On the preconditioning of the Navier–Stokes equations with applications in arterial blood flow

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Outline

Model problem

Variational formulation and discretization
- Stabilized FEM

Preconditioners
- Pressure mass
- Pressure convection–diffusion
- Least–square commutator
- Augmented Lagrangian based approach

Application to arterial blood flow

Conclusion and Outlook

Model problem

Simulation of arterial blood flow

- Velocity
- Pressure
- Vortex generation
- Hemodynamical indicators
  - Wall shear stress
  - Oscillatory shear index
- ...

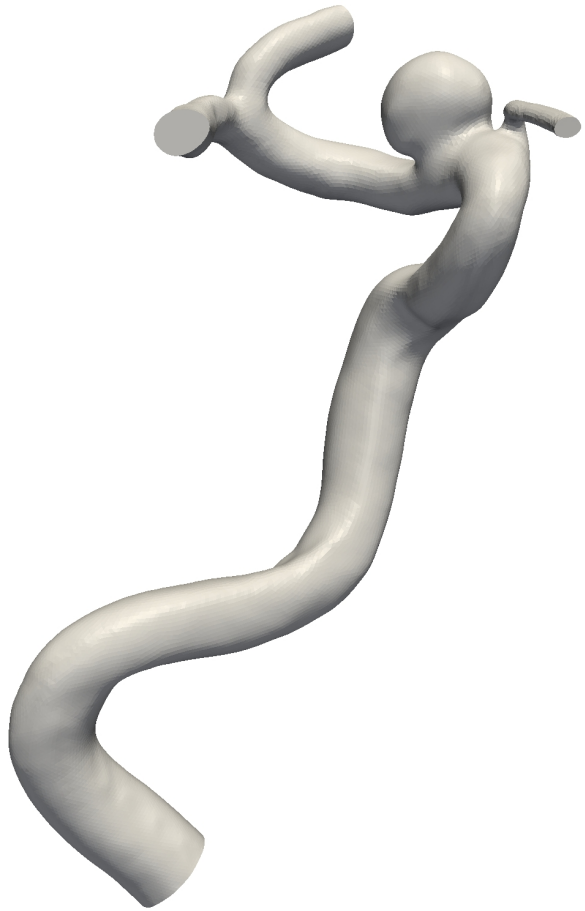

Figure: Cerebral aneurysm at internal carotis.

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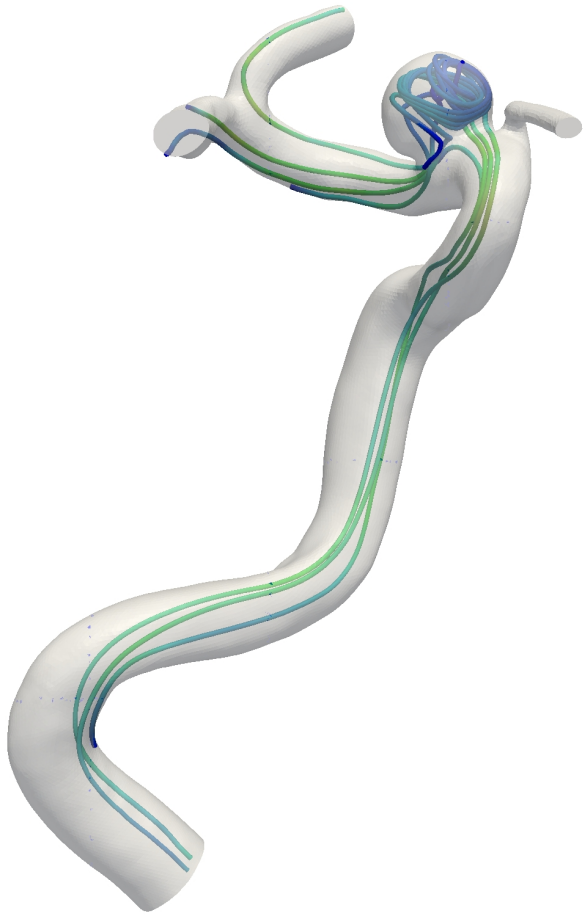

Figure: Cerebral aneurysm at internal carotis.

Model problem

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ bounded Lipschitz domain, with boundary $\Gamma = \partial\Omega$.

- $\overline{\Gamma} = \overline{\Gamma}_D \cup \overline{\Gamma}_N$

Navier–Stokes equations

$$
\begin{aligned}
-\nu\Delta\underline{u} + (\underline{u}\cdot\nabla)\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega,\\
\nabla\cdot\underline{u} &= 0 && \text{in } \Omega,\\
\underline{u} &= \underline{g} && \text{on } \Gamma_D,\\
\nu(\nabla\underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_N,
\end{aligned}
$$

viscosity constant $\nu > 0$.

Boundary conditions

- Γ_D for inflow and no–slip boundary
- Γ_N for outflow boundary

Variational formulation and discretization

$\rightarrow$ Apply integration by parts

Variational formulation

Find $(\underline{u}, p) \in [H^1(\Omega)]^n \times L_2(\Omega)$ with $\underline{u} = \underline{g}$ on Γ_D:

$$\begin{aligned} a(\underline{u}, \underline{v}) + a_1(\underline{u}, \underline{u}, \underline{v}) - b(\underline{v}, p) &= \langle \underline{f}, \underline{v} \rangle_\Omega, \\ b(\underline{u}, q) &= 0 \end{aligned}$$

for all $(\underline{v}, q) \in [H_0^1(\Omega, \Gamma_D)]^n \times L_2(\Omega)$.

Bilinear forms

$$a(\underline{u}, \underline{v}) = \int_\Omega \nabla \underline{u} : \nabla \underline{v} \; dx, \qquad a_1(\underline{w}, \underline{u}, \underline{v}) = \int_\Omega [(\underline{w} \cdot \nabla)\underline{u}] \cdot \underline{v} \; dx,$$

$$b(\underline{v}, q) = \int_\Omega q \; \nabla \cdot \underline{v} \; dx$$

Variational formulation and discretization

Finite dimensional subspaces $V_h \subset [H_0^1(\Omega, \Gamma_D)]^n$ and $Q_h \subset L_2(\Omega)$

- $n_u = \dim V_h$, $n_p = \dim Q_h$

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Stabilized FEM

Find $(\underline{u}_h, p_h) \in \widetilde{V}_h \times Q_h$ with $\underline{u}_h = I_h \underline{g}$ on Γ_D:

$$a(\underline{u}_h, \underline{v}_h) + a_1(\underline{u}_h, \underline{u}_h, \underline{v}_h) - b(\underline{v}_h, p_h) = \langle \underline{f}, \underline{v}_h \rangle_\Omega,$$
$$b(\underline{u}_h, q_h) + c(q_h, p_h) = 0$$

for all $(\underline{v}_h, q_h) \in V_h \times Q_h$.

Variational formulation and discretization

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$$b(\underline{u}_h, q_h) + c(q_h, p_h) = 0$$

for all $(\underline{v}_h, q_h) \in V_h \times Q_h$.

Dohrmann–Bochev stabilization

[Dohrmann, Bochev 2004]

$$c(q_h, p_h) := \frac{1}{\nu} \int_\Omega (p_h - \Pi_h p_h)(q_h - \Pi_h q_h)\, dx$$

with $L_2(\Omega)$–projection $\Pi_h : L_2(\Omega) \to Q_h^{l-1}$.

Remark

- *easy realization (esp. for $\mathcal{P}_1$–$\mathcal{P}_1$)*
- *no stabilization parameter*

Preconditioners

Linear system

$$\begin{pmatrix} A & -B^\top \\ B & C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix}$$

with $A \in \mathbb{R}^{n_u \times n_u}$ (depending on ν), $B \in \mathbb{R}^{n_p \times n_u}$ and $C \in \mathbb{R}^{n_p \times n_p}$.

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Remark

- *A is block diagonal in case of Picard iterations*
- *not the case for Newton iterations*

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Aim

Robust preconditioner wrt. discretization, mesh size and viscosity.

Preconditioners

Block diagonal preconditioner

[Elman, Wathen, Silvester 2005]

$$\mathcal{P}_1 = \begin{pmatrix} \mathcal{P}_A & \\ & \mathcal{P}_S \end{pmatrix} \quad \text{and} \quad \mathcal{P}_2 = \begin{pmatrix} \mathcal{P}_A & -B^\top \\ B & \mathcal{P}_S - B\mathcal{P}_A^{-1}B^\top \end{pmatrix}$$

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Block diagonal preconditioner

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Solving $\mathcal{P}_2(\underline{x}, \underline{y})^\top = (\underline{f}_1, \underline{f}_2)^\top$ reduces to

1. $\underline{w} = \mathcal{P}_A^{-1}\underline{f}_1$
2. $\underline{y} = \mathcal{P}_S^{-1}(\underline{f}_2 - B\underline{w})$
3. $\underline{x} = \mathcal{P}_A^{-1}(\underline{f}_1 + B^\top \underline{y})$

i.e. 2 solves for $\mathcal{P}_A$ and 1 solve $\mathcal{P}_S$.

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i.e. 2 solves for $\mathcal{P}_A$ and 1 solve $\mathcal{P}_S$.

Question

How to choose $\mathcal{P}_S$?

Preconditioners

1. Pressure mass preconditioner (basic approach)

For Stokes equations the Schur complement is spectrally equivalent to pressure mass matrix.

$$\mathcal{P}_S = \frac{1}{\nu} M_p$$

M_p . . . pressure mass matrix

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- *highly competitive for* $Re \leq 10$ $(\nu \geq 0.1)$
- *does not take the convection into account*
- *higher iteration nr. the higher Re*

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Need

Better reflection of balance of convection and diffusion.

Preconditioners

2. Pressure convection–diffusion preconditioner

Motivated by the discrete Oseen operator on Q_h.

$$\mathcal{P}_S = A_p F_p^{-1} M_p$$

A_p . . . pressure stiffness matrix,

$$F_p[i,j] = \nu \int_\Omega \nabla\psi_j \nabla\psi_i \; dx + \int_\Omega (\underline{u}^k \cdot \nabla\psi_j)\psi_i \; dx$$

$\underline{u}^k$ . . . discrete velocity from previous Newton or Picard iteration.

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Realization

$$\mathcal{P}_S^{-1} = M_p^{-1} F_p A_p^{-1}$$

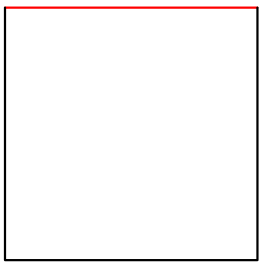
i.e. 1 stiffness matrix solve and 1 mass matrix solve.

Preconditioners

2. Pressure convection–diffusion preconditioner

Numerical results (driven–cavity problem)

$\Omega = (-1,1)^2$, $\Gamma = \Gamma_D$, $\underline{g} = (1 - x_1^4, 0)^\top$ for $x_2 = 1$
else $\underline{g} = \underline{0}$



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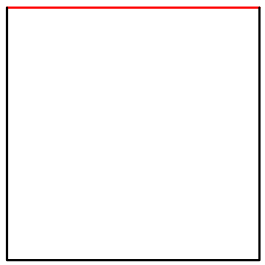

	$L \diagdown \nu$	0.1	0.01	0.001
Picard	4	14	24	75
	5	15	25	76
	6	15	26	65
	7	15	26	55
Newton	4	16	30	83
	5	16	32	139
	6	17	34	141
	7	17	34	128

Table: Iteration numbers of GMRES, Taylor–Hood elements.

Preconditioners

2. Pressure convection–diffusion preconditioner

Remark

Preconditioner also applicable to stabilized case

$$\begin{pmatrix} A & -B^\top \\ B & C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix}$$

[Elman, Wathen, Silvester 2005]

Preconditioners

3. Least–square commutator preconditioner

Assume B has full rank, then

$$B^\top (BA^{-1}B^\top)^{-1}B \qquad = \qquad A$$

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$$W^{-1}B^{\top}(BA^{-1}B^{\top})^{-1}BW^{-1} \quad = \quad W^{-1}AW^{-1}$$

with $W^{-1} \in \mathbb{R}^{n_u \times n_u}$

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$$\begin{aligned} \Rightarrow \quad S^{-1} &= (BA^{-1}B^{\top})^{-1} \\ &= (BW^{-1}B^{\top})^{-1}BW^{-1}AW^{-1}B^{\top}(BW^{-1}B^{\top})^{-1} \end{aligned}$$

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Preconditioner [Elman 1999] [Elman, Silvester, Wathen 2005]

$$\mathcal{P}_S^{-1} = (BW^{-1}B^\top)^{-1}BW^{-1}AW^{-1}B^\top(BW^{-1}B^\top)^{-1}$$

with $W = \mathrm{diag}M_u$, M_u . . . velocity mass matrix

Preconditioners

3. Least–square commutator preconditioner

Numerical results (driven–cavity problem)

	$L \diagdown \nu$	0.1	0.01	0.001
Picard	4	8	13	55
	5	11	16	62
	6	14	21	55
	7	18	27	45
Newton	4	8	18	78
	5	11	21	106
	6	15	27	107
	7	19	34	95

Table: Iteration numbers of GMRES, Taylor–Hood elements.

Preconditioners

4. Augmented Lagrangian based preconditioner

Based on augmented Lagrangian formulation [Fortin, Glowinski 1982]

Linear system

$$\begin{pmatrix} A & -B^\top \\ B & \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix}$$

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W is M_p or its diagonal and $\gamma > 0$.

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Preconditioner [Benzi, Olshanskii 2006]

$$\mathcal{P}_S^{-1} = -\nu(\mathrm{diag} M_p)^{-1} - \gamma W^{-1}$$

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Preconditioner [Benzi, Olshanskii 2006]

$$\mathcal{P}_S^{-1} = -\nu(\mathrm{diag} M_p)^{-1} - \gamma W^{-1}$$

- Robust wrt. h and ν
- Choice of γ by Fourier analysis [Benzi, Wang 2011]

Preconditioners

4. Augmented Lagrangian based preconditioner

Numerical results (driven–cavity problem)

	$L \diagdown \nu$	0.1	0.01	0.001
Picard	4	11	10	8
	5	13	13	13
	6	17	17	17
	7	22	22	22

Table: Iteration numbers of GMRES, Taylor–Hood elements.

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Table: Iteration numbers of GMRES, Taylor–Hood elements.

Remark

- *Modified AL based precond. with more iter. nr. but less comp. time.*
- *Generalization of AL based preconditioner for stabilized FEM*

$$\begin{pmatrix} A & -B^\top \\ B & C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix}.$$

Application to arterial blood flow

Instationary Navier–Stokes equations

$$
\begin{aligned}
\partial_t \underline{u} - 2\nu \operatorname{div} D(\underline{u}) + (\underline{u} \cdot \nabla)\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega \times (0,T),\\
\operatorname{div} \underline{u} &= 0 && \text{in } \Omega \times (0,T),\\
\underline{u} &= \underline{g} && \text{on } \Gamma_D \times (0,T),\\
\nu D(\underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_N \times (0,T),\\
\underline{u} &= \underline{u}_0 && \text{on } \Omega \times \{0\},
\end{aligned}
$$

viscosity constant $\nu > 0$,
symmetric part of velocity gradient $D(\underline{u}) = 1/2(\nabla \underline{u} + (\nabla \underline{u})^\top)$.

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Discretization

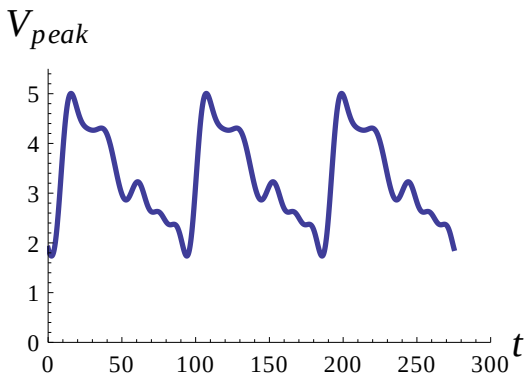
- θ–scheme in time
- $\mathcal{P}_1$–$\mathcal{P}_1$ elements with Dohrmann–Bochev stabilization

Application to arterial blood flow

Solve instationary N–S equations

- Simulation time $[0, 5]$
- $\underline{f} = \underline{0}$, $\nu = 0.04$ ($Re \approx 100$)
- Diameter artery 5
- Inflow (period 0.913)

$$\underline{g} = V_{\text{peak}}(t)(0, 0, 1 - 1/r^2(x_1^2 + x_2^2))^\top$$



$$\frac{a_0}{2} + \sum_{k=1}^{7} a_k \cos(k\omega t) + b_k \sin(k\omega t)$$

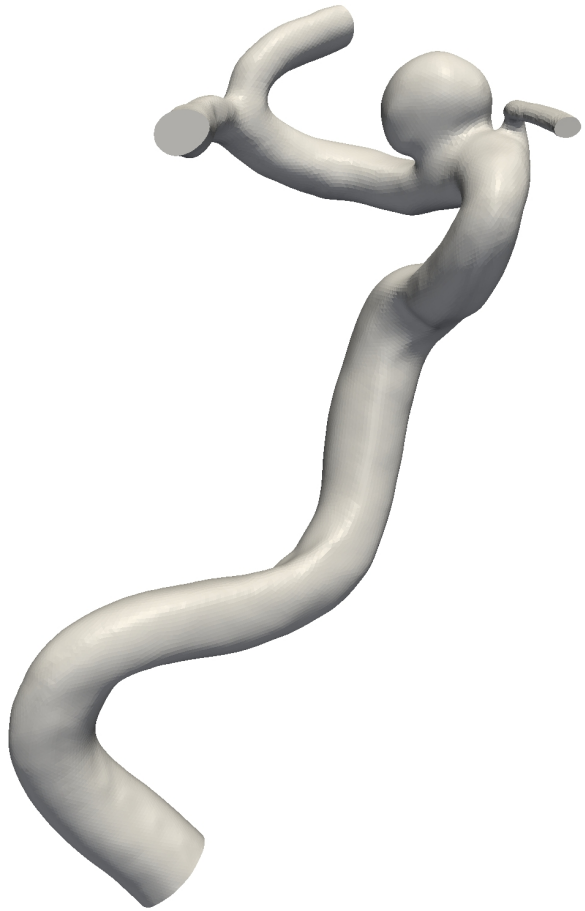


Figure: Cerebral aneurysm at internal carotis.

Application to arterial blood flow

Velocity

velocity

Application to arterial blood flow

Velocity streamlines

velocity streamlines

Conclusion

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Outlook

- Preconditioning of instationary problem
- Hemodynamical indicators $\rightarrow$ Talk by P. Pustejovska
- Non–Newtonian fluids
- Fluid–Structure–Interaction
- Optimal control of hemodynamical indicators, FSI

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