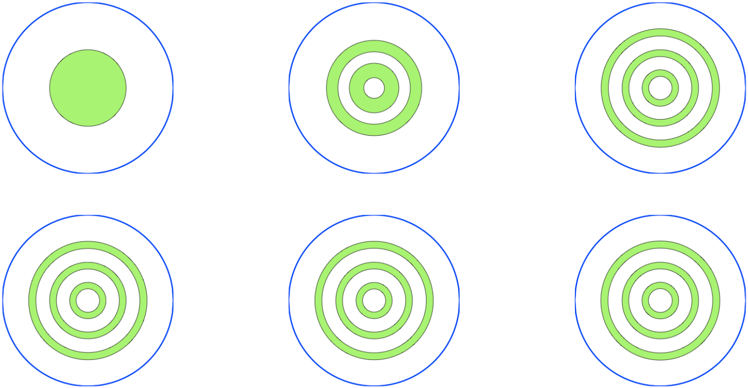
- $\alpha > 1/2 \Rightarrow \omega^* =$ finite number of concentric rings (and $d(\omega, \partial\Omega) > 0$)
- $\alpha < 1/2 \Rightarrow \omega^* =$ infinite number of concentric rings accumulating at $\partial\Omega$!
  (or $\alpha = 1/2$ and T small enough)

The proof is long and very technical. It uses in particular the knowledge of quantum limits in the disk. (L. Hillairet, Y. Privat, E.T.)

Several numerical simulations : Ω = unit disk

$L = 0.2$, $T = 0.05$, for 1, 4, 9, 16, 25 and 36 eigenmodes

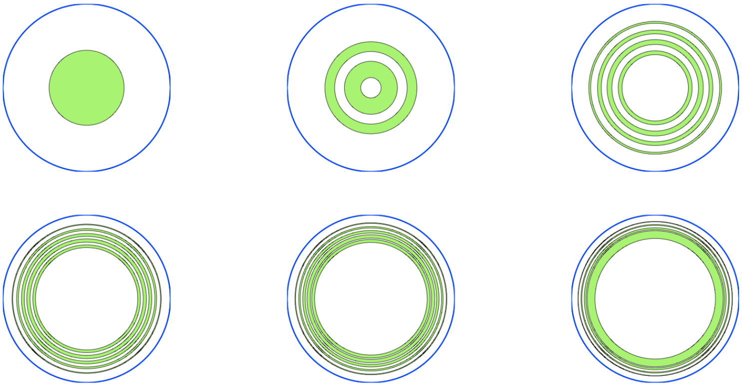
$\alpha = 1$



Several numerical simulations : Ω = unit disk

$L = 0.2$, $T = 0.05$, for 1, 4, 9, 16, 25 and 36 eigenmodes

$\alpha = 0.15$



To sum-up

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j\in\mathbb{N}^*} \gamma_j \int_\Omega \chi_\omega(x)\phi_j(x)^2\, dx$$

	square	disk
wave or Schrödinger	relaxed solution $a = L$ $\exists\omega$ for $L \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ $\nexists$ otherwise (conjecture)	relaxed solution $a = L$ $\exists\omega$ for $L \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ $\nexists$ otherwise (conjecture)
diffusion $(-\triangle)^\alpha$	$\exists!\omega \quad \forall L \quad \forall\alpha > 0$ $\#c.c.(\omega) < +\infty$	$\exists!\omega$ (radial) $\quad \forall L \quad \forall\alpha > 0$ if $\alpha > 1/2$ then $\#c.c.(\omega) < +\infty$ if $\alpha < 1/2$ then $\#c.c.(\omega) = +\infty$

Conclusion of this talk

Ongoing works :

- optimal design for boundary observability. (with P. Jounieaux)
  Ω being assumed bounded connected and its boundary $\mathcal{C}^2$, maximize
  $$\inf_{j\in\mathbb{N}^*} \frac{1}{\lambda_j(\Omega)} \int_\Sigma \left|\frac{\partial\phi_j}{\partial n}\right|^2 dx$$
  over all possible subsets $\Sigma \subset \partial\Omega$ of given Hausdorff measure.
- new strategies to avoid spillover phenomena when solving optimal design problems (Césaro means).
- Same analysis for the optimal design of the domain of control. (effect of the randomization on the HUM operator ?)
- discretization issues. Do the numerical designs converge to the continuous optimal design as the mesh size tends to 0 ?

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Thank you for your attention