Space-Time Methods for Cardiac Electromechanics

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Der Wissenschaftsfonds.

Outline

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Motivation

What my PhD thesis is about:

- Investigate methods for **nonlinear time-dependent** PDEs. Especially the models of cardiac electromechanics.
- Usual approach: Time-stepping procedure. We apply discretization methods for the **whole space-time domain**. This approach was already used for Navier-Stokes equations in [Neumueller, 2013].
- Investigate the **numerical properties** of those methods (convergence behavior, stability,. . .).
- Look at the different possibilities of **coupling** cardiac electrics with cardiac mechanics.
- Numerical simulations.

The Bidomain Equations

Find $(V_{\mathrm{tm}}, u_e, \boldsymbol{v})$ such that:

$$\begin{aligned}
\frac{\partial}{\partial t} V_{\mathrm{tm}} + I_{\mathrm{ion}}(V_{\mathrm{tm}}, \boldsymbol{v}) - \operatorname{div}(\mathbf{M}_i \operatorname{grad} V_{\mathrm{tm}}) - \operatorname{div}(\mathbf{M}_i \operatorname{grad} u_e) &= s_i, \\
- \operatorname{div}(\mathbf{M}_i \operatorname{grad} V_{\mathrm{tm}}) - \operatorname{div}((\mathbf{M}_i + \mathbf{M}_e) \operatorname{grad} u_e) &= 0, \\
\frac{\partial}{\partial t} \boldsymbol{v} + \boldsymbol{H}(V_{\mathrm{tm}}, \boldsymbol{v}) &= 0,
\end{aligned}$$

in $Q_T := \Omega \times (0, T)$. Boundary and Initial conditions:

$$\begin{aligned}
\boldsymbol{n} \cdot \mathbf{M}_i \operatorname{grad}(V_{\mathrm{tm}} + u_e) &= 0 && \text{on } \Sigma_N := \partial\Omega \times (0, T), \\
\boldsymbol{n} \cdot \mathbf{M}_e \operatorname{grad} u_e &= 0 && \text{on } \Sigma_N, \\
V_{\mathrm{tm}}(0, \boldsymbol{x}) &= V_{\mathrm{tm}}^0(\boldsymbol{x}) && \text{on } \Sigma_0 := \Omega \times \{0\}, \\
\boldsymbol{v}(0, \boldsymbol{x}) &= \boldsymbol{v}^0(\boldsymbol{x}) && \text{on } \Sigma_0.
\end{aligned}$$

Nonlinear Elasticity
Quasi-stationary, Lagrangian coordinates

Find $\boldsymbol{U}$ such that:

$$
\begin{aligned}
-\operatorname{Div}(\mathbf{F}(\boldsymbol{U})\mathbf{S}(\boldsymbol{U})) &= \mathbf{0} && \text{in } \Omega_r, \\
\boldsymbol{U} &= \boldsymbol{G}_D && \text{on } \Gamma_{D,r}, \\
\mathbf{F}(\boldsymbol{U})\mathbf{S}(\boldsymbol{U})\boldsymbol{N} &= \boldsymbol{T}_N(\boldsymbol{U}) && \text{on } \Gamma_{N,r}
\end{aligned}
$$

where

- $\mathbf{F}(\boldsymbol{U}) = \mathbf{I} + \operatorname{Grad} \boldsymbol{U}$.
- $\mathbf{S}$ second Piola stress tensor
- Ω_r is the reference configuration with boundary $\Gamma_{D,r}$ and $\Gamma_{N,r}$ resp.

Assume hyperelastic material $\Rightarrow$ $\mathbf{S} = 2\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}}$ where $\mathbf{C} = \mathbf{F}^\top \mathbf{F}$.

Myocardium Model
[Holzapfel and Ogden, 2009]

Isochoric part $\overline{\mathbf{C}} = J^{-2/3}\mathbf{C}$ with $J = \det(\mathbf{F})$.

$$\Psi(\mathbf{C}) := \frac{\kappa}{2} U(J) + \overline{\Psi}_{\text{iso}}(\overline{\mathbf{C}}) + \overline{\Psi}_{\text{trans}}(\overline{\mathbf{C}}) + \overline{\Psi}_{\text{ortho}}(\overline{\mathbf{C}})$$

- **isotropic part**

$$\overline{\Psi}_{\text{iso}}(\overline{\mathbf{C}}) := \frac{a}{2b}\left(\exp\left(b(\bar{I}_1 - 3)\right) - 1\right),$$

- **transversely isotropic part**

$$\overline{\Psi}_{\text{trans}}(\overline{\mathbf{C}}) := \frac{a_f}{2b_f}\left\{\exp\left[b_f(\bar{I}_{4f} - 1)^2\right] - 1\right\} + \frac{a_s}{2b_s}\left\{\exp\left[b_s(\bar{I}_{4s} - 1)^2\right] - 1\right\}$$

Myocardium Model
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$$\Psi(\mathbf{C}) := \frac{\kappa}{2}U(J) + \overline{\Psi}_{\text{iso}}(\overline{\mathbf{C}}) + \overline{\Psi}_{\text{trans}}(\overline{\mathbf{C}}) + \overline{\Psi}_{\text{ortho}}(\overline{\mathbf{C}})$$

- **orthotropic part**

$$\overline{\Psi}_{\text{ortho}}(\overline{\mathbf{C}}) := \frac{a_{fs}}{2b_{fs}}\left\{\exp\left[b_{fs}\bar{I}_{8fs}^{2}\right] - 1\right\}$$

where

$$\bar{I}_1 = \operatorname{tr}(\overline{\mathbf{C}}) \qquad \bar{I}_{4f} = \left(\overline{\mathbf{C}}\boldsymbol{f}_0, \boldsymbol{f}_0\right)$$

$$\bar{I}_{4s} = \left(\overline{\mathbf{C}}\boldsymbol{s}_0, \boldsymbol{s}_0\right) \qquad \bar{I}_{8fs} = \left(\overline{\mathbf{C}}\boldsymbol{f}_0, \boldsymbol{s}_0\right)$$

and $\kappa \gg 1, a, b, a_f, b_f, a_s, b_s, a_{fs}, b_{fs} > 0$.

Outline

Active Stress

Introduce new variable $\mathcal{S}_a$. Add active part to stress tensor (see e.g. [Ambrosi and Pezzuto, 2012; Eriksson et al., 2013; Nash and Panfilov, 2004]):

$$\mathbf{S} + = \mathbf{S}_{\text{act}}$$

where

$$\mathbf{S}_{\text{act}} = \mathbf{S}_{\text{act}} \left(V_{\mathrm{tm}}, \boldsymbol{U}, \lambda, \frac{\partial}{\partial t}\lambda, [\text{Ca}] \ldots \right)$$

Behavior of $\mathbf{S}_{\text{act}}$ is described with additional set of ODEs.
$(\lambda^2 = (\mathbf{C}\boldsymbol{f}_0, \boldsymbol{f}_0))$

Coupling Mechanics to Bidomain
Overview

From literature (e.g.: [Ambrosi et al., 2011; Göktepe and Kuhl, 2010; Nordsletten et al., 2011] we may distinguish two approaches:

- Introduce coupling through geometry changes
- Modify the ionic currents I_{ion}
- Combining both approaches

Questions

- What are the differences ?
- What is necessary ?
- How do the resulting PDEs/ODEs look like?

Coupling Through Geometry

General thoughts:

- What is the role of the local fibre coordinate system $\boldsymbol{f}_0, \boldsymbol{s}_0, \boldsymbol{n}_0$?
- In general: deformation destroys orthogonality of coordinate system
- Conductivity tensors $\mathbf{M}_i, \mathbf{M}_e$ may change due to deformation
- What is the right description for the conductivity tensors?

Coupling Through Geometry

Assume that Bidomain equations are stated in **Eulerian** coordinates. Introduce coupling through transformation to **Lagrangian** coordinates. Example:

$$\int_\Omega \operatorname{div}(\mathbf{M}_i \operatorname{grad} V_{\mathrm{tm}})\ \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} (\mathbf{M}_i \operatorname{grad} V_{\mathrm{tm}}, \boldsymbol{n})\ \mathrm{ds}_{\boldsymbol{x}}\,.$$

use Nanson's Formula $\boldsymbol{n} = J\mathbf{F}^{-\top}\boldsymbol{N}$

$$\begin{aligned}\int_{\partial\Omega} (\mathbf{M}_i \operatorname{grad} V_{\mathrm{tm}}, \boldsymbol{n})\ \mathrm{ds}_{\boldsymbol{x}} &= \int_{\partial\Omega_r} \left(J\mathbf{M}_i \operatorname{grad} V_{\mathrm{tm}}, \mathbf{F}^{-\top}\boldsymbol{N}\right)\ \mathrm{ds}_{\boldsymbol{X}} \\ &= \int_{\partial\Omega_r} \left(J\mathbf{F}^{-1}\mathbf{M}_i \operatorname{grad} V_{\mathrm{tm}}, \boldsymbol{N}\right)\ \mathrm{ds}_{\boldsymbol{X}}\,.\end{aligned}$$

Coupling Through Geometry

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$$\int_\Omega \operatorname{div}(\mathbf{M}_i \operatorname{grad} V_{\mathrm{tm}})\ \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} (\mathbf{M}_i \operatorname{grad} V_{\mathrm{tm}}, \boldsymbol{n})\ \mathrm{ds}_{\boldsymbol{x}}\,.$$

Finally we use $\operatorname{grad} V_{\mathrm{tm}} = \mathbf{F}^{-\top} \operatorname{Grad} V_{\mathrm{tm}}$ and obtain

$$\begin{aligned}\int_{\partial\Omega_r} \left(J\mathbf{F}^{-1}\mathbf{M}_i \operatorname{grad} V_{\mathrm{tm}}, \boldsymbol{N}\right)\ \mathrm{ds}_{\boldsymbol{X}} &= \int_{\partial\Omega_r} \left(J\mathbf{F}^{-1}\mathbf{M}_i\mathbf{F}^{-\top} \operatorname{Grad} V_{\mathrm{tm}}, \boldsymbol{N}\right)\ \mathrm{ds}_{\boldsymbol{X}} \\ &= \int_{\Omega_r} \operatorname{Div}\left(J\mathbf{F}^{-1}\mathbf{M}_i\mathbf{F}^{-\top} \operatorname{Grad} V_{\mathrm{tm}}\right)\ \mathrm{d}\boldsymbol{X}\,.\end{aligned}$$

Need description of $\mathbf{M}_i$ in Eulerian and Lagrangian coordinates

Illustration

- For orthonormal fibre coordinate system

$$\mathbf{M}_i = m_i^f \boldsymbol{f}_0 \otimes \boldsymbol{f}_0 + m_i^s \boldsymbol{s}_0 \otimes \boldsymbol{s}_0 .$$

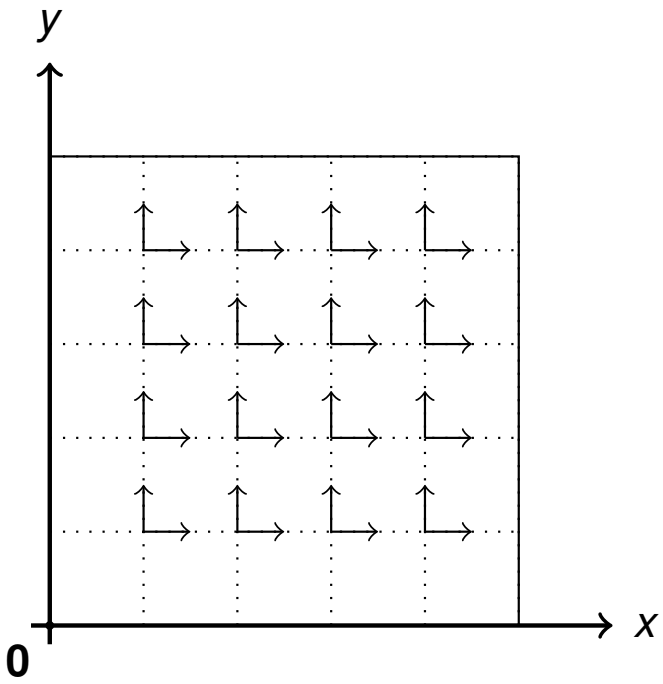


Figure: Initial Geometry

Illustration

- For orthonormal fibre coordinate system

  $$\mathbf{M}_i = m_i^f \boldsymbol{f}_0 \otimes \boldsymbol{f}_0 + m_i^s \boldsymbol{s}_0 \otimes \boldsymbol{s}_0.$$

- After a deformation we get

  $$\mathbf{F}\mathbf{M}_i\mathbf{F}^\top = m_i^f(\mathbf{F}\boldsymbol{f}_0) \otimes (\mathbf{F}\boldsymbol{f}_0) + m_i^s(\mathbf{F}\boldsymbol{s}_0) \otimes (\mathbf{F}\boldsymbol{s}_0).$$

- If $(\boldsymbol{f}_0, \boldsymbol{s}_0) = 0$ m_i^f, m_i^s are eigenvalues of $\mathbf{M}_i$ and $\boldsymbol{f}_0, \boldsymbol{s}_0$ are eigenvectors.

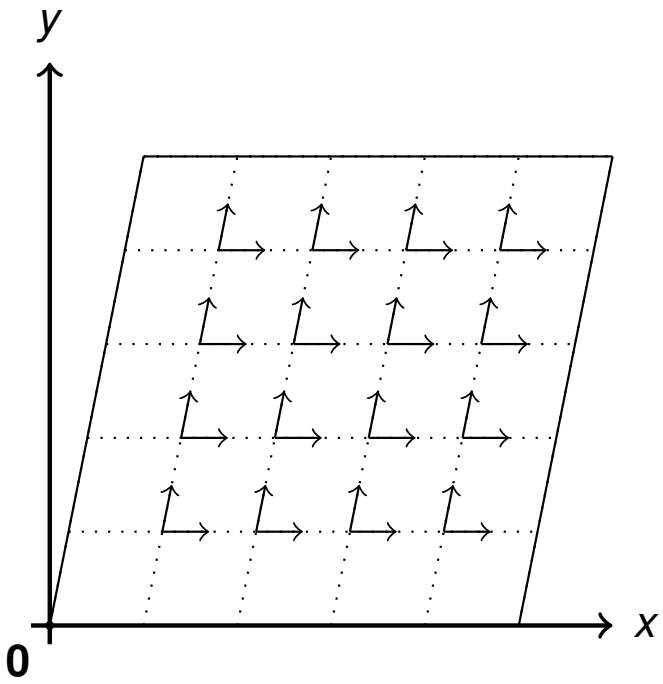


Figure: Deformed Geometry

Illustration
cont.

In the undeformed configuration we have

$$\mathbf{M}_i \operatorname{Grad} u = m_i^f(\boldsymbol{f}_0, \operatorname{Grad} u)\boldsymbol{f}_0 + m_i^s(\boldsymbol{s}_0, \operatorname{Grad} u)\boldsymbol{s}_0$$

This motivates that we use $\mathbf{F}\mathbf{M}_i\mathbf{F}^\top$ as description of $\mathbf{M}_i$ in the deformed configuration.

$$\mathbf{F}\mathbf{M}_i\mathbf{F}^\top \operatorname{grad} u = m_i^f(\mathbf{F}\boldsymbol{f}_0, \operatorname{grad} u)\mathbf{F}\boldsymbol{f}_0 + m_i^s(\mathbf{F}\boldsymbol{s}_0, \operatorname{grad} u)\mathbf{F}\boldsymbol{s}_0$$

Coupling through Geometry
further aspects

Some other issues (see [Ambrosi et al., 2011])

- Not clear which time derivatives one uses in Bidomain (total or partial)
- For total time derivative (eulerian configuration), convective term:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\bullet) = \frac{\partial}{\partial t}(\bullet) + \left(\frac{\partial}{\partial t}\boldsymbol{u}, \mathrm{grad}(\bullet)\right).$$

  This introduces a coupling also for the ODEs (they become PDEs).

Coupling by Adding Nonlinearities

Another approach: add new nonlinearities to the Bidomain equations.

$$I_{\text{ion}}(V_{\mathrm{tm}}, \boldsymbol{v}) + = I_{\text{elast}}(V_{\mathrm{tm}}, \boldsymbol{w}, \lambda_f, \frac{\partial}{\partial t}\lambda_f)$$

Example (taken from [Göktepe and Kuhl, 2010]) :

$$I_{\text{elast}}(V_{\mathrm{tm}}, \boldsymbol{w}, \boldsymbol{U}) := \begin{cases} c_1(\lambda_f^2 - 1)(V_{\mathrm{tm}} - c_2) & \text{if } \lambda_f > 1 \\ 0 & \text{else} \end{cases}$$

Different choices possible, active research topic.

Fully Coupled System

Monodomain, Lagrangian coordinates

$$\frac{\partial}{\partial t}(JV_{\mathrm{tm}}) + J^{-1}\,\mathrm{Div}(J\mathbf{M}_i^{\mathrm{lag}}\,\mathrm{Grad}\,V_{\mathrm{tm}}) + J\overline{I_{\mathrm{ion}}}(V_{\mathrm{tm}}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{U}) = Js_0,$$

$$\frac{\partial}{\partial t}(Jv_i) + JH_i(V_{\mathrm{tm}}, v_i) = 0,$$

$$\frac{\partial}{\partial t}(Jw_j) + JG_j(V_{\mathrm{tm}}, w_j, \boldsymbol{u}) = 0,$$

$$\mathrm{Div}(\mathbf{F}(\mathbf{S}_{\mathrm{pass}} + \mathbf{S}_{\mathrm{act}})) = \mathbf{0}$$

plus suitable initial conditions and boundary conditions.

Outline

Broken Function Spaces

- Consider triangulation of $\overline{Q_T} := \overline{\mathcal{T}_N} = \biguplus_{l=1}^{N} \overline{\tau_l}$
- It holds: $u \in \mathrm{H}^{r,s}(Q_T) \Rightarrow u \in \mathrm{H}^{\min\{r,s\}}(Q_T)$.

Define broken function space

$$\mathrm{H}^s(\mathcal{T}_N) := \left\{ v \in \mathrm{L}^2(Q_T) \colon v\big|_{\tau_l} \in \mathrm{H}^s(\tau_l) \; \forall \tau_l \in \mathcal{T}_N \right\}.$$

Analogously we define for given $p \in \mathbb{N}$

$$S_h^p(\mathcal{T}_N) := \left\{ v_h \in \mathrm{L}^2(Q_T) \colon v_h\big|_{\tau_l} \in \mathbb{P}^p(\tau_l) \forall \tau_l \in \mathcal{T}_N \right\}.$$

Tools for DG Formulation

- Let $\tau_k, \tau_l \in \mathcal{T}_N$. Define **interior facet** $\Gamma_{kl} := \overline{\tau_k} \cap \overline{\tau_l}$. The set of all interior facets is denoted by $\mathcal{I}_N$.
- Let $\Gamma_{kl} \in \mathcal{I}_N$ with normal $\boldsymbol{n}_k := (\boldsymbol{n}_{k,\boldsymbol{x}}, n_{k,t})^\top$ resp. $\boldsymbol{n}_l := -\boldsymbol{n}_k$. Define

$$[\![v]\!]_{kl,\boldsymbol{x}} := v\big|_{\tau_k} \boldsymbol{n}_{k,\boldsymbol{x}} + v\big|_{\tau_l} \boldsymbol{n}_{l,\boldsymbol{x}}, \qquad \text{(space jump)}$$

$$[\![v]\!]_{kl,t} := v\big|_{\tau_k} n_{k,t} + v\big|_{\tau_l} n_{l,t}, \qquad \text{(time jump)}$$

$$\langle v \rangle_{\omega,kl} := \omega_k v\big|_{\tau_k} + \omega_l v\big|_{\tau_l}, \qquad \text{(weighted average)}$$

$$\{v\}_{kl}^{\mathrm{up}} := \begin{cases} v\big|_{\tau_k} & \text{if } n_{k,t} > 0, \\ 0 & \text{if } n_{k,t} = 0, \\ v\big|_{\tau_l} & \text{if } n_{k,t} < 0 \end{cases} \qquad \text{(upwind)}$$

where

$$\omega_{\{k,l\}} := \frac{\left(\mathbf{M}_i\big|_{\tau_{\{l,k\}}} \boldsymbol{n}_k, \boldsymbol{n}_k\right)}{\left(\mathbf{M}_i\big|_{\tau_k} \boldsymbol{n}_k, \boldsymbol{n}_k\right) + \left(\mathbf{M}_i\big|_{\tau_l} \boldsymbol{n}_k, \boldsymbol{n}_k\right)} \qquad \gamma_{kl} := \frac{2\left(\mathbf{M}_i\big|_{\tau_k} \boldsymbol{n}_k, \boldsymbol{n}_k\right)\left(\mathbf{M}_i\big|_{\tau_l} \boldsymbol{n}_l, \boldsymbol{n}_l\right)}{\left(\mathbf{M}_i\big|_{\tau_k} \boldsymbol{n}_k, \boldsymbol{n}_k\right) + \left(\mathbf{M}_i\big|_{\tau_l} \boldsymbol{n}_l, \boldsymbol{n}_l\right)}$$

DG Formulation of the Bidomain Bilinear Forms

$$b_t^{\mathrm{DG}}(u,v) := -\sum_{l=1}^{N} \int_{\tau_l} u\partial_t v \,\mathrm{d}\boldsymbol{x} + \sum_{\Gamma_{kl}\in\mathcal{I}_N} \int_{\Gamma_{kl}} \{u\}_{kl}^{\mathrm{up}} [\![v]\!]_{kl,t} \,\mathrm{d}s_{\boldsymbol{x}} + \int_{\Sigma_T} uv \,\mathrm{d}s_{\boldsymbol{x}}$$

$$\begin{aligned} a_i^{\mathrm{DG}}(u,v) := & \sum_{l=1}^{N} \int_{\tau_l} \mathbf{M}_i \nabla_{\boldsymbol{x}} u \cdot \nabla_{\boldsymbol{x}} v \,\mathrm{d}\boldsymbol{x} - \sum_{\Gamma_{kl}\in\mathcal{I}_N} \int_{\Gamma_{kl}} \langle \mathbf{M}_i \nabla_{\boldsymbol{x}} u \rangle_{\omega,kl} \cdot [\![v]\!]_{kl,\boldsymbol{x}} \,\mathrm{d}s_{\boldsymbol{x}} \\ & - \sum_{\Gamma_{kl}\in\mathcal{I}_N} \int_{\Gamma_{kl}} [\![u]\!]_{kl,\boldsymbol{x}} \cdot \langle \mathbf{M}_i \nabla_{\boldsymbol{x}} v \rangle_{\omega,kl} \,\mathrm{d}s_{\boldsymbol{x}} \\ & + \sum_{\Gamma_{kl}\in\mathcal{I}_N} \frac{\sigma_{kl}\gamma_{kl}}{\overline{h}_{kl}} \int_{\Gamma_{kl}} [\![u]\!]_{kl,\boldsymbol{x}} \cdot [\![v]\!]_{kl,\boldsymbol{x}} \,\mathrm{d}s_{\boldsymbol{x}} + \alpha_i \int_{\Sigma_R} uv ds_q \end{aligned}$$

DG Formulation of the Bidomain Bilinear Forms

$$I_{\text{ion}}^{\text{DG}}((u,v);\phi) := \sum_{l=1}^{N} \int_{\tau_l} I_{\text{ion}}(u,v)\phi dq$$

$$H^{\text{DG}}((u,v);\zeta) := \sum_{l=1}^{N} \int_{\tau_l} H(u,v)\zeta dq$$

Space-Time DG Formulation

Find $(V_{\mathrm{tm}}^h, u_e^h, v^h)$ s.t.:

$$b_t^{\mathrm{DG}}(V_{\mathrm{tm}}^h, \phi^h) + a_i^{\mathrm{DG}}(V_{\mathrm{tm}}^h, \phi^h) + a_i^{\mathrm{DG}}(u_e^h, \phi^h) + I_{\mathrm{ion}}^{\mathrm{DG}}((V_{\mathrm{tm}}^h, v^h); \phi^h) = \left\langle f_1, \phi^h \right\rangle$$

$$a_i^{\mathrm{DG}}(V_{\mathrm{tm}}^h, \psi^h) + a_{i+e}^{\mathrm{DG}}(u_e^h, \psi^h) = \left\langle f_2, \psi^h \right\rangle$$

$$b_t^{\mathrm{DG}}(v^h, \zeta^h) + H^{\mathrm{DG}}((V_{\mathrm{tm}}^h, v^h); \zeta^h) = \left\langle f_3, \zeta^h \right\rangle$$

Nonlinear System $\Rightarrow$ Newton's Method.

Comments on Newton's Method

- No good initial guess $(V_{\mathrm{tm}}^0, u_e^0, v^0)$ available due to dominance of nonlinearities I_{ion}, H
- Need global Newton algorithm to find initial guess
  - Load-stepping strategies
  - Backtracking strategies
  - Use simpler models as initial guess
- We applied load-stepping + backtracking methods.

Linear Bidomain Equations

As a first step study the linear problem: find V_{tm}^h, u_e^h such that

$$b_t^{\mathrm{DG}}(V_{\mathrm{tm}}^h, w^h) + a_i^{\mathrm{DG}}(V_{\mathrm{tm}}^h, w^h) + a_i^{\mathrm{DG}}(u_e^h, w^h) = \left\langle f_1, w^h \right\rangle,$$
$$a_i^{\mathrm{DG}}(V_{\mathrm{tm}}^h, v^h) + a_{i+e}^{\mathrm{DG}}(u_e^h, v^h) = \left\langle f_2, v^h \right\rangle.$$

This corresponds to a block system of the form

$$\begin{pmatrix} B_T + K_i & K_i \\ K_i & K_i + K_e \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{pmatrix}.$$

What properties can be expected of such a system?

Definition of Norms

We define the following norms:
Norms involving time

$$|||u|||^2_{\mathrm{time}} := \sum_{l=1}^{N} h_l \left\| \frac{\partial}{\partial t} u \right\|^2_{\mathrm{L}^2(\tau_l)} + \|u\|^2_{\mathrm{L}^2(\Sigma_0)} + \|u\|^2_{\mathrm{L}^2(\Sigma_T)}$$
$$+ \sum_{\Gamma_{kl} \in \mathcal{I}_N} \left\| [\![u]\!]_{kl,t} \right\|^2_{\mathrm{L}^2(\Gamma_{kl})},$$

$$|||u|||^2_{\mathrm{time},*} := \sum_{l=1}^{N} h_l^{-1} \|u\|^2_{\mathrm{L}^2(\tau_l)} + \|u\|^2_{\mathrm{L}^2(\Sigma_T)} + \sum_{\Gamma_{kl} \in \mathcal{I}_N} \|\{u\}^{\mathrm{up}}_{kl}\|^2_{\mathrm{L}^2(\Gamma_{kl})}.$$

Definition of Norms

We define the following norms:
Norms involving space

$$|||u|||^2_{\mathrm{space},\{i,e\}} := \sum_{l=1}^{N} \left\| \mathbf{M}^{\frac{1}{2}}_{\{i,e\}} \nabla_{\boldsymbol{x}} u \right\|^2_{\mathrm{L}^2(\tau_l)} + \sum_{\Gamma_{kl} \in \mathcal{I}_N} \frac{\gamma_{kl}\sigma_{kl}}{\overline{h}_{kl}} \left\| [\![u]\!]_{kl,\boldsymbol{x}} \right\|^2_{\mathrm{L}^2(\Gamma_{kl})}$$
$$+ \alpha_{\{i,e\}} \int_{\Sigma_R} |u|^2 \; ds_q,$$

$$|||u|||^2_{\mathrm{space},\{i,e\},*} := |||u|||^2_{\mathrm{space},\{i,e\}} + \sum_{\Gamma_{kl} \in \mathcal{I}_N} \overline{h}_{kl} \left\| \left\langle \mathbf{M}_{\{i,e\}} \nabla_x u \right\rangle_{\omega,kl} \right\|^2_{\mathrm{L}^2(\Gamma_{kl})}.$$

Definition of Norms

We define the following norms:

Compound Norms

For studying the compund bilinear form

$$c^{\mathrm{DG}}((u_h, v_h), (\phi_h, \psi_h)) := b_T^{\mathrm{DG}}(u_h, \phi_h) + a_i^{\mathrm{DG}}(u_h + v_h, \phi_h + \psi_h) + a_e^{\mathrm{DG}}(v_h, \psi_h)$$

we define

$$|||(u_h, v_h)|||_{DG}^2 := |||u_h|||_{\mathrm{time}}^2 + |||u_h|||_{\mathrm{space},i}^2 + |||v_h|||_{\mathrm{space},e}^2,$$

$$|||(u_h, v_h)|||_{DG,*}^2 := |||u_h|||_{\mathrm{time},*}^2 + |||u_h|||_{\mathrm{space},i,*}^2 + |||v_h|||_{\mathrm{space},e,*}^2.$$

Properties

Lemma (Boundedness)

For $u, v \in H^s(\mathcal{T}_N)$ with $s > \frac{3}{2}$ there holds

$$\left| c^{\mathrm{DG}}((u,v),(\phi_h,\psi_h)) \right| \leq c_1^C \, |||(u,v)|||_{DG,*} \, |||(\phi_h,\psi_h)|||_{DG},$$

for all $(\phi_h,\psi_h) \in S_h^p(\mathcal{T}_N) \times S_h^p(\mathcal{T}_N)$.

Properties

Lemma (Boundedness)

For $u, v \in H^s(\mathcal{T}_N)$ with $s > \frac{3}{2}$ there holds

$$\left| c^{\mathrm{DG}}((u,v),(\phi_h,\psi_h)) \right| \leq c_1^C \, |||(u,v)|||_{DG,*} \, |||(\phi_h,\psi_h)|||_{DG},$$

for all $(\phi_h,\psi_h) \in S_h^p(\mathcal{T}_N) \times S_h^p(\mathcal{T}_N)$.

Theorem (Stability)

Let $(u_h, v_h) \in S_h^p(\mathcal{T}_N) \times S_h^p(\mathcal{T}_N)$. Then there exists a constant $c_S^C > 0$ independent of h such that

$$\sup_{\substack{(\phi_h,\psi_h) \in S_h^p(\mathcal{T}_N) \times S_h^p(\mathcal{T}_N) \\ (\phi_h,\psi_h) \neq \mathbf{0}}} \frac{c^{\mathrm{DG}}((u_h,v_h),(\phi_h,\psi_h))}{|||(\phi_h,\psi_h)|||_{DG}} \geq c_S^C \, |||(u_h,v_h)|||_{DG}$$

for all $(u_h, v_h) \in S_h^p(\mathcal{T}_N) \times S_h^p(\mathcal{T}_N)$.

Error Estimate

With the discrete stability one can prove an error estimate of the form

$$|||(u,v)-(u_h,v_h)|||_{DG} \leq Ch^{\min\{s,p+1\}-1}$$

for $u,v \in H^s(\mathcal{T}_N) \times H^s(\mathcal{T}_N)$ solutions to the linear problem.

Convergence Study
3D

We do a convergence study to support our theoretical results. For the calculations we use NESHMET developed by M. Neumüller. The setup is

- $Q_T := [0,1]^3$
- The anisotropic diffusion tensors were chosen as

$$\mathbf{M}_i := \begin{pmatrix} \frac{3}{4} & \frac{3}{20} \\ \frac{3}{20} & \frac{3}{4} \end{pmatrix} \qquad \mathbf{M}_e := \begin{pmatrix} \frac{5}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{4} \end{pmatrix}.$$

- On the boundary $\Sigma_R := \partial[0,1]^2 \times [0,1]$ we apply Robin boundary conditions with $\alpha_i = 1$ and $\alpha_e = 1$.
- The given data s_i,s_e,$g_{i,R}$, $g_{e,R}$ and V^0_{tm} are chosen such that the exact solutions of the linear Bidomain equations are given as

$$V_{\mathrm{tm}}(\boldsymbol{x},t) = x(1-x)y(1-y)t(1-t),$$
$$u_e(\boldsymbol{x},t) = \sin(\pi x)\sin(\pi y)\sin(\pi t).$$

Results

Table: Energy error $|||\boldsymbol{z} - \boldsymbol{z}_h|||_{DG}$ for $p = 1$ where $\boldsymbol{z} = (V_{\mathrm{tm}}, u_e)$

level	elements	dof	$\|\|\|\boldsymbol{z} - \boldsymbol{z}_h\|\|\|_{\mathrm{DG}}$	eoc
0	6	48	$1.3723E+0$	–
1	48	384	$1.0833E+0$	0.34
2	384	3072	$6.6351E-1$	0.71
3	3072	24576	$3.5900E-1$	0.89
4	24576	196608	$1.8443E-1$	0.96
5	196608	1572864	$9.3071E-2$	0.99
6	1572864	12582912	$4.6690E-2$	1.00
Theory:				1.00

Results

Table: Error $\left\| V_{\mathrm{tm}} - V_{\mathrm{tm}}^h \right\|_{\mathrm{time+space},i}$ for $p = 1$

level	elements	dof	$\left\| V_{\mathrm{tm}} - V_{\mathrm{tm}}^h \right\|_*$	eoc
0	6	24	$9.2459E-2$	–
1	48	192	$4.7700E-2$	0.95
2	384	1536	$2.2604E-2$	1.08
3	3072	12288	$1.0229E-2$	1.14
4	24576	98304	$4.7466E-3$	1.11
5	196608	786432	$2.2789E-3$	1.06
6	1572864	6291456	$1.1135E-3$	1.03
Theory:				1.00

Results

Table: Error $\left\| u_e - u_e^h \right\|_{\mathrm{space},e}$ for $p = 1$

level	elements	dof	$\left\| u_e - u_e^h \right\|_{\mathrm{space},e}$	eoc
0	6	24	$1.3692E+0$	–
1	48	192	$1.0823E+0$	0.34
2	384	1536	$6.6313E-1$	0.71
3	3072	12288	$3.5886E-1$	0.89
4	24576	98304	$1.8436E-1$	0.96
5	196608	786432	$9.3043E-2$	0.99
6	1572864	6291456	$4.6677E-2$	1.00
Theory:				1.00

Results

Table: Energy error $|||\boldsymbol{z} - \boldsymbol{z}_h|||_{DG}$ for $p = 2$ where $\boldsymbol{z} = (V_{\mathrm{tm}}, u_e)$

level	elements	dof	$\lvert\lvert\lvert\boldsymbol{z} - \boldsymbol{z}_h\rvert\rvert\rvert_{\mathrm{DG}}$	eoc
0	6	120	$8.9235E-1$	–
1	48	960	$3.9336E-1$	1.18
2	384	7680	$1.2610E-1$	1.64
3	3072	61440	$3.4188E-2$	1.88
4	24576	491520	$8.8077E-3$	1.96
5	196608	3932160	$2.2279E-3$	1.99
Theory:				2.00

Results

Table: Error $\left\| V_{\mathrm{tm}} - V_{\mathrm{tm}}^h \right\|_{\mathrm{time+space},i}$ for $p = 2$

level	elements	dof	$\left\| V_{\mathrm{tm}} - V_{\mathrm{tm}}^h \right\|_*$	eoc
0	6	60	$2.7700E-2$	–
1	48	480	$1.2879E-2$	1.10
2	384	3840	$3.4286E-3$	1.91
3	3072	30720	$7.9743E-4$	2.10
4	24576	245760	$1.8905E-4$	2.08
5	196608	1966080	$4.5158E-5$	2.07
Theory:				2.00

Results

Table: Error $\left\| u_e - u_e^h \right\|_{\mathrm{space},e}$ for $p = 2$

level	elements	dof	$\left\| u_e - u_e^h \right\|_{\mathrm{space},e}$	eoc
0	6	60	$8.9192E-1$	–
1	48	480	$3.9315E-1$	1.18
2	384	3840	$1.2605E-1$	1.64
3	3072	30720	$3.4179E-2$	1.88
4	24576	245760	$8.8056E-3$	1.96
5	196608	1966080	$2.2274E-3$	1.99
Theory:				2.00

Convergence Study
4D

We can also do 4D:

- $Q_T := [0,1]^4$
- The anisotropic diffusion tensors were chosen as

$$\mathbf{M}_i := \begin{pmatrix} \frac{3}{4} & \frac{1}{10} & \frac{1}{20} \\ \frac{1}{10} & \frac{7}{10} & \frac{1}{10} \\ \frac{1}{20} & \frac{1}{10} & \frac{3}{4} \end{pmatrix} \qquad \mathbf{M}_e := \begin{pmatrix} \frac{23}{12} & \frac{1}{6} & -\frac{7}{12} \\ \frac{1}{6} & \frac{7}{6} & \frac{1}{6} \\ -\frac{7}{12} & \frac{1}{6} & \frac{23}{12} \end{pmatrix}.$$

- On the boundary $\Sigma_R := \partial[0,1]^3 \times [0,1]$ we apply Robin boundary conditions with $\alpha_i = 1$ and $\alpha_e = 1$.
- The given data s_i,s_e,$g_{i,R}$, $g_{e,R}$ and V_{tm}^0 are chosen such that the exact solutions of the linear Bidomain equations are given as

$$V_{\mathrm{tm}}(\boldsymbol{x},t) = x(1-x)y(1-y)z(1-z)t(1-t),$$
$$u_e(\boldsymbol{x},t) = \sin(\pi x)\sin(\pi y)\sin(\pi z)\sin(\pi t).$$

Results

Table: Energy error $|||\boldsymbol{z} - \boldsymbol{z}_h|||_{DG}$ for $p = 1$

level	elements	dof	$\|\|\|\boldsymbol{z} - \boldsymbol{z}_h\|\|\|_{\mathrm{DG}}$	eoc
0	16	240	1.5223E+0	–
1	256	3840	1.3572E+0	0.17
2	4096	61440	9.5583E−1	0.51
3	65536	983040	6.0298E−1	0.66
4	1048576	15728640	3.2097E−1	0.91
Theory:				1.00

Results

Table: Individual Error components $\left\| V_{\mathrm{tm}} - V_{\mathrm{tm}}^h \right\|_{\mathrm{time}} + \left\| V_{\mathrm{tm}} - V_{\mathrm{tm}}^h \right\|_{\mathrm{space},i}$ and $\left\| u_e - u_e^h \right\|_{\mathrm{space},e}$ for $p = 1$.

			Error V_{tm}		Error u_e	
level	elements	dof	error	eoc	error	eoc
0	16	240	1.5211E−1	–	1.5147E+0	–
1	256	3840	1.8473E−1	−0.28	1.3446E+0	0.17
2	4096	61440	1.3742E−1	0.43	9.4590E−1	0.51
3	65536	983040	8.2419E−2	0.74	5.9732E−1	0.66
4	1048576	15728640	4.1799E−2	0.98	3.1839E−1	0.91
Observed:				1.00		1.00

Basic Assumptions

We will assume the following:

1. There holds

$$\mathrm{I}_{\mathrm{ion}}(V_{\mathrm{tm}}, v), G(V_{\mathrm{tm}}, v) \in \mathcal{C}^1(Q)$$

2. Upon defining

$$\mathbf{A}(u, w) := \begin{pmatrix} \frac{\partial \mathrm{I}_{\mathrm{ion}}}{\partial V_{\mathrm{tm}}}(u, w) & \frac{\partial \mathrm{I}_{\mathrm{ion}}}{\partial v}(u, w) \\ \frac{\partial G}{\partial V_{\mathrm{tm}}}(u, w) & \frac{\partial G}{\partial v}(u, w) \end{pmatrix}$$

   there holds

$$\lambda_{\min}(\mathrm{sym}(\mathbf{A}(u_h, w_h))) \geq c_{nl} > 0 \qquad \text{for all } (u_h, v_h)^\top \in \mathcal{C}(\mathcal{T}_h)$$

3. There holds $\|\mathbf{A}(u_h, w_h))\| \leq c^{nl} < \infty$ for all $(u_h, v_h)^\top \in \mathcal{C}(\mathcal{T}_h)$
4. There holds $\frac{(c^{nl})^2}{4c_{nl}} < 1$.

Linearized Problem

The space-time DG formulation of the linearized nolinear problem reads as find $(\delta_h^{\mathrm{tm}}, \delta_h^e, \delta_h^v) \in [S_h^p(\mathcal{T}_h)]^3$ such that

$$
\begin{aligned}
&c^{\mathrm{DG}}((u_h, v_h); (\delta_h^{\mathrm{tm}}, \delta_h^e, \delta_h^v), (\phi_h, \psi_h, \zeta_h)) := \\
&b_T^{\mathrm{DG}}(\delta_h^{\mathrm{tm}}, \phi_h) + b_T^{\mathrm{DG}}(\delta_h^v, \zeta_h) + a_i^{\mathrm{DG}}(\delta_h^{\mathrm{tm}} + \delta_h^e, \phi_h + \psi_h) + a_e^{\mathrm{DG}}(\delta_h^e, \psi_h) \\
&\qquad\qquad + d_{nl}((u_h, w_h); (\delta_h^{\mathrm{tm}}, \delta_h^v), (\phi_h, \zeta_h)) = rhs
\end{aligned}
$$

where

$$
d_{nl}((u_h, w_h); (\delta_h^{\mathrm{tm}}, \delta_h^v), (\phi_h, \zeta_h)) := \sum_{l=1}^{N} \int_{\tau_l} \left(\mathbf{A}(u_h, w_h) \begin{pmatrix} \delta_h^{\mathrm{tm}} \\ \delta_h^v \end{pmatrix}, \begin{pmatrix} \phi_h \\ \zeta_h \end{pmatrix} \right) dq.
$$

Suitable Norms

For the numerical analysis we need to define appropriate norms :

$$
\begin{aligned}
|||(u,v,w)|||^2_{\mathrm{DG}} &:= |||u|||^2_{\mathrm{time}} + |||w|||^2_{\mathrm{time}} + |||u|||^2_{\mathrm{space},i} + |||v|||^2_{\mathrm{space},e} \\
&\quad + \|u\|^2_{\mathrm{L}^2(Q)} + \|w\|^2_{\mathrm{L}^2(Q)} \\
|||(u,v,w)|||^2_{\mathrm{DG},*} &+= |||u|||^2_{\mathrm{time},*} + |||w|||^2_{\mathrm{time},*} + |||u|||^2_{\mathrm{space},i;*} + |||v|||^2_{\mathrm{space},e,*} \\
&\quad + \|u\|^2_{\mathrm{L}^2(Q)} + \|w\|^2_{\mathrm{L}^2(Q)}
\end{aligned}
$$

Boundedness & Stability

Theorem (Boundedness)

Given $(u_h, v_h) \in \mathcal{C}(\mathcal{T}_h)$. *Under the assumptions stated earlier there holds:*

$$\left| c^{\mathrm{DG}}((u_h, v_h); (\delta_h^{tm}, \delta_h^{e}, \delta_h^{v}), (\phi_h, \psi_h, \zeta_h)) \right| \leq c_2^{B,nl}(u_h, v_h) \left\| (\delta_h^{tm}, \delta_h^{e}, \delta_h^{v}) \right\|_{\mathrm{DG},*} \left\| (\phi_h, \psi_h, \zeta_h) \right\|_{\mathrm{DG}}$$

Theorem (Stability)

Given $(u_h, v_h) \in \mathcal{C}(\mathcal{T}_h)$. *Under the assumptions stated earlier there holds:* $\exists c_s > 0$ *not dependening on h such that*

$$c_s \left\| (\delta_h^{tm}, \delta_h^{e}, \delta_h^{v}) \right\|_{\mathrm{DG}} \leq \sup_{\substack{(\phi_h, \psi_h, \zeta_h) \in [S_h^p(\mathcal{T}_h)]^3 \\ \|(\phi_h, \psi_h, \zeta_h)\|_{\mathrm{DG}} \neq 0}} \frac{c^{\mathrm{DG}}((u_h, v_h); (\delta_h^{tm}, \delta_h^{e}, \delta_h^{v}), (\phi_h, \psi_h, \zeta_h))}{\|(\phi_h, \psi_h, \zeta_h)\|_{\mathrm{DG}}}$$

Convergence Study

We do a convergence study to support our theoretical results. The setup is

- $Q_T := [0,1]^3$
- The anisotropic diffusion tensors were chosen as

$$\mathbf{M}_i := \begin{pmatrix} \frac{3}{4} & \frac{3}{20} \\ \frac{3}{20} & \frac{3}{4} \end{pmatrix} \qquad \mathbf{M}_e := \begin{pmatrix} \frac{5}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{4} \end{pmatrix}.$$

- On the boundary $\Sigma_R := \partial[0,1]^2 \times [0,1]$ we apply Robin boundary conditions with $\alpha_i = 1$ and $\alpha_e = 1$.
- Ionic model given by the FitzHugh-Nagumo model
- The given data s_i,s_e,$g_{i,R}$, $g_{e,R}$ and V^0_{tm}, v^0 are chosen such that the exact solutions of the Bidomain equations are given as

$$V_{\mathrm{tm}}(\boldsymbol{x},t) = x(1-x)y(1-y)t(1-t), \; u_e(\boldsymbol{x},t) = \sin(\pi x)\sin(\pi y)\sin(\pi t),$$
$$v(\boldsymbol{x},t) = \cos(\pi x)\cos(\pi y)\cos(\pi t)$$

- Standard Newton's method

Results

Table: Energy error $|||\boldsymbol{z} - \boldsymbol{z}_h|||_{DG}$ for $p = 1$ where $\boldsymbol{z} = (V_{\mathrm{tm}}, u_e, v)$

level	elements	dof	$\|\|\|\boldsymbol{z} - \boldsymbol{z}_h\|\|\|_{\mathrm{DG}}$	eoc
0	6	72	$1.64859E+0$	$-$
1	48	576	$1.15989E+0$	0.51
2	384	4608	$6.81509E-1$	0.77
3	3072	36864	$3.63150E-1$	0.91
4	24576	294912	$1.85401E-1$	0.97
5	196608	2359296	$9.33066E-2$	0.99
Theory:				1.00

Results

Table: Error $\left\| V_{\mathrm{tm}} - V_{\mathrm{tm}}^h \right\|_{\mathrm{time+space},i}$ for $p = 1$

level	elements	dof	$\left\| V_{\mathrm{tm}} - V_{\mathrm{tm}}^h \right\|_*$	eoc
0	6	24	$9.01159E-2$	–
1	48	192	$4.72390E-2$	0.93
2	384	1536	$2.24687E-2$	1.08
3	3072	12288	$1.01991E-2$	1.14
4	24576	98304	$4.74161E-3$	1.10
5	196608	786432	$2.27850E-3$	1.06
Theory:				1.00

Results

Table: Error $\left\| u_e - u_e^h \right\|_{\mathrm{space},e}$ for $p = 1$

level	elements	dof	$\left\| u_e - u_e^h \right\|_{\mathrm{space},e}$	eoc
0	6	24	$1.3692E+0$	$-$
1	48	192	$1.0823E+0$	0.34
2	384	1536	$6.6313E-1$	0.71
3	3072	12288	$3.5886E-1$	0.89
4	24576	98304	$1.8436E-1$	0.96
5	196608	786432	$9.3043E-2$	0.99
Theory:				1.00

Results

Table: Error $\left\| v - v^h \right\|_{\text{time}}$ for $p = 1$

level	elements	dof	$\left\| v - v^h \right\|_{\text{time}}$	eoc
0	6	24	$9.13903E-1$	–
1	48	192	$4.14408E-1$	1.14
2	384	1536	$1.55578E-1$	1.41
3	3072	12288	$5.47269E-2$	1.50
4	24576	98304	$1.89946E-2$	1.53
5	196608	786432	$6.628E-3$	1.52
Observed:				1.50

Outline

Variational Formulation

Find $\boldsymbol{U}$ s.t.:

$$\int_{\Omega_r} \mathbf{F}\mathbf{S}(\boldsymbol{U}) : \operatorname{Grad} \phi \, \mathrm{d}\boldsymbol{X} = \int_{\Omega_r} \boldsymbol{f} \cdot \phi \, \mathrm{d}\boldsymbol{X} \, .$$

Linearized problem: find $\delta \boldsymbol{U}$ s.t.:

$$\begin{aligned} a_E^k(\delta \boldsymbol{U}, \phi) := & \int_{\Omega_r} \operatorname{Grad}(\delta \boldsymbol{U}) \mathbf{S}_k : \operatorname{Grad}(\phi) \, \mathrm{d}\boldsymbol{X} \\ & + \int_{\Omega_r} \operatorname{sym}(\mathbf{F}_k^\top \operatorname{Grad} \delta \boldsymbol{U}) : \mathbb{C}_k : \operatorname{sym}(\mathbf{F}_k^\top \operatorname{Grad} \phi) \, \mathrm{d}\boldsymbol{X} \\ & = \int_{\Omega_r} \boldsymbol{f} \cdot \phi \, \mathrm{d}\boldsymbol{X} - \int_{\Omega_r} \mathbf{F}_k \mathbf{S}_k : \operatorname{Grad} \phi \, \mathrm{d}\boldsymbol{X} =: l_E^k(\phi) \end{aligned}$$

with the 4th-order **elasticity** tensor $\mathbb{C}_k := 4 \frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C} \partial \mathbf{C}} \Big|_{\boldsymbol{U} = \boldsymbol{U}_k}$.

DG Formulation

Taken ideas from [Ortner and Süli, 2007; Ten Eyck and Lew, 2006]. Similar procedure as for Poisson Equation (add consistency and stabilization terms). Find $\delta_h \boldsymbol{U}$ s. t.:

$$\begin{aligned} a_{E,DG}(\delta_h \boldsymbol{U}, \phi) &:= \sum_{\tau \in \mathcal{T}_h} a_E^k\big|_\tau(\delta_h \boldsymbol{U}, \phi_h) - \sum_{\Gamma_{ij} \in IE} b^{ij}_{\mathrm{cons}_1}(\delta_h \boldsymbol{U}, \phi_h) \\ &\quad - \sum_{\Gamma_{ij} \in IE} b^{ij}_{\mathrm{cons}_2}(\delta_h \boldsymbol{U}, \phi_h) + \sum_{\Gamma_{ij} \in IE} s_{ij}(\delta_h \boldsymbol{U}, \phi_h) \\ &= \sum_{\tau \in \mathcal{T}_h} l_E^k\big|_\tau(\phi_h) - \sum_{\Gamma_{ij} \in IE} b^{ij}_{\mathrm{cons}_3}(\phi_h) \end{aligned}$$

where

$$b^{ij}_{\mathrm{cons}_1}(\delta_h \boldsymbol{U}, \phi_h) := \int_{\Gamma_{ij}} \langle \mathbf{F}_k(\mathbb{C}_k : \mathrm{sym}(\mathbf{F}_k^\top \operatorname{Grad} \delta_h \boldsymbol{u}))\rangle_{ij} : \underline{[\phi]}_{ij,\boldsymbol{x}} \; \mathrm{d}s_{\boldsymbol{x}}$$

DG Formulation

Taken ideas from [Ortner and Süli, 2007; Ten Eyck and Lew, 2006]. Similar procedure as for Poisson Equation (add consistency and stabilization terms). Find $\delta_h \boldsymbol{U}$ s. t.:

$$\begin{aligned} a_{E,DG}(\delta_h \boldsymbol{U}, \phi) &:= \sum_{\tau \in \mathcal{T}_h} a_E^k\big|_\tau(\delta_h \boldsymbol{U}, \phi_h) - \sum_{\Gamma_{ij} \in IE} b^{ij}_{\text{cons}_1}(\delta_h \boldsymbol{U}, \phi_h) \\ &\quad - \sum_{\Gamma_{ij} \in IE} b^{ij}_{\text{cons}_2}(\delta_h \boldsymbol{U}, \phi_h) + \sum_{\Gamma_{ij} \in IE} s_{ij}(\delta_h \boldsymbol{U}, \phi_h) \\ &= \sum_{\tau \in \mathcal{T}_h} l_E^k\big|_\tau(\phi_h) - \sum_{\Gamma_{ij} \in IE} b^{ij}_{\text{cons}_3}(\phi_h) \end{aligned}$$

where

$$b^{ij}_{\text{cons}_2}(\delta_h \boldsymbol{U}, \phi_h) := \int_{\Gamma_{ij}} \langle \text{Grad}(\delta_h \boldsymbol{U}) \mathbf{S}_k \rangle_{ij} : \underline{[\phi]}_{ij,\boldsymbol{x}} \; \mathrm{d}s_{\boldsymbol{x}}$$

DG Formulation

Taken ideas from [Ortner and Süli, 2007; Ten Eyck and Lew, 2006]. Similar procedure as for Poisson Equation (add consistency and stabilization terms). Find $\delta_h \boldsymbol{U}$ s. t.:

$$\begin{aligned} a_{E,DG}(\delta_h \boldsymbol{U}, \phi) &:= \sum_{\tau \in \mathcal{T}_h} a_E^k\big|_\tau(\delta_h \boldsymbol{U}, \phi_h) - \sum_{\Gamma_{ij} \in IE} b^{ij}_{\text{cons}_1}(\delta_h \boldsymbol{U}, \phi_h) \\ &\quad - \sum_{\Gamma_{ij} \in IE} b^{ij}_{\text{cons}_2}(\delta_h \boldsymbol{U}, \phi_h) + \sum_{\Gamma_{ij} \in IE} s_{ij}(\delta_h \boldsymbol{U}, \phi_h) \\ &= \sum_{\tau \in \mathcal{T}_h} l_E^k\big|_\tau(\phi_h) - \sum_{\Gamma_{ij} \in IE} b^{ij}_{\text{cons}_3}(\phi_h) \end{aligned}$$

where

$$b^{ij}_{\text{cons}_3}(\phi_h) := \int_{\Gamma_{ij}} \langle \mathbf{F}_k \mathbf{S}_k \rangle_{ij} : \underline{[\phi_h]}_{ij,\boldsymbol{x}} \, \mathrm{d}s_{\boldsymbol{x}}$$

DG Formulation

Taken ideas from [Ortner and Süli, 2007; Ten Eyck and Lew, 2006]. Similar procedure as for Poisson Equation (add consistency and stabilization terms). Find $\delta_h \boldsymbol{U}$ s. t.:

$$\begin{aligned} a_{E,DG}(\delta_h \boldsymbol{U}, \phi) &:= \sum_{\tau \in \mathcal{T}_h} a_E^k\big|_\tau(\delta_h \boldsymbol{U}, \phi_h) - \sum_{\Gamma_{ij} \in IE} b^{ij}_{\text{cons}_1}(\delta_h \boldsymbol{U}, \phi_h) \\ &\quad - \sum_{\Gamma_{ij} \in IE} b^{ij}_{\text{cons}_2}(\delta_h \boldsymbol{U}, \phi_h) + \sum_{\Gamma_{ij} \in IE} s_{ij}(\delta_h \boldsymbol{U}, \phi_h) \\ &= \sum_{\tau \in \mathcal{T}_h} l_E^k\big|_\tau(\phi_h) - \sum_{\Gamma_{ij} \in IE} b^{ij}_{\text{cons}_3}(\phi_h) \end{aligned}$$

where

$$s_{ij}(\delta_h \boldsymbol{U}, \phi_h) := \frac{\sigma_{ij}}{h_{ij}} \int_{\Gamma_{ij}} \underline{[\delta_h \boldsymbol{U}]}_{ij,\boldsymbol{x}} : \underline{[\phi]}_{ij,\boldsymbol{x}} \, \mathrm{d}s_{\boldsymbol{x}} \qquad \sigma_{ij} > 0$$

Results for DG-Elasticity

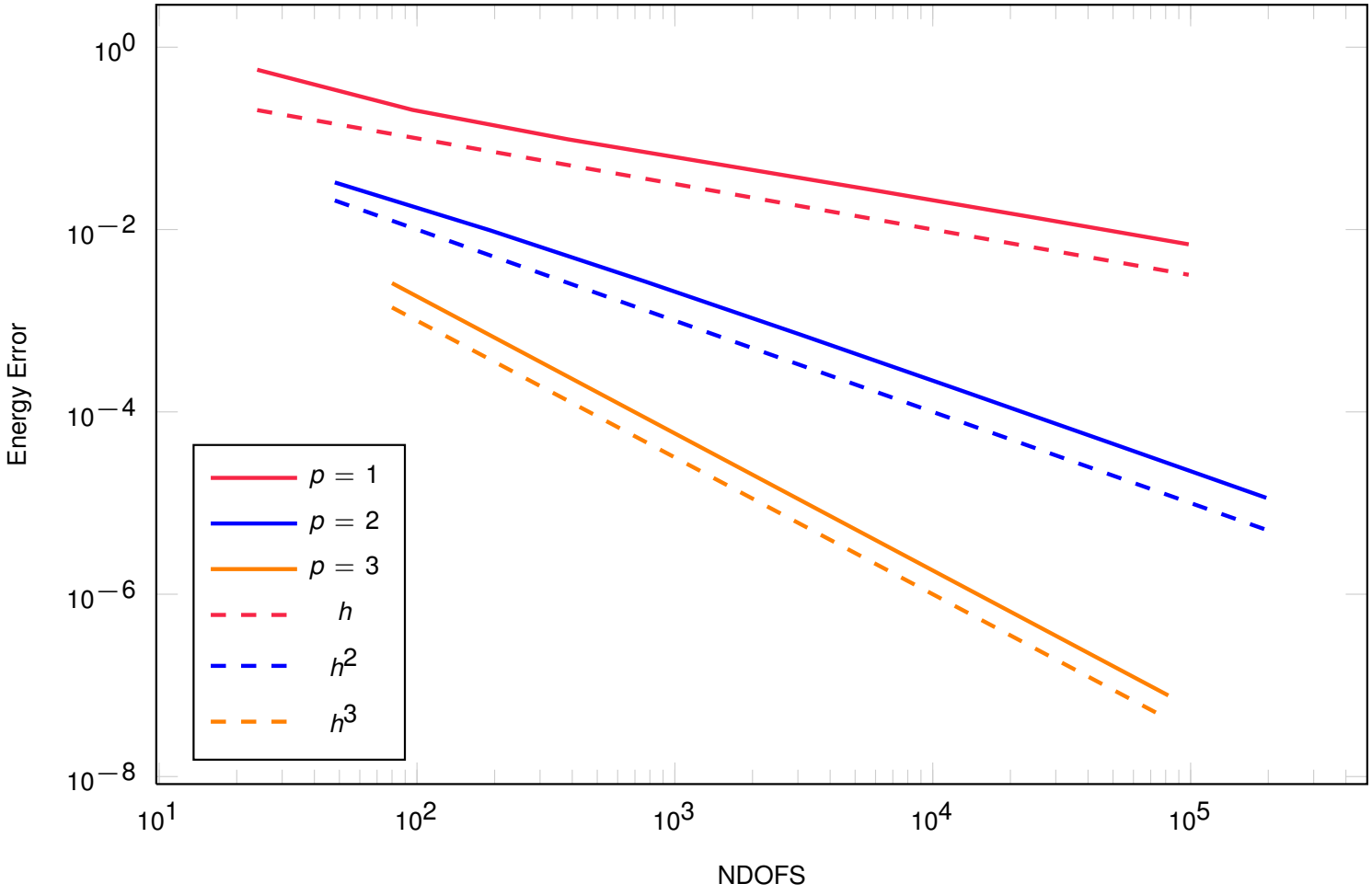
The presented DG formulation was analyzed in [Ortner and Süli, 2007]:

- Unique solvability of linearized problem holds under suitable growth conditions of the elastic material
- Numerical analysis can be found in the paper

Numeric example: Take unit cube $[0,1]^2$ plus compressible Neo-Hookean elasticity model. Choose Dirichlet data and right hand side such that exact solution

$$\boldsymbol{U}(\boldsymbol{X}) = 0.25(0, x(1-x)y^2)^\top .$$

Error Plots


Energy Error
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
NDOFS
$p = 1$
$p = 2$
$p = 3$
h
h^2
h^3

Outline

Blocksystems arising in Newton's Method

Independent of the discretization technique one arrives at the following block system

$$
\begin{array}{ccccc}
V_{\mathrm{tm}} & u_e & \boldsymbol{v} & \boldsymbol{S}_a & \boldsymbol{U}
\end{array}
$$

$$
\begin{pmatrix}
\mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{0} & \mathbf{F}_{14} \\
\mathbf{F}_{21} & \mathbf{F}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{F}_{31} & \mathbf{0} & \mathbf{F}_{33} & \mathbf{0} & \mathbf{0} \\
\mathbf{F}_{41} & \mathbf{0} & \mathbf{0} & \mathbf{F}_{44} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F}_{54} & \mathbf{F}_{55}
\end{pmatrix}
\begin{pmatrix}
\delta V_{\mathrm{tm}} \\ \delta u_e \\ \delta \boldsymbol{v} \\ \delta \boldsymbol{S}_a \\ \delta \boldsymbol{U}
\end{pmatrix}
=
\begin{pmatrix}
\mathrm{Res}_1 \\ \mathrm{Res}_2 \\ \mathrm{Res}_3 \\ \mathrm{Res}_4 \\ \mathrm{Res}_5
\end{pmatrix}
$$

Numerical Example — Setup
coupled

- Geometry is given by $Q_T = (-3,3)^2 \times (0,10)$
- Aliev-Panfilov Model

$$I_{\text{ion}}(V_{\text{tm}}, v) := c_2 V_{\text{tm}} v - c_1 V_{\text{tm}}(1 - V_{\text{tm}})(V_{\text{tm}} - c_3),$$

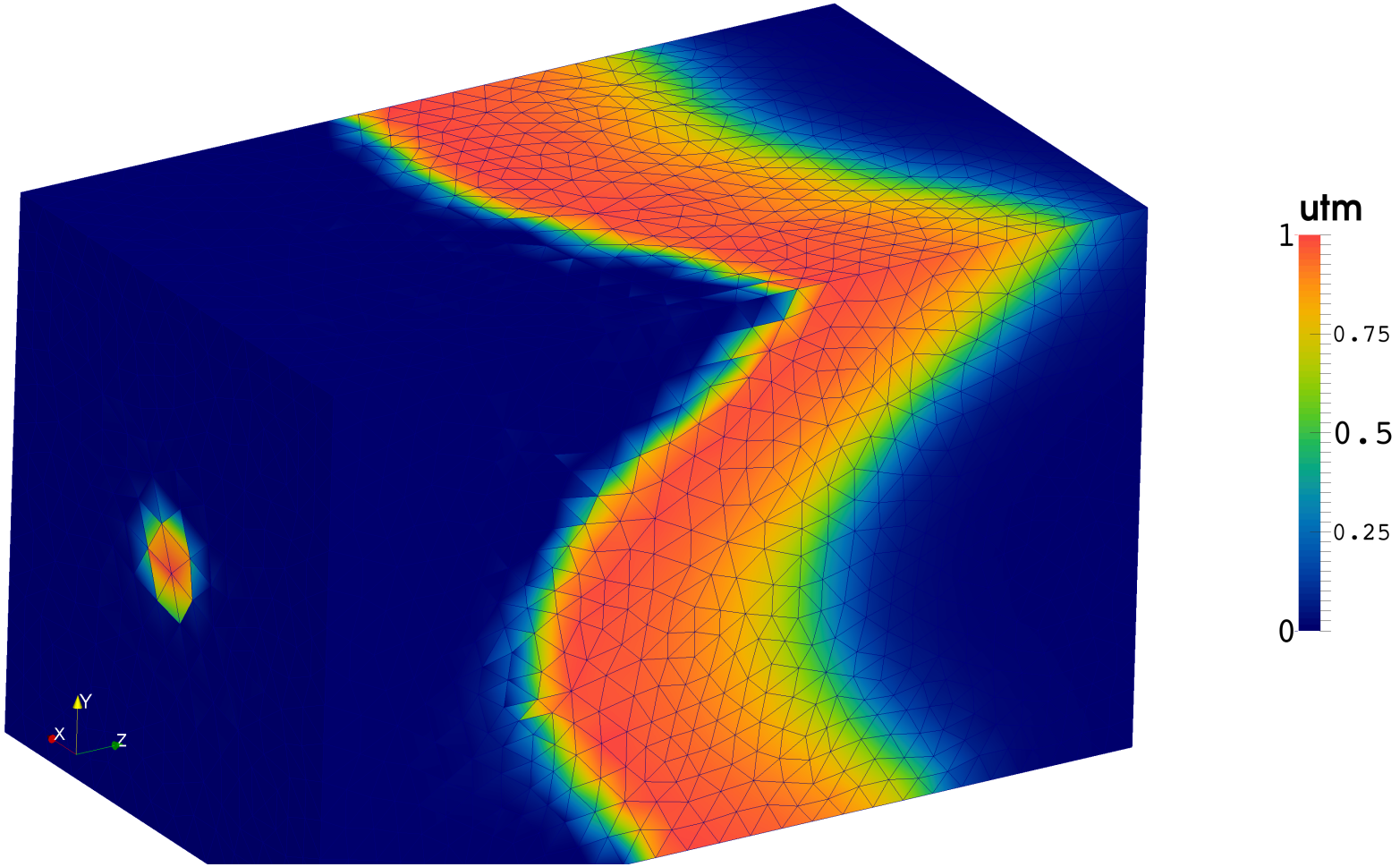
$$H(V_{\text{tm}}, v) := \left(\epsilon_0 + \frac{\mu_1 v}{\mu_2 + V_{\text{tm}}}\right)(c_1 V_{\text{tm}}(1 + c_3 - V_{\text{tm}}) - v).$$

- The anisotropies are given by

$$\boldsymbol{f}_0 = \left(3/\sqrt{10}, 1/\sqrt{10}\right)^\top, \boldsymbol{s}_0 = \left(-1/\sqrt{10}, 3/\sqrt{10}\right)^\top$$

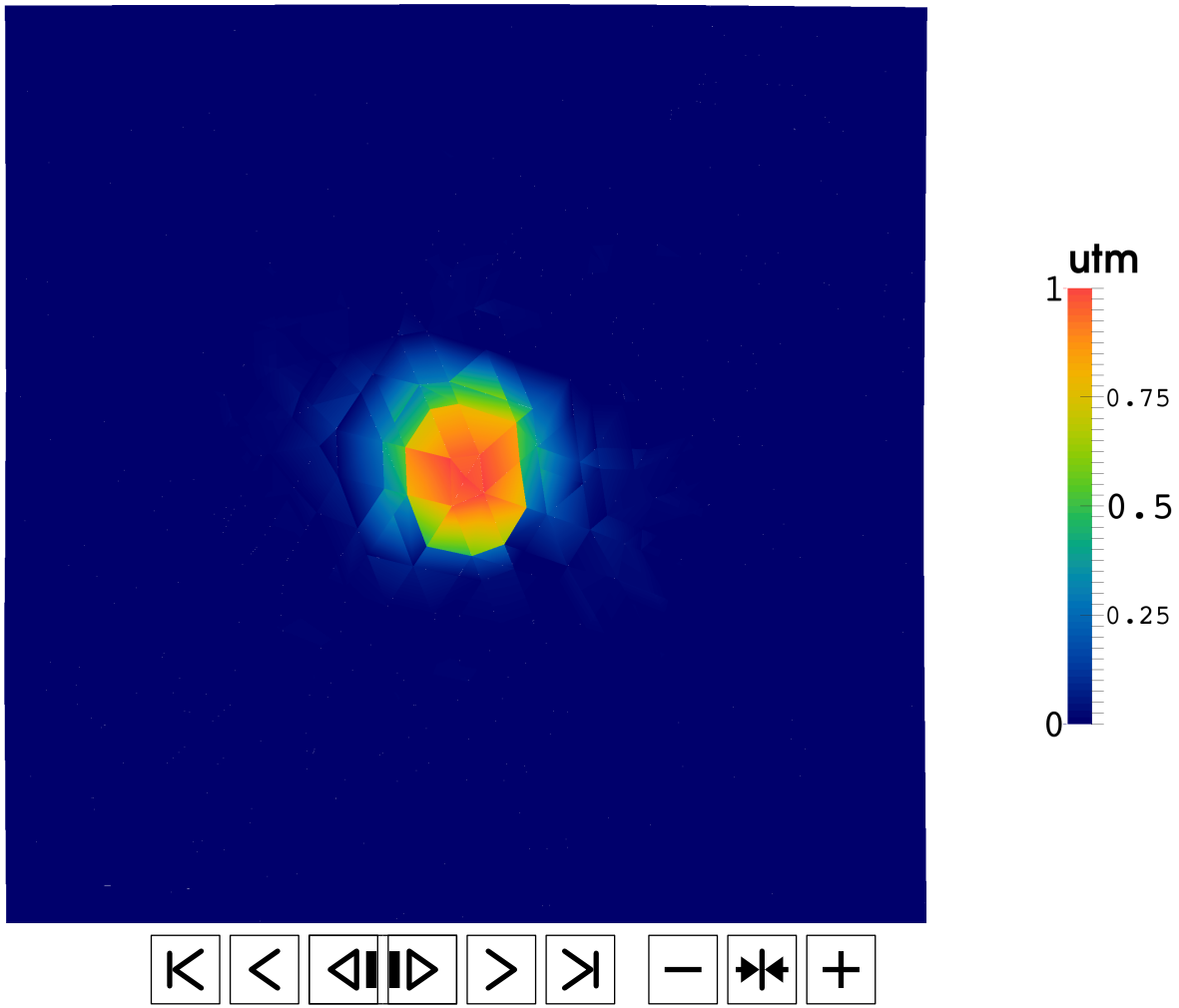
$$m_i^f = 0.2, m_i^s = 0.3, m_e^f = 1.1, m_e^s = 1.3$$

- Use Newton's Method with backtracking and loadstepping for solving nonlinear problem
- Homogeneous Neumann BC's on Σ_N, no sources
- On $x = -3$: homogeneous Dirichlet BC's for elastic behavior
- Initial Condition given by $V_{\text{tm}}^0(\boldsymbol{x}) := 0.9\chi_{B_{0.2}((0,0))}$



V_{tm} (dimensionless)

Numerical Example — Results



Summary & Outlook

We presented

- Full space-time DG formulation for coupled electro-mechanical behavior on unstructured grids
- Presented and investigated the individual components
- Numerical Experiments

Summary & Outlook

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What con be considered in the future:

- Numerical analysis for coupled system
- Find better initial guess for Newton's Method
- Space-Time adaptivity
- Parallelization (Domain decomposition methods)
- Solvers & Preconditioners
- 3D Space $+$ 1D Time for coupled nonlinear problem

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