Sensitivity-based NMPC Strategies for Dynamic Real-time Optimization

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Outline

- **Introduction to NMPC**
- **Stability Properties of NMPC**
- **Reducing On-line computation: asNMPC and amsNMPC**
- **Economic NMPC: ensuring stability and robustness**
- **Conclusions**

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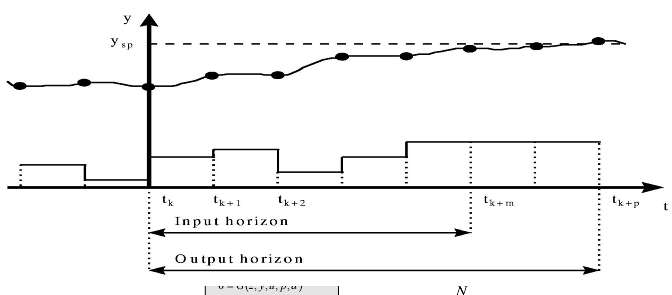
Nonlinear Model Predictive Control (NMPC)



$$\min_u \; J(x(k)) = \sum_{l=0}^{N} \psi(z_l, u_l) + \Psi(z_N)$$

$$s.t. \quad z_{l+1} = f(z_l, u_l)$$

$$z_0 = x(k)$$

Bounds

Lyapunov stability of NMPC

- Nominal stability
  - Basic idea
    - Perfect model, no uncertainty: $x^+ = f(x, u)$
    - Remain bounded and eventually achieve desired state

A function V: $R^n \to R_{\geq 0}$ is a Lyapunov function for a system

$$x(k+1) = f(x(k), u(k))$$

if there exist a set X, three $\mathcal{K}_\infty$ functions α_1,α_2,α_3 such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$V(f(x,u)) - V(x) \leq -\alpha_3(|x|)$$
$$\forall x \in X$$

- Can the objective function serve as $V(x)$?

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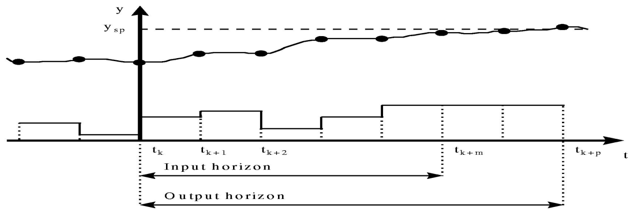
MPC Stability – Infinite Horizon
(Keerthi and Gilbert (1988))



$$J_k = \sum_{l=k}^{\infty} \| y(l) - y^{sp} \|^2_{Q^y} + \sum_{l=k}^{\infty} \| \boldsymbol{u}(l) - \boldsymbol{u}(l-1) \|^2_{Q^u}$$

$$J_k - J_{k+1} = \| y(k) - y^{sp} \|^2_{Q^y} + \| \boldsymbol{u}(k) - \boldsymbol{u}(k-1) \|^2_{Q^u}$$

$$J_1 \geq \sum_{k=1}^{\infty} (J_k - J_{k+1})$$

$$= \sum_{k=1}^{\infty} (\| y(k) - y^{sp} \|^2_{Q^y} + \| \boldsymbol{u}(k) - \boldsymbol{u}(k-1) \|^2_{Q^u})$$

$$\Rightarrow y(k) \to y_{sp}, u(k) \to u(k-1)$$

Nominal stability – perfect model
- Based on discrete Lyapunov arguments with J(x) as Lyapunov function

- Infinite time horizon, ideal case
- Finite time horizon - need endpoint constraint → *z(k+p)=0*
- Choice of terminal cost gives additional stability properties
- Often m (input) < p (output)

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NMPC Nominal Stability with Terminal Cost
(Rawlings and Mayne, 2009)

$$\min_{u} \quad J(x(k)) = \sum_{l=0}^{N} \psi(z_l, u_l) + \Psi(z_N)$$

$$s.t. \quad z_{l+1} = f(z_l, u_l))$$
$$z_0 = x(k)$$
$$Bounds$$

Assumptions:
- *f(x, u)* is Lipschitz continuous *(will assume smooth)*
- The terminal cost Ψ(•) satisfies Ψ(x)>0
- There exists a local control law u = κ_f(x) for all x ϵ X_f such that *Ψ(f(x, κ_f(x))) - Ψ(x) ≤ -ψ(x, κ_f(x))*
- ψ(x, u) satisfies α_p(|x|) ≤ ψ(x, u) ≤ α_q(|x|) where α_p(•), α_q(•) are $\mathcal{K}$ functions.
- N sufficiently long (Grüne, 2013) or *Ψ(x)* sufficiently long (Pannocchia, Rawlings, 2011): no terminal constraints

Nominal Stability Proof for NMPC

$$J(x(k+1)) - J(x(k))$$

$$\leq \Psi(f(z_N, \kappa_f(z_N))) + \sum_{l=1}^{N-1} \psi(z_l, v_l) + \psi(z_N, \kappa_f(z_N)) - [\Psi(z_N) + \sum_{l=0}^{N-1} \psi(z_l, v_l)]$$

$$= \underbrace{\Psi(f(z_N, \kappa_f(z_N))) - \Psi(z_N) + \psi(z_N, \kappa_f(z_N))}_{\leq 0} - \psi(z_0, v_0)$$

$$\leq -\psi(z_0, v_0) = -\psi(x(k), u(k))$$

$$J(x(0)) \geq \sum_{k=0}^{\infty} (J(x(k+1)) - J(x(k)) = \sum_{k=1}^{\infty} \psi(x(k), u(k))$$

$$\Rightarrow x(k) \to 0, u(k) \to 0$$

- Nominal case – no noise: perfect model
- Terminal cost, *Ψ(z_N)* – upper bound on control cost for z → 0.
- Robust case – keep (*J(x(k) – J(x(k+1))* > α(x) > 0 even with noise/ mismatch

Lyapunov stability of NMPC

- Robust stability
  - Basic idea
    - Uncertainty in model: $x^+ = f(x, u, w)$, where *w* could be additive disturbance or uncertain parameters;
    - Remain stable in the presence of disturbances.

A function V(.) is called an ISS-Lyapunov function for a system

$$x(k+1) = f(x(k), u(k)) + q(x(k), w(k))$$

if there exist a set X, $\mathcal{K}_\infty$ functions α_1,α_2,α_3 and a $\mathcal{K}$ function σ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$V(f(x,u,w)) - V(x) \leq -\alpha_3(|x|) + \sigma(|w|)$$
$$\forall x \in X, \ \forall \omega \in W$$

Proof of robust stability for NMPC

- Addditional Assumptions:
  - $|q(x,w)| \le |q(x,0)| + L_g|w|$
  - $|q(x,0)| \le \varrho/\zeta\ \alpha_p(|x|)$, and $|q(x,0)| \le q_{max,}$ where $\varrho \in (0,1)$.

$$
\begin{aligned}
& J(x(k+1)) - J(x(k)) \\
= \; & J(f(x(k),u(k))) - J(x(k)) + J(x(k+1)) - J(f(x(k),u(k))) \\
\le \; & -\psi(x(k),u(k)) + L_J\,|q(x(k),w(k))| \\
\le \; & -\alpha_p(|x(k)|) + L_J \frac{\rho}{\varsigma}\alpha_p(|x(k)|) + L_J L_g\,|w(k)| \\
\le \; & (\rho-1)\alpha_p(|x(k)|) + \sigma\,|w(k)|
\end{aligned}
$$

Nonlinear programming (NLP) formulation for NMPC

$$
\begin{aligned}
J_N(x(k)) := \; & \min_{v_l, z_l} \Psi(z_N) + \sum_{l=0}^{N-1} \psi(z_l, v_l) \\
s.t. \quad & z_{l+1} = f(z_l, v_l), l = 0,\ldots,N-1 \\
& z_0 = x(k) \\
& g(z_l) \le 0, l = 0,\ldots,N \\
& v_l \in U, l = 0,\ldots,N-1
\end{aligned}
$$

z_0 – initial value

$x(k)$ – measurement of state at t_k

ψ, Ψ - (quadratic) stage and terminal costs

v_l - predicted controlled variable

z_l - predicted manipulated variable

How will NLP formulation satisfy assumptions of NMPC stability properties?

Reformulating the NLP

- Disturbances may lead to infeasibility of the NLP
- Dependent active sets make system unstable under perturbations

- Formulation
  - If $\varepsilon^* = 0$ the stability of the mixed constraint problem is the same as the hard constraint-only problem.

$$
\begin{aligned}
J(x(k)) := \; & \min_{v_l, z_l} \Psi(z_N) + \sum_{l=0}^{N-1} \psi(z_l, v_l) + \boxed{\sum_{l=0}^{N-1} \rho \varepsilon_l^T e} \\
s.t. \quad & z_{l+1} = f(z_l, v_l) \\
& z_0 = x(k) \\
& g(z_l) \le \boxed{\varepsilon_l, \varepsilon_l} \ge 0 \\
& v_l \in U, l = 0,\ldots,N-1 \\
& e = [1,1,1,1\ldots1]^T
\end{aligned}
$$

Add barrier terms (developed later)

If $g(z)$ is linear, MFCQ, CRCQ are satisfied at KKT point.

➔ Continuity of $J(x(k))$ with data perturbations

Reformulated NMPC Problem

$$
\begin{aligned}
J(x(k)) := \; & \min_{v_l, z_l} (\Psi(z_N) - \mu \sum_j \ln(\varepsilon_N^{(j)} - g^{(j)}(z_N)) \\
s.t. \quad & + \sum_{l=0}^{N-1} (\psi(z_l, v_l) + \sum_{l=0}^{N-1} \rho \varepsilon_l^T e - \mu \sum_j \ln(\varepsilon_l^{(j)} - g^{(j)}(z_l))) \\
& z_{l+1} = f(z_l, v_l), l = 0,\ldots,N-1 \\
& z_0 = x(k) \\
& \varepsilon_l \ge 0, v_l \in U, l = 0,\ldots,N-1
\end{aligned}
$$

$$v_l := [v_l^T, \varepsilon_l^T]^T$$

$$\psi(z_l, v_l) := \psi(z_l, v_l) + \sum_{l=0}^{N-1} \rho \varepsilon_l^T e - \mu \sum_j \ln(\varepsilon_l^{(j)} - g^{(j)}(z_l))$$

$$
\begin{aligned}
J(x(k)) := \; & \min_{v_l, z_l} \Psi(z_N) + \sum_{l=0}^{N-1} \psi(z_l, v_l) \\
s.t. \quad & z_{l+1} = f(z_l, v_l), z_0 = x(k), l = 0,\ldots,N-1 \\
& v_l \in U, l = 0,\ldots,N-1
\end{aligned}
$$

- Reformulated problem satisfies LICQ for:
  - Bounded multipliers
  - Continuous solutions w.r.t. $x(k)$
  - Sensitivity

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Nonlinear Model Predictive Control – Air Separation Unit (Huang, B., 2011)

Objective: maintain product specifications under ramping demands

4 manipulated variables.
4 output variables.

Horizon: 100 minutes in 20 finite elements.
Sampling time: 5 minutes.

DAEs: 1520

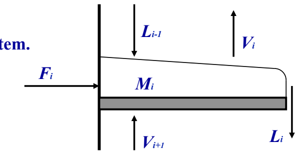
After Discretization:
Variables: 117,140
Constraints: 116,900

Case Study: Basic Air Separation Unit

•Mesh Equations for Distillation Column

Assumption:

- **Vapor holdups are negligible. Index 2 system.**
- **Ideal vapor phases.**
- **Well mixed entering streams.**
- **Constant pressure drop.**
- **Equilibrium stage model.**



Mass balance: $\frac{dM_i}{dt} = L_{i-1} + V_{i+1} - L_i - V_i + F_i$

Component balance: $\frac{d(M_i x_{i,j})}{dt} = L_{i-1}x_{i-1,j} + V_{i+1}y_{i+1,j} - L_i x_{i,j} - V_i y_{i,j} + F_i x^f_{i,j}$

Energy balance: $\frac{d(M_i h^L_i)}{dt} = L_{i-1}h^L_{i-1} + V_{i+1}h^V_{i+1} - L_i h^L_i - V_i h^V_i + F_i h^f_i$

Phase equilibrium: $y_{i,j}p_i = \gamma_{i,j} x_{i,j} p^{sat}_{i,j}$

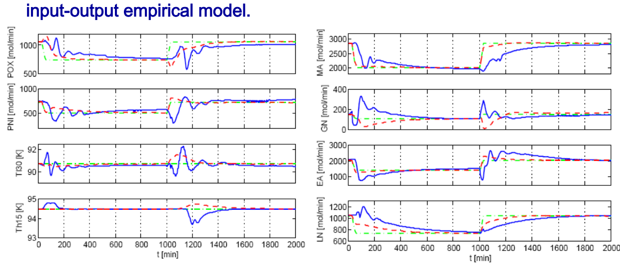
Summation: $1 = \sum_{j \in COMP} y_{i,j}$

Hydrodynamics: $L_i = k_d M_i$

Reformulated index 1 system contains 320 ODEs, 1200 AEs.

ASU Nonlinear MPC - Case 1

t = 30-60 min, product rates are ramped down by 30%. t =1000-1030 min, they are ramped back. NMPC is compared to MPC with linear input-output empirical model.



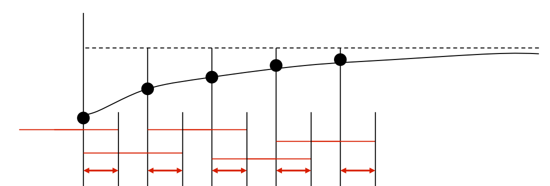
Output Variables — Manipulated Variables

The green dot-dashed lines are the set-points, the blue dashed lines are the linear controller profiles and red solid lines are NMPC profile.

All the tuning parameters are favored to the linear controller.
Horizon Solution Time: 200 CPUs, 6 IPOPT iters.

What about Fast NMPC?

- Fast NMPC is not just NMPC with a fast solver (Engell, 2007)
- ***Computational delay*** – between receipt of process measurement and injection of control, determined by cost of dynamic optimization
- Leads to loss of ***performance*** and ***stability*** (see Rawlings and Mayne, 2009; Findeisen and Allgöwer, 2004; Santos et al., 2001)

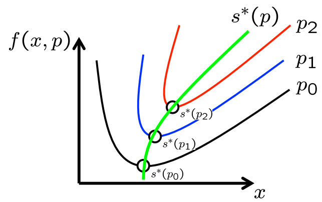


Can computational delay be overcome?
- Fast Newton-based NMPC
- Cheap NLP Sensitivity

NLP Sensitivity

Parametric Program

$$\left.\begin{array}{ll} \min & f(x,p) \\ \text{s.t.} & c(x,p)=0 \\ & x \geq 0 \end{array}\right\} \mathrm{P}(p)$$



Solution Triplet

$s^*(p)^T = [x^{*T}\ \lambda^{*T}\ \nu^{*T}]$

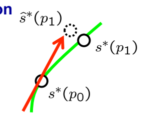
Optimality Conditions $\mathrm{P}(p)$

$$\begin{aligned} \nabla_x f(x,p) + \nabla_x c(x,p)\,\lambda - \nu &= 0 \\ c(x,p) &= 0 \\ XVe &= 0 \end{aligned}$$

NLP Sensitivity → Rely upon Existence and Differentiability of $s^*(p)$

→ Main Idea: Obtain $\left.\frac{\partial s}{\partial p}\right|_{p_0}$ **and find** $\tilde{s}^*(p_1)$ **by Taylor Series Expansion**

$$\tilde{s}^*(p_1) \approx s^*(p_0) + \left.\frac{\partial s}{\partial p}^T\right|_{p_0} (p_1 - p_0)$$



KKT Properties and Constraint Qualifications for Sensitivity

- LICQ, SOSC, SC → (*ds*/dp*) - derivatives can be calculated (Fiacco, 1983)
- MFCQ, GSSOSC, CRCQ → ($D_{\Delta p}x^*$) - directional derivatives calculated with additional LP and QP steps (Ralph and Dempe, 1995)
- MFCQ, GSSOSC – continuity of objective functions and primal variables with respect to *p*. (Kojima, 1985)

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NLP Sensitivity with IPOPT
(Pirnay, Lopez Negrete, B., 2011)

Obtaining $\left.\frac{\partial s}{\partial p}\right|_{p_0}$

Optimality Conditions of $\mathrm{P}(p)$

$$\left.\begin{aligned} \nabla_x \mathcal{L} = \nabla_x f(x,p) + \nabla_x c(x,p)\,\lambda - \nu &= 0 \\ c(x,p) &= 0 \\ XVe &= 0 \end{aligned}\right\} \mathrm{Q}(s,p) = 0$$

Apply Implicit Function Theorem to $\mathrm{Q}(s,p)=0$ **around** $(p_0, s^*(p_0))$

$$\frac{\partial \mathrm{Q}(s^*(p_0),p_0)}{\partial s}\left.\frac{\partial s}{\partial p}\right|_{p_0} + \frac{\partial \mathrm{Q}(s^*(p_0),p_0)}{\partial p} = 0$$

$$\begin{bmatrix} W(s^*(p_0)) & A(x^*(p_0)) & -I \\ A(x^*(p_0))^T & 0 & 0 \\ V^*(p_0) & 0 & X^*(p_0) \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial p} \\ \frac{\partial \lambda}{\partial p} \\ \frac{\partial \nu}{\partial p} \end{bmatrix} + \begin{bmatrix} \nabla_{x,p}\mathcal{L}(s^*(p_0)) \\ \nabla_p\, c(x^*(p_0)) \\ 0 \end{bmatrix} = 0$$

KKT Matrix IPOPT

$$\begin{bmatrix} W(x_k,\lambda_k) & A(x_k) & -I \\ A(x_k)^T & 0 & 0 \\ V_k & 0 & X_k \end{bmatrix}$$

- **→ Already Factored at Solution**
- **→ Sensitivity Calculation from Single Backsolve**
- **→ Approximate Solution Retains Active Set**

Updates for active set changes

- **Fix-relax (in sIPOPT)**
  - Bounds are violated → fix $x^* + \Delta x = x^B$ → $\Delta x = x^B - x^*$
  - Multipliers become negative → fix $\upsilon^* + \Delta\upsilon = 0$ → $\Delta\upsilon = -\upsilon^*$
- **Clipping in first interval** (CFI)
  - Find step length α such that

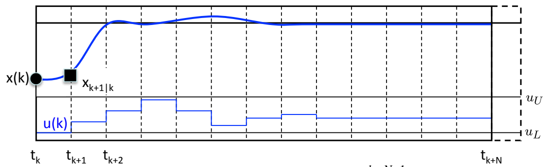
    $$v^L \leq v_0 + \alpha\Delta v_0 \leq v^U$$

    $$z^L \leq z_1 + \alpha\Delta z_1 \leq z^U$$

  - Easiest to implement in asNMPC
- **Specialized Linear and Quadratic Programming** (Jäschke et al., 2014)
  - Determine directional derivatives
  - Track family of solutions through homotopy

Advanced Step Nonlinear MPC (Zavala, B., 2009)

Solve NLP in background (between steps, not on-line)
Update using sensitivity on-line



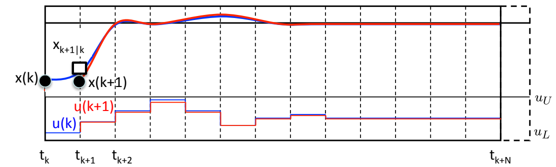
$$\min \quad J(x(k), u(k)) = F(x_{k+N|k}) + \sum_{l=k+1}^{k+N-1} \psi(x_{l|k}, v_{l|k})$$

$$\begin{aligned} s.t. \quad & x_{k+1|k} = \boxed{f(x(k), u(k))} \\ & x_{l+1|k} = f(x_{l|k}, v_{l|k}), \quad l = k+1, \ldots k+N-1 \\ & x_{l|k} \in X, \quad v_{l|k} \in U, \quad x_{k+N|k} \in X_f \end{aligned}$$

Solve NLP(k) in background (between t_k and t_{k+1})

Advanced Step Nonlinear MPC (Zavala, B., 2009)

Solve NLP in background (between steps, not on-line)
Update using sensitivity on-line

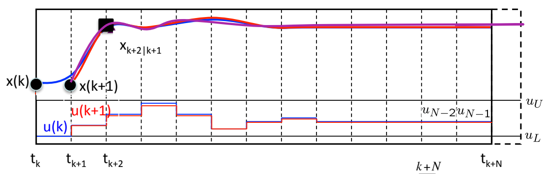


$$\begin{bmatrix} W_k & A_k & -I \\ A_k^T & 0 & 0 \\ Z_k & 0 & X_k \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x} \\ \Delta \lambda \\ \Delta z \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ x_{k+1|k} - x(k+1) \\ 0 \end{bmatrix}$$

Solve NLP(k) in background (between t_k and t_{k+1})
Sensitivity to update problem on-line to get (u(k+1))

Advanced Step Nonlinear MPC (Zavala, B., 2009)

Solve NLP in background (between steps, not on-line)
Update using sensitivity on-line



$$\min \quad J(x(k+1), u(k+1)) = F(x_{k+N+1|k+1}) + \sum_{l=k+2}^{k+N} \psi(x_{l|k+1}, v_{l|k+1})$$

$$\begin{aligned} s.t. \quad & x_{k+2|k+1} = f(\boxed{x(k+1), u(k+1)}) \\ & x_{l+1|k+1} = f(x_{l|k}, v_{l|k}), \quad l = k+2, \ldots k+N \\ & x_{l|k+1} \in X, \quad v_{l|k+1} \in U, \quad x_{k+N+1|k+1} \in X_f \end{aligned}$$

Solve NLP(k) in background (between t_k and t_{k+1})
Sensitivity to update problem on-line to get (u(k+1))
Solve NLP(k+1) in background (between t_{k+1} and t_{k+2})

Nominal Stability: ideal NMPC and asNMPC

$$\min_{u} \quad J_k = \sum_{l=0}^{N} \psi(z_l, u_l) + F(z_N)$$

$$\begin{aligned} s.t. \quad & z_{l+1} = f(z_l, u_l)) \\ & z_0 = x(k) \end{aligned}$$

Bounds

$$J_{k-1} - J_k \geq \psi(x(k-1), u(k-1)) + \overbrace{\{F(z_{N|k-1}) - \psi(z_{N|k}, u_{N|k}) - F(z_{N|k})\}}^{\geq 0}$$
$$\geq \psi(x(k-1), u(k-1))$$

$$J_0 \geq \sum_{k=1}^{\infty} (J_{k-1} - J_k) = \sum_{k=1}^{\infty} \psi(x(k-1), u(k-1))$$
$$\Rightarrow x(k) \to 0, u(k) \to 0$$

- *Assumptions:* *f*() is Lipschitz continuous, ψ(), α() is a $\mathcal{K}_\infty$ function in ||x||
- Nominal case – no noise: perfect model
- asNMPC yields identical solutions (no sensitivity perturbation)
  → Identical NMPC stability property

Robust Stability Margins

Plant

$$x_{k+1} = \bar{f}(x_l,u_l)) = f(x_k,u_k)) + q(x_k,u_k,w_k))$$

$$\| q(x_k,u_k,w_k)) \| \le L_J \| x_k \| + \sigma(\| w_k \|)$$

Model

$$z_{l+1} = f(z_l,u_l)), z_0 = x_k$$

$$\bar{J}(x_{k-1}) - \bar{J}(x_k) = (\bar{J}(x_{k-1}) - J(z_{k|k-1}) + (J(z_{k|k-1}) - J(x_k)) + (J(x_k) - \bar{J}(x_k))$$

$$\ge \psi(x_{k-1},u_{k-1}) - \varepsilon(x_k) - \varepsilon_{as}(x_k)$$

$$\ge \psi(x_{k-1},u_{k-1}) - \varepsilon(x_k) - \varepsilon_{as}(x_k)$$

$$\ge \psi(x_{k-1},u_{k-1}) - (L_J + L_J\kappa_{as})[\sigma(\| w \|)] + \alpha_g(| x_{k-1} |)]$$

Assume above noise and model mismatch model

- Advanced step NMPC is ISS and tolerates some model mismatch (see Jiang and Wang, 2001; Magni and Scattolini, 2005; Zavala, B., 2009)

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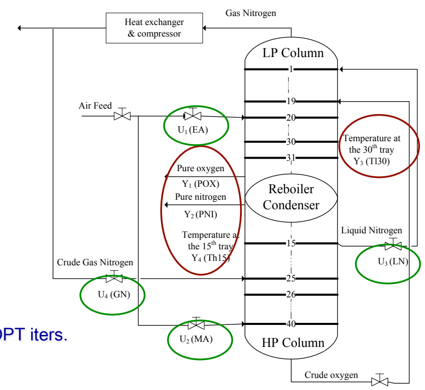
Nonlinear Model Predictive Control – Air Separation Unit (Huang, B., 2011)

Objective: *minimize operating cost subject to demand specifications*

4 manipulated variables.
4 output variables.

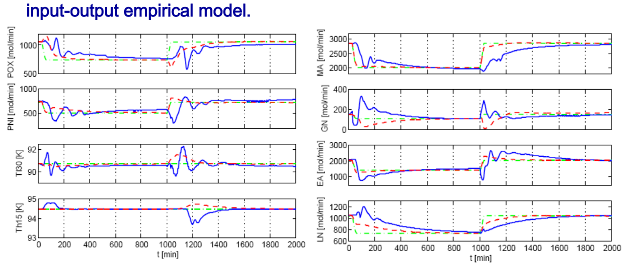
Horizon: 100 minutes in 20 finite elements.
Sampling time: 5 minutes.

DAEs: 1520
Variables: 117,140
Constraints: 116,900
Background: 200 CPUs, 6 IPOPT iters.
Online: 1 CPUs



ASU Nonlinear MPC - Case 1

t = 30-60 min, product rates are ramped down by 30%. t =1000-1030 min, they are ramped back. asNMPC is compared to MPC with linear input-output empirical model.



Output Variables **Manipulated Variables**

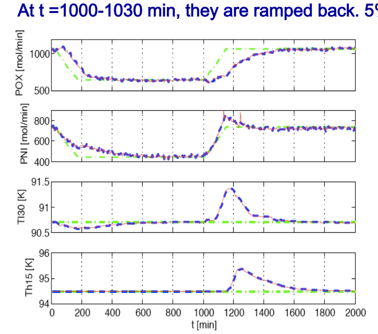
The green dot-dashed lines are the set-points, the blue dashed lines are the linear controller profiles and red solid lines are NMPC profile.

All the tuning parameters are favored to the linear controller.

NMPC of Air Separation Unit – Case 2

(Huang, B., 2009)

At t = 30-60 min, product rates are ramped down by 40%.
At t =1000-1030 min, they are ramped back. 5% disturbance is added to M_l.



N = 20, K = 3
320 ODEs, 1200 AEs.
Variables: 117,140
Constraints: 116,900

400 NLPs solved
Background: 200 CPUs, 6 iters.
Online: 1 CPUs

Computational Feedback Delay
Reduced from 200 → 1 second.

Blue dashed lines are ideal NMPC profile
Red lines are AS-NMPC profile.
In contrast, linearized controller is unstable

Advanced Multi-step NMPC (amsNMPC)

- What if NLP solution exceeds one sampling time?
  - Use longer sampling time? Degrades performance
  - Define N_s = solution time/sampling time.
  - Do model linearization N_s steps behind.

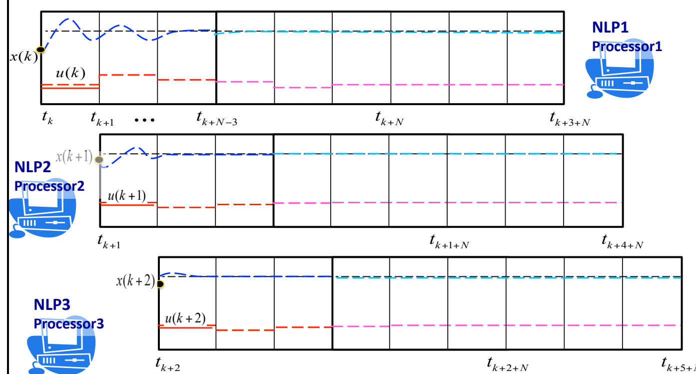
Background: predict state at $t(k+N_s)$ and solve NLP using prediction as initial value.
On-line: update u(k) based on NLP sensitivity at each t(k).

Serial approach: solve NLP every N_s sampling times, using one processor.
Parallel approach: solve NLP on N_s processors at t(k).

As $N_s \rightarrow \infty$, amsNMPC becomes neighboring extremal controller

X. Yang, L. T. Biegler, Advanced-multi-step nonlinear model predictive control, Journal of Process Control, 23, 2013, pp. 1116-1128

Parallel amsNMPC, N_s = 3

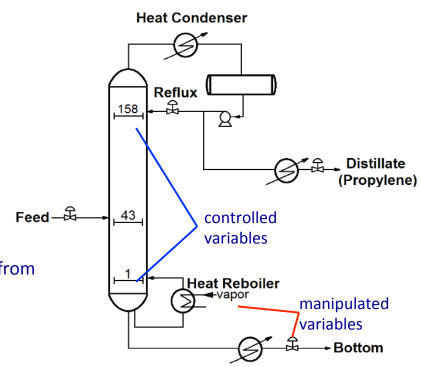


amsNMPC Case study
C3 splitter distillation column

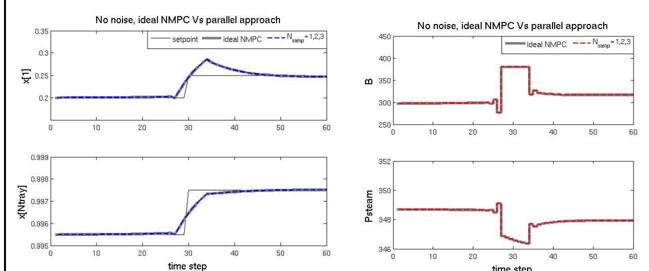
B., Yang, Fischer (2015)

- 158 trays
- 111650 variables
- 111580 constraints
- Horizon length=35
- Sampling time: 1 min

- Assumptions:
Pressure in the column is controlled ;
Pressure drop is constant from tray to tray.



Nominal case – no noise



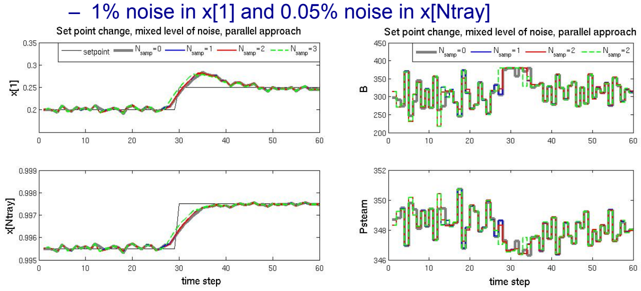
States of iNMPC Vs parallel approach

Control profile of iNMPC Vs parallel approach

- Parallel approach behaves identically to ideal NMPC and asNMPC.
- NLP takes 50 CPUs, Sensitivity update takes less than 0.5 CPUs
- Increasing N_s does not change performance.

Noisy case for amsNMPC

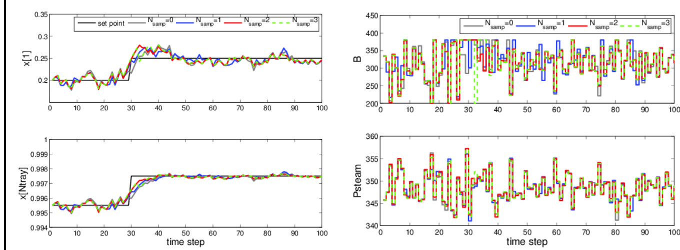
– 1% noise in x[1] and 0.05% noise in x[Ntray]



States with mixed level of noise | **control profile with mixed level of noise**

- Parallel amsNMPC can handle a small level of noise with set point change, and for small level of noise N_s makes no difference.
- As noise increases, more deviation of the states with N_s =3 than with smaller N_s.
- Robust performance deteriorates with increasing N_s

5% noise in x[1] and 0.1% noise in x[Ntray]



- Performance loss with increasing N_s does not occur because model response to noise is not highly nonlinear.

Economic NMPC

- NLP formulation

$$\min_{v_l, z_l} \quad \Psi(z_N) + \sum_{l=0}^{N-1} \psi(z_l, v_l)$$
$$s.t. \quad z_{l+1} = f(z_l, v_l), l = 0, \ldots, N-1$$
$$z_0 = x(k)$$
$$z_l \in X, v_l \in U, l = 0, \ldots, N-1, z_N \in X_N$$

where $\Psi(z_N), \psi(z_l, v_l)$ are economic terms
Steady state set point is not known -> no terminal region
(see Grüne, 2013; Grüne and Stieler, 2014)

- Challenge: normally, J does not satisfy $\alpha_1(|x|) \leq J(x) \leq \alpha_2(|x|)$

Challenges with D-RTO

Replace regulation objective with economic objective in NMPC?

Bartusiak, Young et al. (2007)
Dadhe and Engell (2008), Engell (2007, 2009)
Wuerth, Marquardt, Rawlings (2009)
Angeli and Rawlings (2010)
Angeli, Amrit and Rawlings (2011)
Diehl, Amrit and Rawlings (2011)
Wolf, Würth, Marquardt (2012)

Robust Stability of Lyapunov function → must be $\mathcal{K}_\infty$ function (e.g., strong convexity of stage cost)

$$Max \sum_i \{\text{Profit}_i\} + \text{Profit}_N \qquad N$$

Open Questions

- should D-RTO optimum go to a steady state?
- how do we enforce optimal steady state?

Remedy: Regularize economic stage cost?

Economic NMPC Stability-1

- Economic stage costs generally do not satisfy assumptions for Lyapunov stability e.g., $\alpha_p(|x|) \leq \psi(x,u) \leq \alpha_q(|x|)$
- Obtain steady state solution

$$\min \psi(z,v),\ \text{s.t. } z = f(z,v), z \in \mathbb{X}, v \in \mathbb{U}$$

- Define transformed states and controls

$$\bar{z}_l = z_l - z^*, \quad \bar{v}_l = v_l - v^*$$

- Define rotated stage cost (Diehl, Amrit, Rawlings, 2011)

$$L(\bar{z}_l, \bar{v}_l) = \bar{\psi}(\bar{z}_l, \bar{v}_l) + \lambda^{*,T}(\bar{z}_l - \bar{f}(\bar{z}_l, \bar{v}_l))$$

- If rotated stage cost is strongly convex at all feasible steady state points, then Lyapunov assumptions are satisfied:

$$L(\bar{z}, \bar{v}) \geq 1/2 \int_0^1 [\bar{z}^T\ \bar{v}^T] \nabla^2 L(\tau\bar{z}, \tau\bar{v}) \begin{bmatrix} \bar{z} \\ \bar{v} \end{bmatrix} d\tau$$
$$\geq \gamma_L(|\bar{z}|^2 + |\bar{v}|^2) \geq \beta_\infty(|\bar{z}|)$$

Economic NMPC Stability-2

- If rotated stage cost is not strongly convex, add regularization terms to original stage cost

$$\psi(z_l, v_l) \Rightarrow \psi(z_l, v_l) + 1/2 \left\| (z_l - z^*, v_l - v^*) \right\|_Q^2$$
$$\Psi(z_N) \Rightarrow \Psi(z_N) + 1/2 \left\| (z_N - z^*) \right\|_{Q_N}^2$$

- Property 1: There exists Q sufficiently large to satisfy Lyapunov assumptions for rotated stage cost
- Q threshold determined from steady state solution
- Property 2: The NMPC controller with rotated stage costs yields the same control trajectory as with original stage costs. (Diehl, Amrit, Rawlings, 2011)
- Stability and Robustness properties extended to cyclic steady states with k steps (Huang, Harinath, B; 2012).

Apply Gershgorin's theorem (Jäschke, Yang, B., 2014)

Recall at steady state optimum

$$L(\bar{z}, \bar{v}) \geq 1/2 \int_0^1 [\bar{z}^T\ \bar{v}^T] \nabla^2 L(\tau\bar{z}, \tau\bar{v}) \begin{bmatrix} \bar{z} \\ \bar{v} \end{bmatrix} d\tau$$
$$\geq \gamma_L(|\bar{z}|^2 + |\bar{v}|^2) \geq \beta_\infty(|\bar{z}|)$$

- For a matrix $A = (a_{i,j}) = \nabla^2[\psi(z,v) + \lambda^{*,T}(z - f(z,v))]$

$$a_{i,i} - \sum_{i \neq j} |a_{i,j}| \leq \mu_i \leq a_{i,i} + \sum_{i \neq j} |a_{i,j}|$$

where μ_i is the i^{th} eigenvalue of A.

- To ensure that all the eigenvalues of A+Q are positive, we require

$$0 < q_i + a_{i,i} - \sum_{i \neq j} |a_{i,j}| \leq \mu_i + q_i$$

therefore

$$q_i > \sum_{i \neq j} |a_{i,j}| - a_{i,i}$$

- A+Q should be positive definite for **every z and v**!

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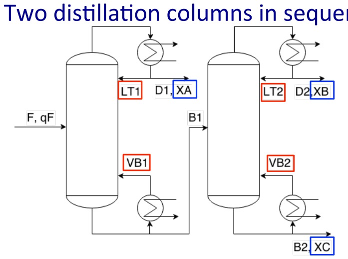
Economic NMPC: Distillation Case study

- Two distillation columns in sequence

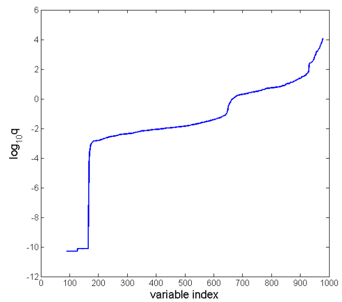


- 41 trays
- Feed enters at stage 21
- Manipulated variables: LT1, VB1, LT2, VB2
- Additive noise
- Without regularization, the Hessian matrix of Lagrange function of the steady state problem is not positive definite with $\lambda_{min} = -1.414$

$$\min_u\ J(u) = p_F F + p_V(VB1 + VB2) - (p_A D1 + p_B D2 + p_C B2)$$
$$s.t.\ Massbalance,\ Equilibrium$$
$$x_A \geq x_{A,min},\ x_B \geq x_{B,min},\ x_C \geq x_{C,min}$$
$$0 \leq LT1, LT2 \leq LT_{max},\ 0 \leq VB1, VB2 \leq VB_{max}$$

R. B. Leer. Self-optimizing control structures for active constraint regions of a sequence of distillation columns. Master's thesis, Norweign University of Science and Technology, 2012. Url: http://www.divaportal.org/smash/get/diva2:629241/FULLTEXT01.pdf.

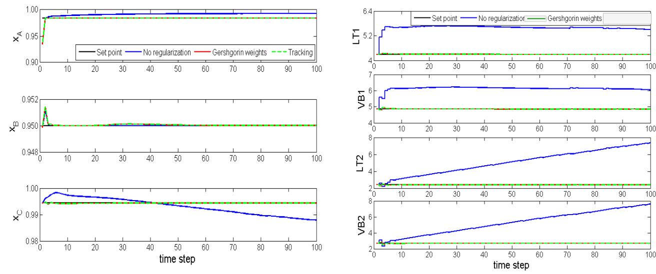
Elements of Regularization matrix (Q = diag{q}) for Distillation Case Study



$$\psi(z,v) \Rightarrow \psi(z,v) + 1/2 \left\| (z - z^*, v - v^*) \right\|_Q^2$$

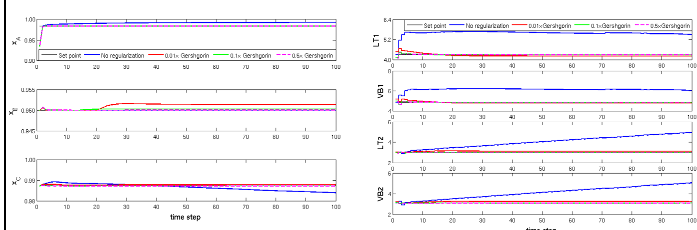
- ~100 q_i are 0, ~600 q_i are <1, ~50 q_i are > 100

Economic NMPC without Noise



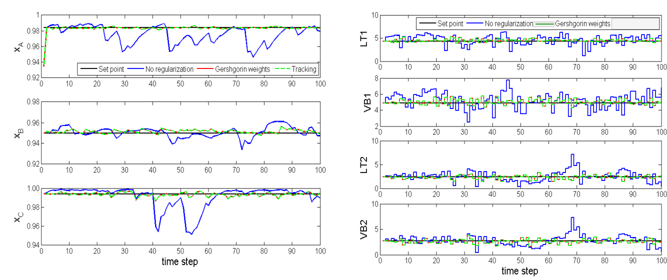
Case 1: no regularization—x_A cannot reach optimum steady state; x_C diverges.
Case 2: economic term +regularization—optimum steady state reached quickly.

Economic NMPC without Noise: Partial Gershgorin weights



0.01 x Gershgorin—offset in x_B.
0.1 x Gershgorin—offset becomes very small.
0.5 x Gershgorin—offset becomes even smaller.
Since Hessian is still not positive definite, positive definiteness is just **sufficient** conditions for stability of Economic NMPC.

Economic NMPC with 5% noise



No regularization—very large oscillations in x_A, x_B and x_C.
Gershgorin Regularization—states remain within a small region around optimum steady state.

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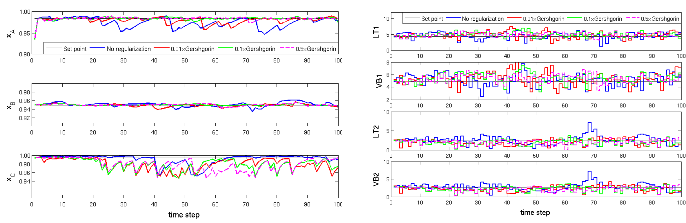
Economic NMPC with 5% noise: partial Gershgorin weights



0.01 x Gershgorin—large oscillations in x_A.
0.1 x Gershgorin, 0.5 x Gershgorin—behavior very close to full value of Gershgorin weights.

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Comparison of Time Average Economic Costs with Regularization

$$Min \sum_{l=1}^{100} p_F F + p_V (VB1_l + VB2_l) - (p_A D1_l + p_B D2_l + p_c B2_l)$$

Cost with	No noise	5% noise
No regularization	2.476	-19.97
0.01 x Gershgorin	-20.4	-21.77
0.1 x Gershgorin	-22.09	-22.97
0.5 x Gershgorin	-22.26	-20.1
Gershgorin	-22.27	-22.03

- Cost is worst without regularization.
- CPU time also suffers
  - Without regularization ~6mins/NLP
  - With regularization ~50s/NLP

Summary of Results

- **Stability Properties for asNMPC** (Zavala, B., 2009)
  - Nominal stability – no disturbances nor model mismatch
  - Input to State Stability (ISS)
  - RPI set = $\mathfrak{R}^n$ due to soft constraints
- **Advanced Multi-step NMPC** (amsNMPC, Yang, B., 2013)
  - Coordinate background NLP solves over multiple time steps
  - Extend NLP sensitivity with Schur complement approach
- **Extension to economic objectives** (Huang, Harinath, B., 2012; Jaeschke, Yang, B., 2014)
  - Nominal and ISS stability based on rotated stage costs
  - Gershgorin analysis leads to sufficient regularization
  - Extended to cyclic processes
  - Development of unbiased regularized stage costs?
- **Moving Horizon Estimation** (Lopez Negrete, Huang, B., 2010, 2011)
  - Fast sensitivity-based smoothed covariance of arrival cost
  - Robust stability for embedded asMHE and asNMPC?
  - Statistical properties of arrival cost formulations?

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- Center for Advanced Process Decision-Making (CAPD)
- For more information: *numero.cheme.cmu.edu*

Conclusions

Sensitivity-based Nonlinear Estimation & Control

- MHE → asMHE
- NMPC → asNMPC
- RTO → D-RTO

Bigger NLPs are not harder to solve

- Embrace and exploit size, sparsity and structure
- Exact first and second derivatives are essential
- Newton-based optimization is fast
- Optimal sensitivity is (nearly) free

NMPC and MHE Computational Strategies

- Full-Discretization + Fast Sensitivity Calculations
- Large-scale distillation with nonlinear DAE models

From NMPC Tracking to Economic Optimization

- Direct optimization in real-time
- Maintain stability and exploit uncertainties
- Still many open questions

A Nonrobust NMPC Example (Grimm et al., 2004)

$$\min_{x,u} \; g(\mathbf{x}_{10}) + \rho_g s_N + \sum_{l=0}^{10-1} l(\mathbf{x},u) + \rho_l s_l$$

$$s.t. \quad x_1(k+1) = f_1(\mathbf{x},u) = \frac{-(x_1^2(k)+x_2^2(k))^{1/2}u(k) + x_1(k)}{1+(x_1^2(k)+x_2^2(k))u^2(k) - 2x_1(k)u(k)}$$

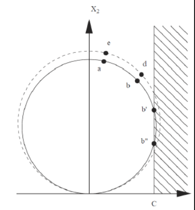
$$x_2(k+1) = f_2(\mathbf{x},u) = \frac{x_2(k)}{1+(x_1^2(k)+x_2^2(k))u^2(k) - 2x_1(k)u(k)}$$

$$x_{1,i} \le c + s_i, i = 0,\dots,N-1, |\mathbf{x}_N| \le \varepsilon + s_N$$

$$u \in [-1,1], \kappa_f(\mathbf{x}) = -1$$

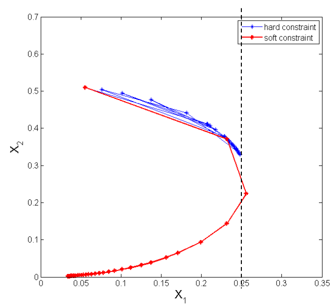
$$g(x) = |\mathbf{x}| \cos^{-1} \frac{(x_2 - |\mathbf{x}|)(-|\mathbf{x}|)}{|\mathbf{x}|\sqrt{x_1^2 + (x_2 - |\mathbf{x}|)^2}}$$

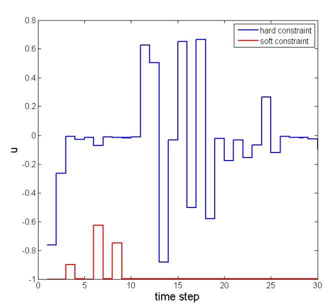
$$l(x,u) = |\mathbf{x}| \cos^{-1} \frac{x_1 f_1(\mathbf{x},-1) + (x_2 - |\mathbf{x}|)(f_2(\mathbf{x},-1) - |\mathbf{x}|)}{\sqrt{x_1^2 + (x_2 - |\mathbf{x}|)^2}\sqrt{f_1(\mathbf{x},-1)^2 + (f_2(\mathbf{x},-1) - |\mathbf{x}|)^2}}$$



Grimm, G., Messina, M. J., Tuna, S. and Teel, A. [2004], 'Examples when nonlinear model predictive control is nonrobust', *Automatica* ***40, 523–533.***

Nonrobust NMPC Example





- The constraint $x_1 \le c$ prevents the trajectory from going beyond $x_1=c$
- Soft constraint allows the trajectory to go beyond $x_1=c$ and then converge

Economic NMPC Case study: CSTR

$$\min \quad -m(2c_B - \frac{1}{2})$$

$$s.t. \quad \frac{dc_A}{dt} = \frac{m}{V}(c_{Af} - c_A) - kc_A$$

$$\frac{dc_B}{dt} = \frac{m}{V}(-c_B) + kc_A$$

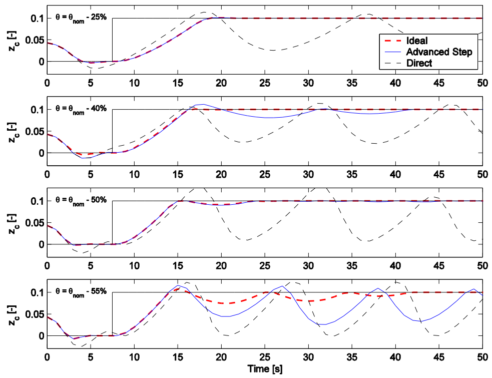
$$10 \le m \le 20$$

$$0.45 \le c_B \le 1$$

- First order reaction A→B
- c_A – concentration of A
- c_B – concentration of B
- m – manipulated input in L/min
- V=10L
- k =1.2L/(mol•min)
- c_{Af} – feed concentration
- c_{Bf} =1mol/L

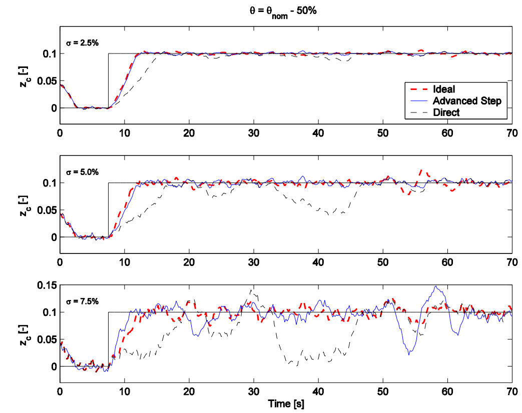
M. Diehl, R. Amrit, J. B. Rawlings, A lyapunov function for economic optimizing model predictive control, IEEE Transactions on Automatic Control 56 (2011) 703–707.

CSTR NMPC Example – Model Mismatch

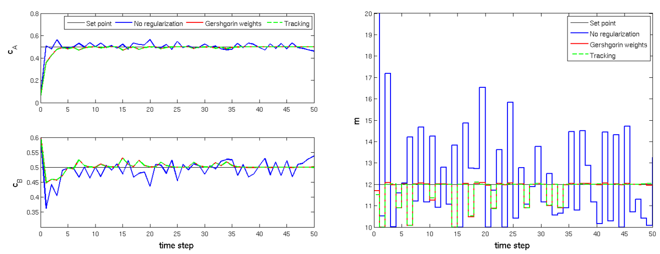


Advanced Step NMPC not as robust as ideal - suboptimal selection of u(k)

CSTR Example: Mismatch + Noise



CSTR Example with 10% noise



Case 1: no regularization—very large oscillations in c_A, c_B and m.
Case 2, Case 3—the state stay within a small region around the optimum steady state.

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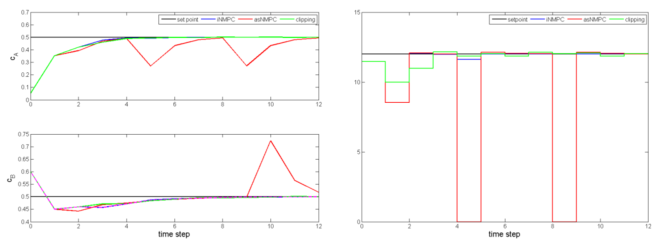
CSTR Case Study: Comparison of economic cost

$$\sum_{l=1}^{50} -m_l (2c_{B,l} - \frac{1}{2})$$

Cost with	10% noise
No regularization	-145.4996
Gershgorin weights	-146.3681
tracking	-146.3506

- Cost is the worst without regularization.
- Cost is the lowest with both economic terms and regularization.

Advanced-step Economic NMPC



Case 1: ideal NMPC—converges to the setpoint quickly
Case 2: asNMPC—control bounds are violated. Strategy not feasible.
Case 3: asNMPC + clipping—Avoids control bounds violation. Similar results to ideal NMPC.

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Recent asNMPC Studies

- Various CSTRs (Zavala, B., 2009)
- NMPC, D-RTO for LDPE reactors (Zavala, B., 2009)
- NMPC, D-RTO for Air Separation Units (Huang, Zavala, B., 2009, 2010; Huang, Patwardhan, B., 2010)
- Large-scale distillation, C3 splitters (Fischer, B., 2011)
- Multi-stage NMPC for power generation cycles (d'Amato, Kumar, Lopez-Negrete, B., 2012)
- Thermo-mechanical pulping (Harinath, B., Dumont, 2010)
- Multi-stage Thermo-mechanical pulping (Harinath, B., Dumont, 2013)
- Advanced multi-step NMPC for distillation control (Yang, Fischer, B., 2013)

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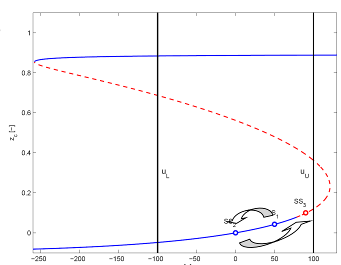
CSTR NMPC Example (Hicks and Ray)

$$\min_{v_{l|k}} \sum_{l=k}^{k+N-1} Q_c(z^c_{l|k})^2 + Q_t(z^t_{l|k})^2 + R(v_{l|k})^2$$

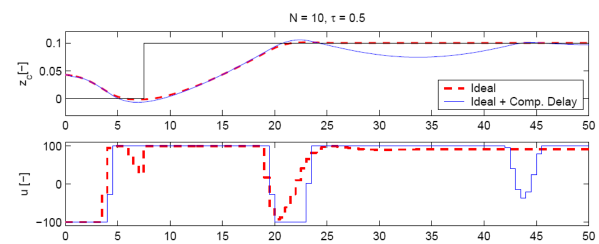
$$\begin{aligned}
\text{s.t.}\quad z^c_{l+1|k} &= \frac{1}{\theta}(1-(z^c_{l|k}+z^c_{ss})) - k_0 exp\left(-\frac{E_a}{z^t_{l|k}+z^t_{ss}}\right)(z^c_{l|k}+z^c_{ss})\\
z^t_{l+1|k} &= \frac{1}{\theta}(t_f-(z^t_{l|k}+z^t_{ss})) + k_0 exp\left(-\frac{E_a}{z^t_{l|k}+z^t_{ss}}\right) z^c_{l|k} - \alpha(v+v_s s)((z^t_{l|k}+z^t_{ss})-t_c)\\
z^c_{k|k} &= x^c(k),\ \ z^t_{k|k} = x^t(k)\\
z^c_{k+N|k} &= 0\ \ z^t_{k+N|k} = 0,\ \ u^U \le v_{l|k} \le u^L.
\end{aligned}$$

Maintain unstable setpoint
Close to bound constraint
Final time constraint for stability
$z_{k+N|k} = 0$, (N is finite)

Study effects of:
- Computational Delay
- Advanced Step NMPC
- Measurement Noise
- Model Mismatch



CSTR NMPC Example – Nominal Case



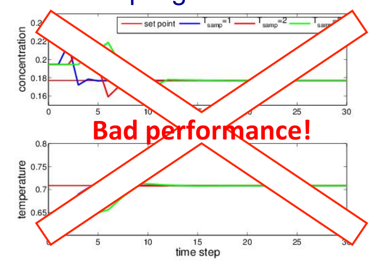
- NMPC applied with N = 10, τ = 0.5 sampling time
- Stable (z = 0) and unstable (z = 0.1) steady states
- u_2* close to upper bound
- Computational delay = 0.5, leads to instabilities
- asNMPC has identical performance as Ideal NMPC

asNMPC: Concepts and Properties

- Interpretation: Fast linear MPC controller using *linearization of nonlinear model at previous step*.
- NLP solved between samples, "instantaneous" sensitivity update at sampling time
- On-line computation 2-3 orders of magnitude faster;
  - ➔ Computational delay virtually eliminated
- Second order errors compared to ideal NMPC
  - ➔ Nominal and ISS stability (Zavala, B., 2009)
- ISpS stability when coupled with embedded state estimators (Huang, Patwardhan, B., 2009a,b, 2010a-c, 2012)

Advanced-multi-step NMPC

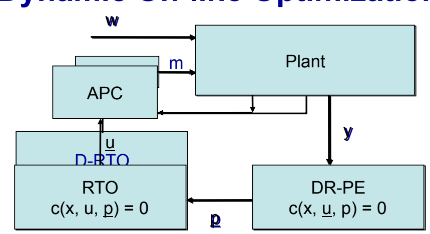
- Motivation: what if the solution of NLP takes more than one sampling time?
  - Slow down sampling?



Summary: Advanced-(multi) step NMPC

- Avoids computational delay
- Allows solving NLP ≥ one sampling time
- Behaves close to ideal NMPC and asNMPC with small level of measurement noise and less nonlinear model
- With increasing noise, process nonlinearity and N_s , memory effect becomes more significant
- Robustness could be obtained by using soft constraints and adding ℓ_1 penalty to the objective. Moreover, robustness is preserved when amsNMPC is applied, under additional assumptions on memory effect

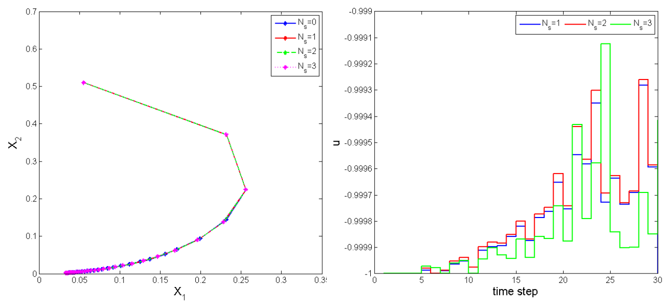
Dynamic On-line Optimization:



Integrate On-line Optimization with APC
- Consistent, first-principle dynamic models
- Consistent, feed-forward optimization
- Increase in computational complexity
- Time-critical calculations

Essential for:
- Feed changes
- Nonstandard operations
- Optimal disturbance rejection

Nonrobust Example



- Robustness will not be lost when the advanced-multi-step strategy is applied.

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Stability Properties of *asNMPC*

x(k+1) = f(x(k), u(k)) – plant and model identical

Nominal Stability Theorem (Zavala, B., 2008)

Assume that the NLP can be solved within one sampling time, nominal stability assumptions hold for ideal NMPC (Mayne, 2000), and the nonlinear model is perfect without measurement noise. Then ideal NMPC controller performance and asNMPC controller performance are identical.
Ideal NMPC stability ➔ asNMPC stability.

x(k+1) = f(x(k), u(k)) + g(x(k), u(k), w(k))
plant and model not identical

Robust Stability Theorem (Zavala, B., 2008)

Assume that the NLP can be solved within one sampling time, and that robust stability assumptions hold for ideal NMPC (Magni, Scattolini, 2007). Then there exist bounds on the noise, *w*, and model mismatch, *g*, for which the cost function $J_{N+1}(x)$, obtained from the asNMPC strategy, is an input-to-state (ISS)-Lyapunov function and the resulting closed-loop system is ISS stable.

NMPC – Can we avoid on-line optimization?

- **Divide Dynamic Optimization Problem** (Diehl, Bock et al., 2002)**:**
  - preparation, feedback response and transition stages
  - solve complete NLP in background ('between' sampling times) as part of preparation and transition stages
  - solve perturbed problem on-line
  - > two orders of magnitude reduction in on-line computation
- **Based on NLP sensitivity of z_0 for dynamic systems**
  - Extended to Collocation approach – Zavala et al. (2008, 2009)
  - Similar approach for MH State and Parameter Estimation – Zavala et al. (2008)
- **Stability Properties** (Zavala et al., 2009)
  - Nominal stability – no disturbances nor model mismatch
    - Lyapunov-based analysis for NMPC
  - Robust stability – some degree of mismatch
    - Input to State Stability (ISS) from Magni et al. (2005)
  - Extension to economic objective functions