Topology optimization and minimal partitions using a gradient-free perimeter approximation

Samuel Amstutz

LMA
Laboratoire de Mathématiques d'Avignon

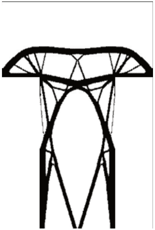
Joint work with A.A. Novotny, N. Van Goethem and M. Zabiba

From open to closed loop control
Graz
June 22-26, 2015

Perimeter penalization in topology optimization: why?

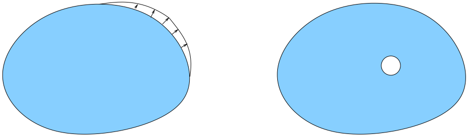
Why is it useful?

- To control the complexity of domains
- To enforce the existence of optimal shapes
  Math. argument: $BV(D) \hookrightarrow L^1(D)$ is compact



Why is it difficult?

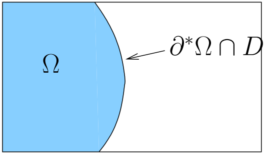
- The perimeter is differentiable w.r.t. smooth shape variations (shape derivative).
- For a topology perturbation of form $\Omega_\varepsilon = \Omega \setminus \overline{B(z, \varepsilon)}$, $\Omega \subset \mathbb{R}^d$, the perimeter varies like ε^{d-1}, while usual cost functions vary like ε^d (no topological derivative).



Shape vs topology perturbation

Perimeter in the sense of geometric measure theory

Let $D \subset \mathbb{R}^d$ open and bounded.



Definition

Let $\Omega \subset D$ measurable. The relative perimeter of Ω is defined by

$$\mathrm{Per}_D(\Omega) := \int_D |D\chi_\Omega| = \sup \left\{ \int_D \chi_\Omega \,\mathrm{div}\varphi, \varphi \in \mathcal{C}_c(D, \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\}.$$

Theorem (De Giorgi, Federer)

If $Per_D(\Omega) < \infty$ (i.e. $\chi_\Omega \in BV(D)$), then

$$Per_D(\Omega) = \mathcal{H}^{d-1}(\partial^*\Omega \cap D),$$

where $\partial^*\Omega$ is the essential boundary of Ω (points of density different from 0 and 1).

Perimeter approximation by Γ-convergence

Γ-convergence (De Giorgi-Franzoni, 1975)

Definition

Let $F_n, F : X \to \mathbb{R}$, X metric space.
One says that $F_n \xrightarrow{\Gamma} F$ at $x \in X$ iif

1. $\forall x_n \to x,\ F(x) \leq \liminf F_n(x)$,
2. $\exists y_n \to x,\ F(x) \geq \limsup F_n(y)$.

Theorem

Suppose that

1. $F_n \xrightarrow{\Gamma} F$ *in* X,
2. $F_n(x_n) \leq \inf_X F_n + \varepsilon_n,\ \varepsilon_n \to 0$,
3. $x_n \to x$.

Then x *is a minimizer of* F *and* $\lim F_n(x_n) = F(x)$.

Remarks

- The convergence of (x_n) is usually obtained from an equicoercivity argument:

$$\sup F_n(x_n) < \infty \Rightarrow (x_n) \text{ is compact.}$$

  This property may be as difficult to prove as the Γ-convergence.

- If $F_n \xrightarrow{\Gamma} F$ and G is continuous then

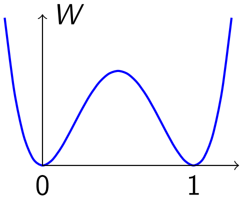
$$F_n + G \xrightarrow{\Gamma} F + G.$$

- The Γ-convergence does not imply the pointwise convergence $F_n(x) \to F(x)$.

A classical perimeter approximation: the Van Der Waals-Cahn-Hiliard functional

For a potential W with wells 0 and 1 define

$$F_\varepsilon(u) = \int_D \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u).$$



Theorem (Modica-Mortola, 1977)

When $\varepsilon \to 0$,

$$F_\varepsilon(u) \xrightarrow{\Gamma} \begin{cases} c\,Per_D(\Omega) & \textit{if } u = \chi_\Omega \in BV(D,\{0,1\}) \\ +\infty & \textit{otherwise} \end{cases}$$

in $L^1(D)$, with $c = \int_0^1 \sqrt{W(t)}dt$.

Advantages

- Approximation of the perimeter in the appropriate sense for optimization
- Intermediate values of u are penalized
  $\rightsquigarrow$ possible combination with relaxation methods

Drawbacks

- The functional does not accept characteristic functions.
- The derivative w.r.t. u involves $-\Delta u$. Hence optimization by an explicit method may be very slow for fine grids (CFL condition).

These drawbacks stem from the term ∇u.

A gradient-free perimeter approximation

For all $u \in L^2(D)$ consider $L_\varepsilon u := v_\varepsilon$ the smoothed version of u by

$$\begin{cases} -\varepsilon^2 \Delta v_\varepsilon + v_\varepsilon = u & \text{in } D, \\ \partial_n v_\varepsilon = 0 & \text{on } \partial D, \end{cases}$$

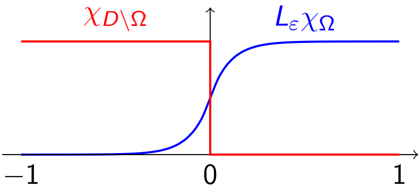
and define

$$F_\varepsilon(\Omega) = \frac{1}{\varepsilon} \langle L_\varepsilon \chi_\Omega, \chi_{D\setminus\Omega} \rangle =: \frac{1}{\varepsilon} \int_D (L_\varepsilon \chi_\Omega) \chi_{D\setminus\Omega}.$$

Example in 1d

$D = (-1, 1)$, $\Omega = (0, 1)$
One finds

$$\lim_{\varepsilon \to 0} F_\varepsilon(\Omega) = \frac{1}{2} = \frac{1}{2} \mathrm{Per}_D(\Omega).$$



More generally, for any $u \in L^\infty(D,[0,1])$, define

$$\tilde{F}_\varepsilon(u) := \frac{1}{\varepsilon}\langle L_\varepsilon u, 1-u\rangle = \frac{1}{\varepsilon}\langle 1 - L_\varepsilon u, u\rangle.$$

Theorem

When $\varepsilon \to 0$ one has in $L^1(D)$

$$\tilde{F}_\varepsilon(u) \xrightarrow{\Gamma} \begin{cases} \frac{1}{2} Per_D(\Omega) & \text{if } u = \chi_\Omega \in BV(D,\{0,1\}) \\ +\infty & \text{otherwise.} \end{cases}$$

Remarks

- *By Legendre-Fenchel transform one obtains*

$$\tilde{F}_\varepsilon(u) = \inf_{v \in H^1(D)} \left\{ \varepsilon \|\nabla v\|^2_{L^2(D)} + \frac{1}{\varepsilon}\left(\|v\|^2_{L^2(D)} + \langle u, 1-2v\rangle \right) \right\}.$$

- *In both expressions there is no ∇u.*
- *$\tilde{F}_\varepsilon$ is weakly-$*$ continuous in $L^\infty(D,[0,1])$.*
- *One also has the pointwise convergence $\tilde{F}_\varepsilon(\chi_\Omega) \to Per_D(\Omega)$.*

Solution of topology optimization problems with perimeter penalization

Let $\tilde{J} : L^1(D, [0,1]) \to \mathbb{R}$ be continuous,

$$\tilde{I}_\varepsilon := \inf_{u \in L^\infty(D,[0,1])} \left\{ \tilde{J}(u) + \alpha \tilde{F}_\varepsilon(u) \right\},$$

$$I := \inf_{\Omega \subset D} \left\{ \tilde{J}(\chi_\Omega) + \frac{\alpha}{2} \mathrm{Per}_D(\Omega) \right\}.$$

Proposition (equicoercivity)

If $\sup_{\varepsilon>0} \tilde{F}_\varepsilon(u_\varepsilon) < \infty$ then (u_ε) is compact in $L^1(D, [0,1])$.

Theorem

Let u_ε be an approximate minimizer of I_ε, i.e.

$$\tilde{J}(u_\varepsilon) + \alpha \tilde{F}_\epsilon(u_\varepsilon) \leq \tilde{I}_\varepsilon + \lambda_\varepsilon, \qquad \lambda_\varepsilon \to 0.$$

Then $\tilde{J}(u_\varepsilon) + \alpha \tilde{F}_\varepsilon(u_\varepsilon) \to I$.
Moreover, (u_ε) admits cluster points, and if u is a cluster point then $u = \chi_\Omega$ where Ω is a minimizer of I.

Algorithms

Objective

Given $J : L^\infty(D, \{0,1\}) \to \mathbb{R}$ solve

$$I = \inf_{\Omega \subset D} \left\{ J(\chi_\Omega) + \frac{\alpha}{2} \mathrm{Per}_D(\Omega) \right\}.$$

Let a sequence $\varepsilon_k \searrow 0$.

At ε_k fixed: two approches

- Consider a continuous extension $\tilde{J} : L^1(D, [0,1]) \to \mathbb{R}$ of J and find a (approximate) minimizer of

$$\tilde{I}_{\varepsilon_k} = \inf_{u \in L^\infty(D,[0,1])} \left\{ \tilde{J}(u) + \alpha \tilde{F}_{\varepsilon_k}(u) \right\}.$$

- Find an approximate minimizer of

$$I_{\varepsilon_k} = \inf_{\Omega \subset D} \left\{ J(\chi_\Omega) + \alpha \tilde{F}_{\varepsilon_k}(\chi_\Omega) \right\}.$$

In practice...

The convexification (of the admissible set) approach

The extension $\tilde{J}$ can be constructed by relaxation (e.g. homogenization), if available.

To minimize one can use:

- general methods for nonlinear optimization with box constraints,
- an alternating algorithm based on

$$\tilde{I}_\varepsilon = \inf_{u \in L^\infty(D,[0,1])} \inf_{v \in H^1(D)} \left\{ \tilde{J}(u) + \alpha \left[\varepsilon \|\nabla v\|^2_{L^2(D)} + \frac{1}{\varepsilon} \left(\|v\|^2_{L^2(D)} + \langle u, 1 - 2v \rangle \right) \right] \right\}.$$

Interest of the latter: if $\tilde{J}$ is linear or can be written as an inf (like the homogenized compliance, through the dual energy).

Minimization w.r.t. v amounts to solving $v = L_\varepsilon u$.

The direct approach

Relies on the concepts of shape and topological derivatives (both exist for $\tilde{F}_\varepsilon$).

Examples

Conductivity maximization

$$J(\chi_\Omega) = \int_{\Gamma_N} gy + \ell|\Omega|, \quad \begin{cases} -\operatorname{div}((\gamma_0\chi_{D\setminus\Omega} + \gamma_1\chi_\Omega)\nabla y) = 0 & \text{in } D \\ (\gamma_0\chi_{D\setminus\Omega} + \gamma_1\chi_\Omega)\nabla y.n = g & \text{on } \Gamma_N \end{cases}$$

The dual energy is

$$\int_{\Gamma_N} gy = \inf_{\substack{-\operatorname{div}\tau=0\\ \tau.n=g}} \int_D (\gamma_0\chi_{D\setminus\Omega} + \gamma_1\chi_\Omega)^{-1}|\tau|^2.$$

Method: relaxation (in the weak-$*$ topology) + alternating algorithm based on

$$\tilde{I}_\varepsilon = \inf_{u\in L^\infty(D,[0,1])} \inf_{v\in H^1(D)} \inf_{\substack{-\operatorname{div}\tau=0\\ \tau.n=g}} \left\{ \int_D (\gamma_0(1-u) + \gamma_1 u)^{-1}|\tau|^2 \right.$$
$$\left. + \ell\int_D u + \alpha\left[\varepsilon\|\nabla v\|^2_{L^2(D)} + \frac{1}{\varepsilon}\left(\|v\|^2_{L^2(D)} + \langle u, 1-2v\rangle\right)\right]\right\}.$$

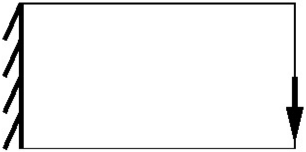
Minimization w.r.t. u is given by

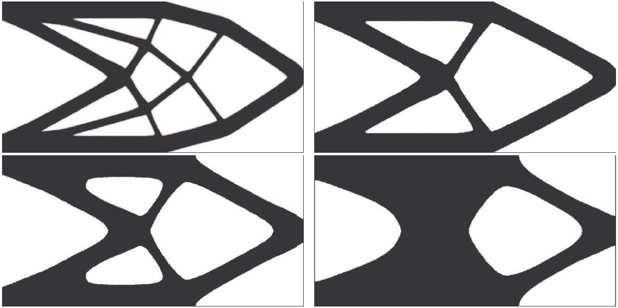
$$u = \begin{cases} 1 \text{ if } \ell + \dfrac{\alpha}{2\varepsilon}(1-2v) \leq 0, \\ P_{[0,1]}\left(\sqrt{\dfrac{|\tau|^2}{(\gamma_1-\gamma_0)\left(\ell + \frac{\alpha}{2\varepsilon}(1-2v)\right)}} - \dfrac{\gamma_0}{\gamma_1-\gamma_0}\right) \text{ else.} \end{cases}$$

Optimal heater for $\alpha = 0.1, 0.5, 2$, respectively
$(\gamma_1 = 1,\ \gamma_0 = 10^{-3})$.

Compliance minimization in linear elasticity

Method: homogenization (rank 2 laminates, cf. Allaire)
+ alternating algorithm (minimization w.r.t. u is again explicit)

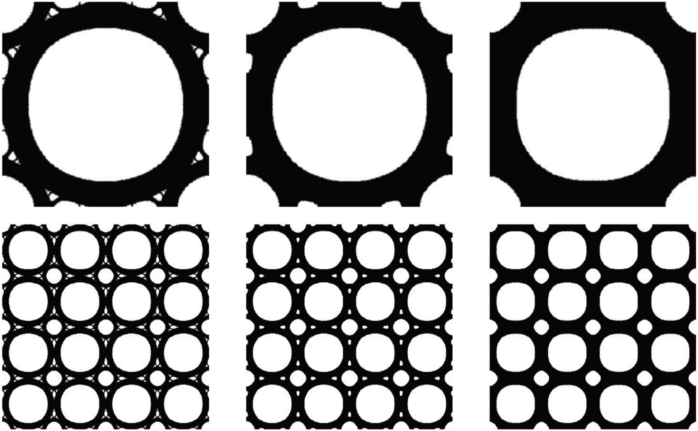




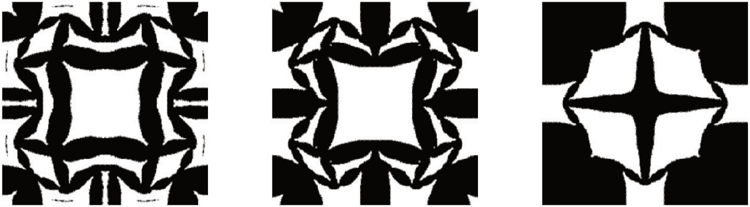
Cantilever for $\alpha = 0.1, 2, 20, 50,$ respectively.

Optimal design of microstructures

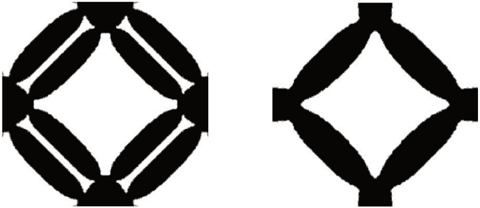
Goal: optimize the Representative Volume Element to obtain desired homogenized properties (periodic model)
Method: topological derivative + level set representation (homogenization is unknown)



Bulk modulus maximization for $\alpha = 0$, $\alpha = 0.1$, $\alpha = 0.5$.
Optimized RVE and associated periodic microstructure.



Poisson ratio minimization for $\alpha = 0$, $\alpha = 0.01$ and $\alpha = 0.02$.
Corresponding Poisson: -0.345, -0.319 and -0.260, respectively.



Poisson ratio maximization for $\alpha = 0$ and $\alpha = 0.1$.
Corresponding Poisson ratios: 0.871 and 0.831, respectively.

Extension to the multiphase case: minimal partitions

Typical problem: find a partition $(\Omega_1, ..., \Omega_N)$ of D which minimizes

$$\mathcal{J}(\Omega_1, ..., \Omega_N) = \sum_{i=1}^{N} \int_{\Omega_i} g_i + \frac{\alpha}{2} \mathrm{Per}_D(\Omega_i).$$

The objective

$$J(\chi_{\Omega_1}, ..., \chi_{\Omega_N}) = \sum_{i=1}^{N} \int_D \chi_{\Omega_i} g_i$$

is relaxed (in the weak-$*$ topology) by

$$\tilde{J}(u_1, ..., u_N) = \sum_{i=1}^{N} \int_D u_i g_i$$

over

$$X = \left\{ (u_1, ..., u_N) \in L^\infty(D, [0,1])^N, \sum_i u_i = 1 \right\}.$$

For Γ-convergence issues, X is endowed with the L^1 distance.

Mathematical issues

We want to approximate

$$\frac{1}{2}\sum_{i=1}^{N} \mathrm{Per}_D(\Omega_i) \qquad \text{by} \qquad \sum_{i=1}^{N} \tilde{F}_\varepsilon(\chi_{\Omega_i}).$$

But the Γ-convergence is not stable upon addition ($\limsup$ inequality), as a collection of recovery sequences (u_i^ε) does not necessarily belong to X. However, the pointwise convergence of $\tilde{F}_\varepsilon$ allows to choose constant recovery sequences.

Theorem

Let $(u_1^\varepsilon, \dots, u_N^\varepsilon)$ be an approximate minimizer of

$$\inf_{(u_1,\dots,u_N)\in X} \mathcal{J}_\varepsilon(u_1,\dots,u_N) := \tilde{J}(u_1,\dots,u_N) + \frac{\alpha}{2}\sum_i \tilde{F}_\varepsilon(u_i).$$

Then $\mathcal{J}(u_1^\varepsilon, \dots, u_N^\varepsilon) \to I := \inf \mathcal{J}(\Omega_1, \dots, \Omega_N)$.
Moreover, $(u_1^\varepsilon, \dots, u_N^\varepsilon)$ admits cluster points, and if $(u_1, \dots, u_N)$ is a cluster point then $u_i = \chi_{\Omega_i}$ where $(\Omega_1, \dots, \Omega_N)$ is a minimizer of I.

Algorithmic issues

At ε fixed, the problem admits the formulation:

$$\inf_{(u_1,\dots,u_N)\in X} \inf_{v_1,\dots,v_N\in H^1(D)} \sum_{i=1}^{N} \Big\{ \langle g_i, u_i\rangle + \alpha \Big[\varepsilon \|\nabla v_i\|^2_{L^2(D)} + \frac{1}{\varepsilon}\Big(\|v_i\|^2_{L^2(D)} + \langle u_i, 1-2v_i\rangle \Big)\Big]\Big\}.$$

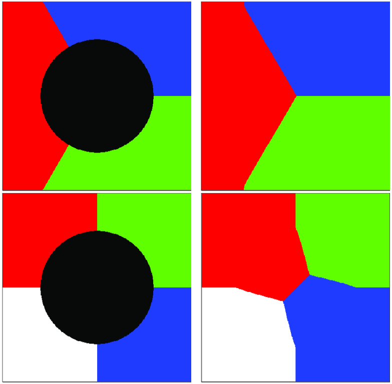
We use an alternating algorithm.

- Minimizing w.r.t. $(v_1, \dots, v_N)$ is achieved by setting $v_i = L_\varepsilon u_i$.
- Minimizing w.r.t. $(u_1, \dots, u_N)$ is linear and spatially uncoupled, while X is a convex polyhedron: setting $\zeta_i = g_i + \frac{\alpha}{\varepsilon}(1-2v_i)$, put $u_i = 1$ on the smallest ζ_i.

Remark: the u_i's are characteristic functions of a partition at every iteration.

Example: triple and quadruple points

Given a partition $(E_0, E_1, ..., E_N)$ of D, define $g_i = -\chi_{E_i}$ in order to favor the phase Ω_i in the set E_i. In the "ocluded part" E_0 no phase is favored.



Triple and quadruple points problems:
g_i's (left) and inpainted image (right).

Application: image classification

Given an image f we search for a partition $\{\Omega_i\}$ of D and an image w constant on each Ω_i which minimize (usually for $p = 1$ or $p = 2$)

$$\mathcal{J}(\{\Omega_i\}, w) = \|w - f\|_{L^p}^p + \frac{\alpha}{2} \sum_{i=1}^{N} \mathrm{Per}_D(\Omega_i).$$

For $w = \sum_i u_i c_i$, $u_i = \chi_{\Omega_i}$ it reads

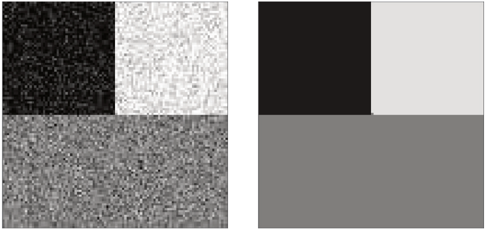
$$\mathcal{J}(\Omega_1, ..., \Omega_N) = \sum_{i=1}^{N} \int_{\Omega_i} \underbrace{|c_i - f|^p}_{g_i} + \frac{\alpha}{2} \mathrm{Per}_D(\Omega_i).$$

Unsupervised classification: the c_i's are updated at each iteration by inserting a 3^{rd} minimization in the alternating algorithm.

- $p = 2$: c_i is the mean of f over Ω_i
- $p = 1$: c_i is the median of f over Ω_i

Color images: $g_i = \sum_{j=1}^{3} |c_{ij} - f_j|^p$

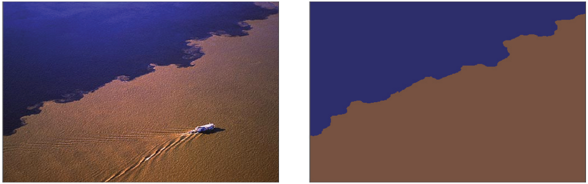
Examples (greylevel)



Unsupervised classification with 3 labels ($p = 2$).

Unsupervised classification with 2 or 3 labels ($p = 1$).

Examples (color)



Unsupervised classification with 2 labels ($p = 1$).

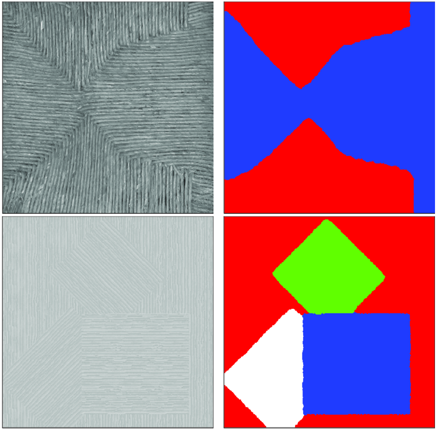
Unsupervised classification with 2 or 5 labels ($p = 1$).

Anisotropy-based classification

The fidelity term $|c_i - f|^p$ is replaced by

$$g_i = -(\nabla\phi.\xi_i)^2$$

where ϕ is a smoothed version of the original image and ξ_i are prescribed unit vectors (basic texture identification model).



Supervised classification of anisotropic textures.

Deblurring (binary case)

Model: minimize

$$\mathcal{J}(\Omega) = \Phi(\chi_\Omega) + \frac{\alpha}{2}\mathrm{Per}_D(\Omega)$$

with

$$\Phi(u) = \|A(uc_1 + (1-u)c_2) - f\|^2_{L^2(D)},$$

$A \in \mathcal{L}(L^2(D))$, $c_1, c_2 \in \mathbb{R}$ known.

Due to the spatial coupling, minimization w.r.t. u is not explicit. Using that $\nabla\Phi$ is λ-Lipschitz in L^2, $\lambda = 2(c_1 - c_2)^2\|A^*A\|$, we have

$$\Phi(u) = \inf_{\hat{u}\in L^2(D)} \Phi(\hat{u}) + \langle\nabla\Phi(\hat{u}), u - \hat{u})\rangle + \frac{\lambda}{2}\|u - \hat{u}\|^2.$$

We obtain the formulation at ε fixed

$$\inf_{u\in L^\infty(D,[0,1])} \inf_{\hat{u}\in L^2(D)} \inf_{v\in H^1(D)} \Bigg\{ \Phi(\hat{u}) + \langle \nabla\Phi(\hat{u}), u-\hat{u}\rangle + \frac{\lambda}{2}\|u-\hat{u}\|^2 + \alpha\left[\varepsilon\|\nabla v\|^2_{L^2(D)} + \frac{1}{\varepsilon}\left(\|v\|^2_{L^2(D)} + \langle u, 1-2v\rangle\right)\right]\Bigg\}.$$

- Minimization w.r.t. v: $v = L_\varepsilon u$
- Minimization w.r.t. $\hat{u}$: $\hat{u} = u$
- Minimization w.r.t. u:

$$u = P_{[0,1]}\left(\hat{u} - \frac{1}{\lambda}\left(\nabla\Phi(\hat{u}) + \frac{\alpha}{\varepsilon}(1-2v)\right)\right)$$

Remark: u is no longer a characteristic function during the iterations.

Examples

Deblurring and denoising: original image, damaged image with blur and noise effects (middle), reconstructed image (right).

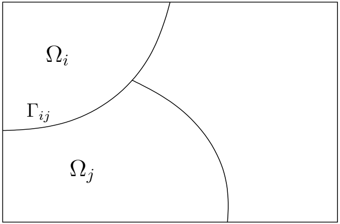
Deblurring: original image (left), blurred image (middle), and restored image (right).

Extension: interface energies (ongoing)

$$\Gamma_{ij} = \partial^*\Omega_i \cap \partial^*\Omega_j$$

Goal: minimize

$$\sum_i \int_{\Omega_i} g_i + \sum_{i<j} \alpha_{ij} \mathcal{H}^1(\Gamma_{ij})$$



We have

$$\begin{aligned}
\mathcal{H}^1(\Gamma_{ij}) &= \frac{1}{2}\left[\mathcal{H}^1(\partial^*\Omega_i) + \mathcal{H}^1(\partial^*\Omega_j) - \mathcal{H}^1(\partial^*\Omega_i \cup \partial^*\Omega_j)\right] \\
&= \lim_{\varepsilon \to 0} \left[\tilde{F}_\varepsilon(\chi_{\Omega_i}) + \tilde{F}_\varepsilon(\chi_{\Omega_j}) - \tilde{F}_\varepsilon(\chi_{\Omega_i} + \chi_{\Omega_j})\right] \\
&= \dots \\
&= \lim_{\varepsilon \to 0} \frac{2}{\varepsilon} \langle L_\varepsilon \chi_{\Omega_i}, \chi_{\Omega_j} \rangle.
\end{aligned}$$

Open questions: Γ-convergence? equicoercivity?

One can still use an alternating algorithm thanks to the formulation

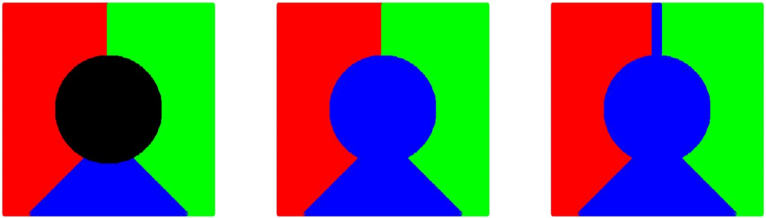
$$2\langle L_\varepsilon u_i, u_j\rangle = \underbrace{\langle L_\varepsilon(u_i+u_j), u_i+u_j\rangle}_{\text{dual}} - \underbrace{\langle L_\varepsilon u_i, u_i\rangle}_{\text{primal}} - \underbrace{\langle L_\varepsilon u_j, u_j\rangle}_{\text{primal}}$$

combined with

$$\begin{aligned}\langle L_\varepsilon u, u\rangle &= \sup_{v\in H^1(D)} 2\langle u, v\rangle - \varepsilon^2\|\nabla v\|^2 - \|v\|^2 \qquad \text{(primal)}\\ &= \inf_{p\in H_0^{div}(D)} \|u+\varepsilon\,\mathrm{div}\, p\|^2 + \|p\|^2 \qquad \text{(dual)}.\end{aligned}$$

The solutions of the latter problems are $v = L_\varepsilon u$, $p = \varepsilon\nabla v$.
The minimization in u is quadratic and spatially uncoupled.

Example



Data (left), initialisation (middle) and result (right) for $\alpha_{red/green} = 100$, $\alpha_{red/blue} = \alpha_{green/blue} = 0$

THANK YOU!