Interface operators, domain decomposition and applications

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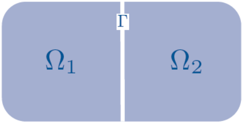
INTRODUCTION AND MOTIVATION

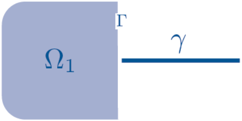
DOMAIN DECOMPOSITION WITHOUT OVERLAP

Original problem:



$Lu_1 = f_1$ in Ω_1
$Lu_2 = f_2$ in Ω_2
$+$ coupling conditions on Γ

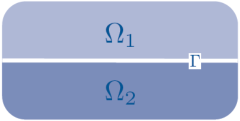
Geometrical multiscale:



$Lu_1 = f_1$ in Ω_1
$L_\gamma u_\gamma = f_\gamma$ on γ
$+$ coupling conditions on Γ

DOMAIN DECOMPOSITION WITHOUT OVERLAP

Multiphysics:



$L_1 u_1 = f_1$ in Ω_1
$L_2 u_2 = f_2$ in Ω_2
$+$ coupling conditions on Γ

AIM

- Write the problem as an equation on a suitable interface
- Set up effective solution techniques

APPLICATIONS

- Multiscale modeling of the circulatory system.
- Blood filtration through tissues (and also hemofiltration devices, blood oxygenators, ...).
- Fluid structure interaction $\rightarrow$ talk of Ulrich Langer on Monday

DD FOR GEOMETRICAL MULTISCALE MODELS

Joint work with P.J. Blanco (LNCC, Brazil)

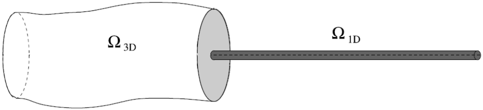
and A. Quarteroni (EPFL and MOX-Milano)

MOTIVATION

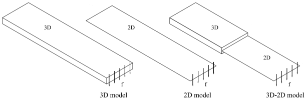
Geometrical multiscale modeling

Examples

- 3D-1D-0D models for the cardiovascular system



- 3D-2D-1D couplings in structural mechanics



- ...

[Blanco, Discacciati, Quarteroni (2011)]

BASIC ASSUMPTIONS

- A given physical system is split into two parts: one of them can be represented using a dimensionally reduced model $\Rightarrow$ we have two kinds of models:

  a complex dimensional or $\mathbb{C}$D-model
  and
  a simple dimensional or $\mathbb{S}$D-model.

- We consider $\mathbb{C} = 1, 2, 3$ and $\mathbb{S} = 0, 1, 2$ with $\mathbb{C} > \mathbb{S}$:

  3D-2D, 3D-1D, 2D-1D, 2D-0D, ...

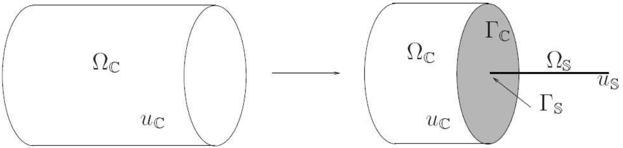
- For the moment, we consider a system of two dimensionally-heterogeneous models with one coupling interface.

VARIATIONAL FORM

Consider the $\mathbb{C}$-problem in $\Omega_{\mathbb{C}}$:

$$\text{find } u_{\mathbb{C}} \in U_{\mathbb{C}} : \quad a_{\mathbb{C}}(u_{\mathbb{C}}, \hat{u}_{\mathbb{C}}) = f_{\mathbb{C}}(\hat{u}_{\mathbb{C}}) \qquad \forall \hat{u}_{\mathbb{C}} \in U_{\mathbb{C}}$$

We reduce part of $\Omega_{\mathbb{C}}$ to $\Omega_{\mathbb{S}}$:

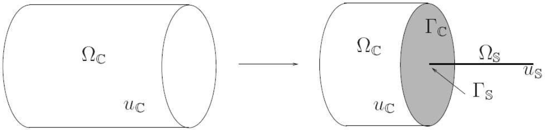


where we have the $\mathbb{S}$-problem:

$$\text{find } u_{\mathbb{S}} \in U_{\mathbb{S}} : \quad a_{\mathbb{S}}(u_{\mathbb{S}}, \hat{u}_{\mathbb{S}}) = f_{\mathbb{S}}(\hat{u}_{\mathbb{S}}) \qquad \forall \hat{u}_{\mathbb{S}} \in U_{\mathbb{S}}$$

Both bilinear forms $a_{\mathbb{C}}(\cdot,\cdot)$ and $a_{\mathbb{S}}(\cdot,\cdot)$ are continuous and coercive.

VARIATIONAL FORM: COUPLING ISSUES



We have two interfaces:

- $\Gamma_{\mathbb{C}}$ (($\mathbb{C}-1$)-dimensional) $\rightarrow$ trace space $\Lambda_{\mathbb{C}}$ and $\Lambda'_{\mathbb{C}}$
- $\Gamma_{\mathbb{S}}$ (($\mathbb{S}-1$)-dimensional) $\rightarrow$ trace space $\Lambda_{\mathbb{S}}$ and $\Lambda'_{\mathbb{S}}$

We consider the linear and continuous

- **restriction operator**

$$\boxed{\mathcal{R}_{\mathbb{S}} : \Lambda_{\mathbb{C}} \rightarrow \Lambda_{\mathbb{S}}, \quad u_{\mathbb{C}|\Gamma_{\mathbb{C}}} \mapsto \mathcal{R}_{\mathbb{S}} u_{\mathbb{C}|\Gamma_{\mathbb{C}}}}$$

  surjective, but not necessarily injective;

- **extension operator**

$$\boxed{\mathcal{E}_{\mathbb{C}} : \Lambda_{\mathbb{S}} \rightarrow \Lambda_{\mathbb{C}}, \quad u_{\mathbb{S}|\Gamma_{\mathbb{S}}} \mapsto \mathcal{E}_{\mathbb{C}} u_{\mathbb{S}|\Gamma_{\mathbb{S}}}}$$

  injective, but not necessarily surjective.

THE AUGMENTED VARIATIONAL FORMULATION

- Let $\alpha \in \{0, 1\}$ be a parameter a priori defined.

- We can write the **augmented variational formulation**:

find $(u_{\mathbb{C}}, u_{\mathbb{S}}, \lambda_{\mathbb{C}}, \lambda_{\mathbb{S}}) \in U_{\mathbb{C}} \times U_{\mathbb{S}} \times \Lambda'_{\mathbb{C}} \times \Lambda'_{\mathbb{S}}$ such that

$$a_{\mathbb{C}}(u_{\mathbb{C}}, \hat{u}_{\mathbb{C}}) + a_{\mathbb{S}}(u_{\mathbb{S}}, \hat{u}_{\mathbb{S}})$$

$$+(1-\alpha)\langle \lambda_{\mathbb{C}}, \hat{u}_{\mathbb{C}} - \mathcal{E}_{\mathbb{C}} \hat{u}_{\mathbb{S}} \rangle_{\mathbb{C}} + (1-\alpha)\langle \hat{\lambda}_{\mathbb{C}}, u_{\mathbb{C}} - \mathcal{E}_{\mathbb{C}} u_{\mathbb{S}} \rangle_{\mathbb{C}}$$

$$+\alpha\langle \lambda_{\mathbb{S}}, \hat{u}_{\mathbb{S}} - \mathcal{R}_{\mathbb{S}} \hat{u}_{\mathbb{C}} \rangle_{\mathbb{S}} + \alpha\langle \hat{\lambda}_{\mathbb{S}}, u_{\mathbb{S}} - \mathcal{R}_{\mathbb{S}} u_{\mathbb{C}} \rangle_{\mathbb{S}}$$

$$= f_{\mathbb{C}}(\hat{u}_{\mathbb{C}}) + f_{\mathbb{S}}(\hat{u}_{\mathbb{S}})$$

for all $(\hat{u}_{\mathbb{C}}, \hat{u}_{\mathbb{S}}, \hat{\lambda}_{\mathbb{C}}, \hat{\lambda}_{\mathbb{S}}) \in U_{\mathbb{C}} \times U_{\mathbb{S}} \times \Lambda'_{\mathbb{C}} \times \Lambda'_{\mathbb{S}}$.

We have introduced the duality pairings:

$$\langle \cdot, \cdot \rangle_{\mathbb{C}} : \Lambda'_{\mathbb{C}} \times \Lambda_{\mathbb{C}} \to \mathbb{R} \quad \text{and} \quad \langle \cdot, \cdot \rangle_{\mathbb{S}} : \Lambda'_{\mathbb{S}} \times \Lambda_{\mathbb{S}} \to \mathbb{R}.$$

VARIATIONAL FORM: ROLE OF α

- α defines how the model represents the physical phenomenon we want to address:
  - $\alpha = 1$: the field u is continuous via the pairing $\langle \cdot, \cdot \rangle_{\mathbb{S}}$ and it can be shown that the dual variable is continuous in $\Lambda'_{\mathbb{C}}$

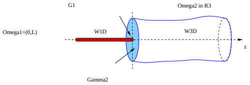
$$+\alpha\langle \lambda_{\mathbb{S}}, \hat{u}_{\mathbb{S}} - \mathcal{R}_{\mathbb{S}}\hat{u}_{\mathbb{C}}\rangle_{\mathbb{S}} + \alpha\langle \hat{\lambda}_{\mathbb{S}}, u_{\mathbb{S}} - \mathcal{R}_{\mathbb{S}} u_{\mathbb{C}}\rangle_{\mathbb{S}}$$

  - $\alpha = 0$: the field u is continuous via $\langle \cdot, \cdot \rangle_{\mathbb{C}}$, while the flux is continuous in $\Lambda'_{\mathbb{S}}$

$$+(1-\alpha)\langle \lambda_{\mathbb{C}}, \hat{u}_{\mathbb{C}} - \mathcal{E}_{\mathbb{C}}\hat{u}_{\mathbb{S}}\rangle_{\mathbb{C}} + (1-\alpha)\langle \hat{\lambda}_{\mathbb{C}}, u_{\mathbb{C}} - \mathcal{E}_{\mathbb{C}} u_{\mathbb{S}}\rangle_{\mathbb{C}}$$

- α is chosen a priori depending upon the problem.
- If the heterogeneous model is a good approximation of the original homogeneous problem, the solution does not depend too much on α.
- We set $\alpha = 1$.

EXAMPLE: 3D-1D LAPLACE COUPLING



- Bilinear forms

$$a_{\mathbb{C}}(u_{\mathbb{C}}, \hat{u}_{\mathbb{C}}) = \int_{\Omega_{\mathbb{C}}} k \nabla u_{\mathbb{C}} \cdot \nabla \hat{u}_{\mathbb{C}} \qquad a_{\mathbb{S}}(u_{\mathbb{S}}, \hat{u}_{\mathbb{S}}) = \int_{\Omega_{\mathbb{S}}} Ak \frac{du_{\mathbb{S}}}{d\xi} \frac{d\hat{u}_{\mathbb{S}}}{d\xi}$$

- Spaces

$$U_{\mathbb{C}} = H^1(\Omega_{\mathbb{C}}) \qquad \Lambda_{\mathbb{C}} = H^{1/2}(\Gamma) \qquad \Lambda'_{\mathbb{C}} = H^{-1/2}(\Gamma)$$

$$U_{\mathbb{S}} = H^1(\Omega_{\mathbb{S}}) \qquad \Lambda_{\mathbb{S}} = \mathbb{R} \qquad \Lambda'_{\mathbb{S}} = \mathbb{R}$$

- Restriction operator $\mathcal{R}_{\mathbb{S}}$

$$\mathcal{R}_{\mathbb{S}} : H^{1/2}(\Gamma_{\mathbb{C}}) \to \mathbb{R}, \quad u_{\mathbb{C}|\Gamma_{\mathbb{C}}} \mapsto u_{\mathbb{C},\mathbb{S}|\Gamma_{\mathbb{S}}} = \frac{1}{|\Gamma_{\mathbb{C}}|} \int_{\Gamma_{\mathbb{C}}} u_{\mathbb{C}}$$

- Extension operator $\mathcal{E}_{\mathbb{C}}$

$$\mathcal{E}_{\mathbb{C}} : \mathbb{R} :\to H^{1/2}(\Gamma_{\mathbb{C}}), \quad u_{\mathbb{S}|\Gamma_{\mathbb{S}}} \mapsto u_{\mathbb{S},\mathbb{C}|\Gamma_{\mathbb{C}}} = u_{\mathbb{S}|\Gamma_{\mathbb{S}}}$$

- Duality pairings

$$\langle \lambda_{\mathbb{C}}, \hat{u}_{\mathbb{C}} - \mathcal{E}_{\mathbb{C}} \hat{u}_{\mathbb{S}} \rangle_{\mathbb{C}} = \int_{\Gamma_{\mathbb{C}}} \lambda_{\mathbb{C}} (\hat{u}_{\mathbb{C}} - \hat{u}_{\mathbb{S},\mathbb{C}}), \qquad \lambda_{\mathbb{C}} \in H^{-1/2}(\Gamma_{\mathbb{C}}) \quad (\alpha = 0)$$

$$\langle \lambda_{\mathbb{S}}, \hat{u}_{\mathbb{S}} - \mathcal{R}_{\mathbb{S}} \hat{u}_{\mathbb{C}} \rangle_{\mathbb{S}} = |\Gamma_{\mathbb{C}}| \lambda_{\mathbb{S}} (\hat{u}_{\mathbb{S}} - \hat{u}_{\mathbb{C},\mathbb{S}})_{|\Gamma_{\mathbb{S}}}, \qquad \lambda_{\mathbb{S}} \in \mathbb{R} \quad (\alpha = 1)$$

- Coupling conditions

if $\alpha = 1$

$$u_{\mathbb{S}} = \frac{1}{|\Gamma_{\mathbb{C}}|} \int_{\Gamma_{\mathbb{C}}} u_{\mathbb{C}} \text{ on } \Gamma_{\mathbb{S}}$$

$$k \frac{du_{\mathbb{S}}}{d\xi} = k \nabla u_{\mathbb{C}} \cdot \mathbf{n} \text{ on } \Gamma_{\mathbb{C}}$$

if $\alpha = 0$

$$u_{\mathbb{S}} = u_{\mathbb{C}} \text{ on } \Gamma_{\mathbb{C}}$$

$$Ak \frac{du_{\mathbb{S}}}{d\xi} = \int_{\Gamma_{\mathbb{C}}} k \nabla u_{\mathbb{C}} \cdot \mathbf{n} \text{ on } \Gamma_{\mathbb{S}}$$

- For a 3D-2D coupling we would have $\Lambda_{\mathbb{S}} = H^{1/2}(\Gamma_{\mathbb{S}})$ instead of being $\Lambda_{\mathbb{S}}$ the finite-dimensional space $\mathbb{R}$ of the 1D case.

EXTENSION OPERATORS FOR INTERFACE DATA

- We want to write interface problems associated to the augmented variational formulation.
- We need to characterize Dirichlet-to-Neumann and Neumann-to-Dirichlet maps for the $\mathbb{C}$D and the $\mathbb{S}$D models.
- According to our choice $\alpha = 1$, we impose the interface conditions only on $\Gamma_{\mathbb{S}} \Rightarrow$ we work in the trace space $\Lambda_{\mathbb{S}}$.

EXTENSION OPERATORS FOR THE $\mathbb{S}$D MODEL

For the $\mathbb{S}$D model we consider the following operators:

1. Extension of Dirichlet data on $\Gamma_{\mathbb{S}}$, $\mathcal{D}_{\mathbb{S}} : \Lambda_{\mathbb{S}} \to \hat{U}_{\mathbb{S}}$

   Given $\mu_{\mathbb{S}} \in \Lambda_{\mathbb{S}}$, find $\mathcal{D}_{\mathbb{S}}\mu_{\mathbb{S}} \in \hat{U}_{\mathbb{S}}$ such that

$$\mathcal{D}_{\mathbb{S}}\mu_{\mathbb{S}} = \mu_{\mathbb{S}} \text{ on } \Gamma_{\mathbb{S}} \quad \text{and} \quad a_{\mathbb{S}}(\mathcal{D}_{\mathbb{S}}\mu_{\mathbb{S}}, \hat{u}^I_{\mathbb{S}}) = 0 \qquad \forall \hat{u}^I_{\mathbb{S}} \in U^0_{\mathbb{S}}$$

2. Extension of Neumann data on $\Gamma_{\mathbb{S}}$, $\mathcal{N}_{\mathbb{S}} : \Lambda_{\mathbb{S}} \to \hat{U}_{\mathbb{S}}$

   Given $\lambda_{\mathbb{S}} \in \Lambda'_{\mathbb{S}}$, find $\mathcal{N}_{\mathbb{S}}\lambda_{\mathbb{S}} \in \hat{U}_{\mathbb{S}}$ such that

$$a_{\mathbb{S}}(\mathcal{N}_{\mathbb{S}}\lambda_{\mathbb{S}}, \hat{u}^J_{\mathbb{S}}) = -\langle \lambda_{\mathbb{S}}, \hat{u}^J_{\mathbb{S}} \rangle_{\mathbb{S}} \qquad \forall \hat{u}^J_{\mathbb{S}} \in U_{\mathbb{S}}.$$

Existence and uniqueness are guaranteed by the Lax-Milgram Lemma.

EXTENSION OPERATORS FOR THE $\mathbb{C}$D MODEL

Let us define the adjoint operator $\mathcal{R}_{\mathbb{S}}^{*}$ as:

$$\langle \lambda_{\mathbb{S}}, \mathcal{R}_{\mathbb{S}} u_{\mathbb{C}} \rangle_{\mathbb{S}} = \langle \mathcal{R}_{\mathbb{S}}^{*} \lambda_{\mathbb{S}}, u_{\mathbb{C}} \rangle_{\mathbb{C}} \qquad \forall (u_{\mathbb{C}}, \lambda_{\mathbb{S}}) \in \Lambda_{\mathbb{C}} \times \Lambda'_{\mathbb{S}}$$

1. Extension of Dirichlet data on $\Gamma_{\mathbb{S}}$

   Given $\mu_{\mathbb{S}} \in \Lambda_{\mathbb{S}}$, we have to impose that the mean value of a suitable function in $\Omega_{\mathbb{C}}$ is equal to $\mu_{\mathbb{S}}$ on $\Gamma_{\mathbb{S}}$:

$$\begin{aligned} a_{\mathbb{C}}(\mathcal{D}_{\mathbb{C}} \mu_{\mathbb{S}}, \hat{u}_{\mathbb{C}}) + \langle \mathcal{R}_{\mathbb{S}}^{*} \lambda_{\mathbb{S}}, \hat{u}_{\mathbb{C}} \rangle_{\mathbb{C}} &= 0 \qquad && \forall \hat{u}_{\mathbb{C}} \in U_{\mathbb{C}} \\ \langle \mathcal{R}_{\mathbb{S}}^{*} \hat{\lambda}_{\mathbb{S}}, \mathcal{D}_{\mathbb{C}} \mu_{\mathbb{S}} \rangle_{\mathbb{C}} &= \langle \hat{\lambda}_{\mathbb{S}}, \mu_{\mathbb{S}} \rangle_{\mathbb{S}} \qquad && \forall \hat{\lambda}_{\mathbb{S}} \in \Lambda'_{\mathbb{S}} \end{aligned}$$

2. Extension of Neumann data on $\Gamma_{\mathbb{S}}$

   Given $\lambda_{\mathbb{S}} \in \Lambda'_{\mathbb{S}}$, find $\mathcal{N}_{\mathbb{C}} \lambda_{\mathbb{S}} \in U_{\mathbb{C}}$ such that

$$a_{\mathbb{C}}(\mathcal{N}_{\mathbb{C}} \lambda_{\mathbb{S}}, \hat{u}_{\mathbb{C}}^{J}) = \langle \mathcal{R}_{\mathbb{S}}^{*} \lambda_{\mathbb{S}}, \hat{u}_{\mathbb{C}} \rangle_{\mathbb{C}} \qquad \forall \hat{u}_{\mathbb{C}} \in U_{\mathbb{C}}$$

INTERFACE VARIATIONAL FORMULATIONS

- We follow the classical approach to derive the Steklov-Poincaré equation
[Quarteroni, Valli (1999); Toselli, Widlund (2005)]
- We decompose the unknowns $u_{\mathbb{S}}$ and $u_{\mathbb{C}}$ as

$$u_{\mathbb{S}} = u^I_{\mathbb{S}} + \mathcal{D}_{\mathbb{S}}\mu_{\mathbb{S}} \quad \text{and} \quad u_{\mathbb{C}} = u^I_{\mathbb{C}} + \mathcal{D}_{\mathbb{C}}\mu_{\mathbb{S}}$$

where $\mu_{\mathbb{S}} \in \Lambda_{\mathbb{S}}$, while $u^I_{\mathbb{S}}$ and $u^I_{\mathbb{C}}$ satisfy problems with homogeneous Dirichlet data on $\Gamma_{\mathbb{S}}$ and depend on the forces and remaining boundary conditions.

- We split the admissible variations as

$$\hat{u}_{\mathbb{S}} = \hat{u}^I_{\mathbb{S}} + \mathcal{D}_{\mathbb{S}}\hat{\mu}_{\mathbb{S}} \quad \text{and} \quad \hat{u}_{\mathbb{C}} = \hat{u}^I_{\mathbb{C}} + \mathcal{D}_{\mathbb{C}}\hat{\mu}_{\mathbb{S}}$$

- We insert the splitting in the augmented variational formulation and we obtain:

find $\mu_{\mathbb{S}} \in \Lambda_{\mathbb{S}}$ such that

$$a_{\mathbb{S}}(\mathcal{D}_{\mathbb{S}}\mu_{\mathbb{S}}, \mathcal{D}_{\mathbb{S}}\hat{\mu}_{\mathbb{S}}) + a_{\mathbb{C}}(\mathcal{D}_{\mathbb{C}}\mu_{\mathbb{S}}, \mathcal{D}_{\mathbb{C}}\hat{\mu}_{\mathbb{S}})$$

$$= f_{\mathbb{S}}(\mathcal{D}_{\mathbb{S}}\hat{\mu}_{\mathbb{S}}) - a_{\mathbb{S}}(u^{l}_{\mathbb{S}}, \mathcal{D}_{\mathbb{S}}\hat{\mu}_{\mathbb{S}}) + f_{\mathbb{C}}(\mathcal{D}_{\mathbb{C}}\hat{\mu}_{\mathbb{S}}) - a_{\mathbb{C}}(u^{l}_{\mathbb{C}}, \mathcal{D}_{\mathbb{C}}\hat{\mu}_{\mathbb{S}})$$

for all $\hat{\mu}_{\mathbb{S}} \in \Lambda_{\mathbb{S}}$

or, in operator form:

$$\text{find } \mu_{\mathbb{S}} \in \Lambda_{\mathbb{S}} : \quad \mathcal{S}_{\Gamma_{\mathbb{S}}}\mu_{\mathbb{S}} = g_{\Gamma_{\mathbb{S}}} \quad \text{in } \Lambda'_{\mathbb{S}}$$

AUGMENTED DIRICHLET-DIRICHLET METHOD

Idea

- Instead of writing an interface equation only for $\mu_{\mathbb{S}}$, we write interface problems for both variables $\mu_{\mathbb{S}}$ and $\lambda_{\mathbb{S}}$ (primal and dual variables).

Derivation

- Consider again the same decomposition of $u_{\mathbb{S}}$ and $u_{\mathbb{C}}$ through contributions defined via Dirichlet sub-problems for the $\mathbb{S}$D-model and the $\mathbb{C}$D-model.
- Consider the admissible variations as

$$\hat{u}_{\mathbb{S}} = \hat{u}_{\mathbb{S}}^{I} + \mathcal{D}_{\mathbb{S}}\hat{\mu}_{\mathbb{S}}^{1} \quad \text{and} \quad \hat{u}_{\mathbb{C}} = \hat{u}_{\mathbb{C}}^{I} + \mathcal{D}_{\mathbb{C}}\hat{\mu}_{\mathbb{S}}^{2}$$

where $\hat{\mu}_{\mathbb{S}}^{1} \neq \hat{\mu}_{\mathbb{S}}^{2}$.

- Substituting in the augmented variational formulation we get

find $(\mu_{\mathbb{S}}, \lambda_{\mathbb{S}}) \in \Lambda_{\mathbb{S}} \times \Lambda'_{\mathbb{S}}$ such that

$$a_{\mathbb{S}}(\mathcal{D}_{\mathbb{S}}\mu_{\mathbb{S}}, \mathcal{D}_{\mathbb{S}}\hat{\mu}^1_{\mathbb{S}}) + a_{\mathbb{C}}(\mathcal{D}_{\mathbb{C}}\mu_{\mathbb{S}}, \mathcal{D}_{\mathbb{C}}\hat{\mu}^2_{\mathbb{S}}) + \langle \lambda_{\mathbb{S}}, \hat{\mu}^1_{\mathbb{S}} - \hat{\mu}^2_{\mathbb{S}} \rangle_{\mathbb{S}}$$
$$= f_{\mathbb{S}}(\mathcal{D}_{\mathbb{S}}\hat{\mu}^1_{\mathbb{S}}) - a_{\mathbb{S}}(u^I_{\mathbb{S}}, \mathcal{D}_{\mathbb{S}}\hat{\mu}^1_{\mathbb{S}})$$
$$+ f_{\mathbb{C}}(\mathcal{D}_{\mathbb{C}}\hat{\mu}^2_{\mathbb{S}}) - a_{\mathbb{C}}(u^I_{\mathbb{C}}, \mathcal{D}_{\mathbb{C}}\hat{\mu}^2_{\mathbb{S}}) \qquad \forall (\hat{\mu}^1_{\mathbb{S}}, \hat{\mu}^2_{\mathbb{S}}) \in \Lambda_{\mathbb{S}} \times \Lambda_{\mathbb{S}}$$

or, equivalently,

find $(\mu_{\mathbb{S}}, \lambda_{\mathbb{S}}) \in \Lambda_{\mathbb{S}} \times \Lambda'_{\mathbb{S}}$ such that

$$s_{\Gamma_{\mathbb{S}},\mathbb{S}}(\mu_{\mathbb{S}}, \hat{\mu}^1_{\mathbb{S}}) + \langle \lambda_{\mathbb{S}}, \hat{\mu}^1_{\mathbb{S}} \rangle_{\mathbb{S}} = g_{\Gamma_{\mathbb{S}},\mathbb{S}}(\hat{\mu}^1_{\mathbb{S}}) \qquad \forall \hat{\mu}^1_{\mathbb{S}} \in \Lambda_{\mathbb{S}}$$
$$s_{\Gamma_{\mathbb{S}},\mathbb{C}}(\mu_{\mathbb{S}}, \hat{\mu}^2_{\mathbb{S}}) - \langle \lambda_{\mathbb{S}}, \hat{\mu}^2_{\mathbb{S}} \rangle_{\mathbb{S}} = g_{\Gamma_{\mathbb{S}},\mathbb{C}}(\hat{\mu}^2_{\mathbb{S}}) \qquad \forall \hat{\mu}^2_{\mathbb{S}} \in \Lambda_{\mathbb{S}}$$

This corresponds to the Dirichlet-Dirichlet augmented system

$$\begin{pmatrix} \mathcal{S}_{\Gamma_{\mathbb{S}},\mathbb{S}} & \mathcal{I}_\lambda \\ \mathcal{S}_{\Gamma_{\mathbb{S}},\mathbb{C}} & -\mathcal{I}_\lambda \end{pmatrix} \begin{pmatrix} \mu_{\mathbb{S}} \\ \lambda_{\mathbb{S}} \end{pmatrix} = \begin{pmatrix} g_{\Gamma_{\mathbb{S}},\mathbb{S}} \\ g_{\Gamma_{\mathbb{S}},\mathbb{C}} \end{pmatrix}$$

where $\mathcal{I}_\lambda$ is the identity operator in $\Lambda'_{\mathbb{S}}$.

The following result holds.

Proposition

There exists a unique solution $(\mu_{\mathbb{S}}, \lambda_{\mathbb{S}}) \in \Lambda_{\mathbb{S}} \times \Lambda'_{\mathbb{S}}$ *such that*

$$\|\mu_{\mathbb{S}}\|_{\Lambda_{\mathbb{S}}} + \|\lambda_{\mathbb{S}}\|_{\Lambda'_{\mathbb{S}}} \leq C(\|g_{\Gamma_{\mathbb{S}},\mathbb{S}}\|_{\Lambda'_{\mathbb{S}}} + \|g_{\Gamma_{\mathbb{S}},\mathbb{C}}\|_{\Lambda'_{\mathbb{S}}}) \qquad C > 0.$$

AUGMENTED DIRICHLET-NEUMANN METHOD

- We change the splitting of $u_{\mathbb{S}}$ and $u_{\mathbb{C}}$ considering a Dirichlet problem for the $\mathbb{S}$D-model and a Neumann problem for the $\mathbb{C}$D-model:

$$\boxed{u_{\mathbb{S}} = u_{\mathbb{S}}^{I} + \mathcal{D}_{\mathbb{S}}\mu_{\mathbb{S}} \quad \text{and} \quad u_{\mathbb{C}} = u_{\mathbb{C}}^{J} + \mathcal{N}_{\mathbb{C}}\lambda_{\mathbb{S}}}$$

where $\mu_{\mathbb{S}} \in \Lambda_{\mathbb{S}}$ and $\lambda_{\mathbb{S}} \in \Lambda'_{\mathbb{S}}$.

- The admissible variations in this case are

$$\hat{u}_{\mathbb{S}} = \hat{u}_{\mathbb{S}}^{I} + \mathcal{D}_{\mathbb{S}}\hat{\mu}_{\mathbb{S}} \quad \text{and} \quad \hat{u}_{\mathbb{C}} = \hat{u}_{\mathbb{C}}^{J} + \mathcal{N}_{\mathbb{C}}\hat{\lambda}_{\mathbb{S}}.$$

- Substituting in the augmented variational formulation we obtain

$$a_{\mathbb{S}}(\mathcal{D}_{\mathbb{S}}\mu_{\mathbb{S}}, \mathcal{D}_{\mathbb{S}}\hat{\mu}_{\mathbb{S}}) + \langle \lambda_{\mathbb{S}}, \hat{\mu}_{\mathbb{S}} \rangle_{\mathbb{S}} + \langle \hat{\lambda}_{\mathbb{S}}, \mu_{\mathbb{S}} - \mathcal{R}_{\mathbb{S}}(u^J_{\mathbb{C}} + \mathcal{N}_{\mathbb{C}}\lambda_{\mathbb{S}}) \rangle_{\mathbb{S}}$$
$$= f_{\mathbb{S}}(\mathcal{D}_{\mathbb{S}}\hat{\mu}_{\mathbb{S}}) - a_{\mathbb{S}}(u^I_{\mathbb{S}}, \mathcal{D}_{\mathbb{S}}\hat{\mu}_{\mathbb{S}}) \qquad \forall (\hat{\mu}_{\mathbb{S}}, \hat{\lambda}_{\mathbb{S}}) \in \Lambda_{\mathbb{S}} \times \Lambda'_{\mathbb{S}}.$$

- This corresponds to the interface problem:

find $(\mu_{\mathbb{S}}, \lambda_{\mathbb{S}}) \in \Lambda_{\mathbb{S}} \times \Lambda'_{\mathbb{S}}$ such that

$$s_{\Gamma_{\mathbb{S}},\mathbb{S}}(\mu_{\mathbb{S}}, \hat{\mu}_{\mathbb{S}}) + \langle \lambda_{\mathbb{S}}, \hat{\mu}_{\mathbb{S}} \rangle_{\mathbb{S}} = g_{\Gamma_{\mathbb{S}},\mathbb{S}}(\hat{\mu}_{\mathbb{S}}) \qquad \forall \hat{\mu}_{\mathbb{S}} \in \Lambda_{\mathbb{S}}$$
$$\langle \hat{\lambda}_{\mathbb{S}}, \mu_{\mathbb{S}} \rangle_{\mathbb{S}} - \langle \hat{\lambda}_{\mathbb{S}}, \mathcal{R}_{\mathbb{S}}(\mathcal{N}_{\mathbb{C}}\lambda_{\mathbb{S}}) \rangle_{\mathbb{S}} = \langle \hat{\lambda}_{\mathbb{S}}, \mathcal{R}_{\mathbb{S}} u^J_{\mathbb{C}} \rangle_{\mathbb{S}} \qquad \forall \hat{\lambda}_{\mathbb{S}} \in \Lambda'_{\mathbb{S}}$$

or to the augmented Dirichlet-Neumann system

$$\begin{pmatrix} \mathcal{S}_{\Gamma_{\mathbb{S}},\mathbb{S}} & \mathcal{I}_\lambda \\ \mathcal{I}_\mu & -\mathcal{T}_{\Gamma_{\mathbb{S}},\mathbb{C}} \end{pmatrix} \begin{pmatrix} \mu_{\mathbb{S}} \\ \lambda_{\mathbb{S}} \end{pmatrix} = \begin{pmatrix} g_{\Gamma_{\mathbb{S}},\mathbb{S}} \\ \mathcal{R}_{\mathbb{S}} u^J_{\mathbb{C}} \end{pmatrix}$$

where $\mathcal{I}_\lambda$ and $\mathcal{I}_\mu$ are the identity operators in $\Lambda'_{\mathbb{S}}$ and $\Lambda_{\mathbb{S}}$.

FEW COMMENTS

- At the continuous level these formulations are equivalent.
- The augmented approach permits to impose conditions of different type on the coupling interfaces.
- We compute at once both the primal and the dual variables.
- This framework can be extended easily to the case of multi-component systems: in case of $N_{\mathbb{C}}$ $\mathbb{C}$D models and $N_{\mathbb{S}}$ $\mathbb{S}$D models with M interfaces, the Dirichlet-Dirichlet augmented method would become:

  find $(\boldsymbol{\mu}_{\mathbb{S}}, \boldsymbol{\lambda}_{\mathbb{S}}) \in \boldsymbol{\Lambda}_{\mathbb{S}} \times \boldsymbol{\Lambda}'_{\mathbb{S}}$ such that

$$\begin{pmatrix} \boldsymbol{\mathcal{S}}_{\Gamma_{\mathbb{S}},\mathbb{S}} & \boldsymbol{\mathcal{I}}_{\boldsymbol{\lambda}} \\ \boldsymbol{\mathcal{S}}_{\Gamma_{\mathbb{S}},\mathbb{C}} & -\boldsymbol{\mathcal{I}}_{\boldsymbol{\lambda}} \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}_{\mathbb{S}} \\ \boldsymbol{\lambda}_{\mathbb{S}} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_{\Gamma_{\mathbb{S}},\mathbb{S}} \\ \mathbf{g}_{\Gamma_{\mathbb{S}},\mathbb{C}} \end{pmatrix}$$

GALERKIN APPROXIMATION & DIRICHLET-DIRICHLET METHOD

- We introduce Galerkin finite element approximations for all the components in the system.
- Let $h = (h_{\mathbb{C}}, h_{\mathbb{S}})$ with $h_{\mathbb{C}}$ and $h_{\mathbb{S}}$ be the discretization parameters for the $\mathbb{C}$D and the $\mathbb{S}$D systems.
- In the Dirichlet-Dirichlet case, we formally obtain the problem

find $(\mu_{\mathbb{S},h}, \lambda_{\mathbb{S},h}) \in \Lambda_{\mathbb{S},h} \times \Lambda'_{\mathbb{S},h}$ such that

$$s_{\Gamma_{\mathbb{S}},\mathbb{S}}(\mu_{\mathbb{S},h}, \hat{\mu}^1_{\mathbb{S},h}) + \langle \lambda_{\mathbb{S},h}, \hat{\mu}^1_{\mathbb{S},h}\rangle_{\mathbb{S}} = g_{\Gamma_{\mathbb{S}},\mathbb{S}}(\hat{\mu}^1_{\mathbb{S},h}) \qquad \forall \hat{\mu}^1_{\mathbb{S},h} \in \Lambda_{\mathbb{S},h}$$

$$s_{\Gamma_{\mathbb{S}},\mathbb{C}}(\mu_{\mathbb{S},h}, \hat{\mu}^2_{\mathbb{S},h}) - \langle \lambda_{\mathbb{S},h}, \hat{\mu}^2_{\mathbb{S},h}\rangle_{\mathbb{S}} = g_{\Gamma_{\mathbb{S}},\mathbb{C}}(\hat{\mu}^2_{\mathbb{S},h}) \qquad \forall \hat{\mu}^2_{\mathbb{S},h} \in \Lambda_{\mathbb{S},h}.$$

CASE $\mathbb{S} = 0, 1$

- In the case $\mathbb{S} = 0, 1$, then $\Lambda_{\mathbb{S}} = \Lambda_{\mathbb{S},h} = \mathbb{R}$ and, for M interfaces, we have the problem

  find $(\boldsymbol{\mu}_{\mathbb{S},h}, \boldsymbol{\lambda}_{\mathbb{S},h}) \in \mathbb{R}^M \times \mathbb{R}^M$ such that

$$\begin{pmatrix} \mathbf{S}_{\Gamma_{\mathbb{S}},\mathbb{S},h} & \mathbf{1} \\ \mathbf{S}_{\Gamma_{\mathbb{S}},\mathbb{C},h} & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}_{\mathbb{S},h} \\ \boldsymbol{\lambda}_{\mathbb{S},h} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_{\Gamma_{\mathbb{S}},\mathbb{S},h} \\ \mathbf{g}_{\Gamma_{\mathbb{S}},\mathbb{C},h} \end{pmatrix}$$

- **Proposition**

  The condition number of the matrix

$$\begin{pmatrix} \mathbf{S}_{\Gamma_{\mathbb{S}},\mathbb{S},h} & \mathbf{1} \\ \mathbf{S}_{\Gamma_{\mathbb{S}},\mathbb{C},h} & -\mathbf{1} \end{pmatrix}$$

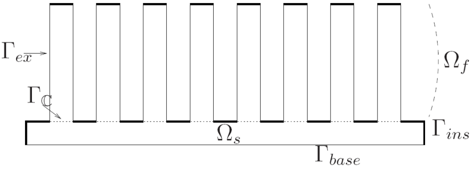
  is independent of $h = (h_{\mathbb{C}}, h_{\mathbb{S}})$.

  Analogous results hold for the Dirichlet-Neumann and for the Neumann-Neumann formulations.

NUMERICAL EXPERIMENTS

2D-1D COUPLED HEAT TRANSFER SYSTEM

We consider a 2D heat sink for the thermal management of high-density electronic components:



The system is described by the following equations:

$$
\begin{aligned}
&-\text{div}\,(k\nabla u_2) = 0 && \text{in } \Omega_s \cup \Omega_f \\
&k\frac{\partial u_2}{\partial n} = 0 && \text{on } \Gamma_{ins} \\
&u_2 = u_2^* && \text{on } \Gamma_{base} \\
&k\frac{\partial u_2}{\partial n} + \text{Bi}\, u_2 = 0 && \text{on } \Gamma_{ex}
\end{aligned}
$$

To reduce the computational cost, we replace the fins by 1D rods.

- Functional spaces: $U_2 = H^1(\Omega_2) + \text{b.c.}$ and $U_1 = H^1(\Omega_1) + \text{b.c.}$, $\Lambda_2 = H^{1/2}(\Gamma_2)$, $\Lambda_2' = H^{-1/2}(\Gamma_2)$, $\Lambda_1 = \Lambda_1' = \mathbb{R}$.
- Bilinear forms:

$$a_2(u_2, \hat{u}_2) = \int_{\Omega_2} k \nabla u_2 \cdot \nabla \hat{u}_2 + \int_{\Gamma_{ex}} \text{Bi}\, u_2 \hat{u}_2$$

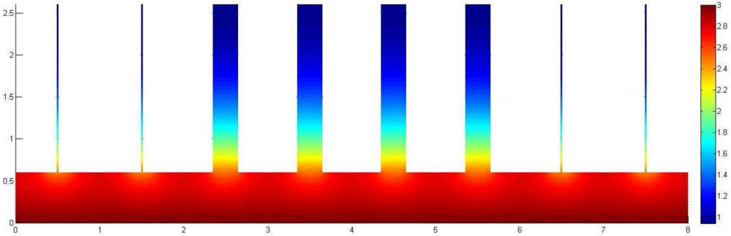
$$a_1(u_1, \hat{u}_1) = \int_{\Omega_1} k\delta \frac{\mathrm{d}u_1}{\mathrm{d}\xi} \frac{\mathrm{d}\hat{u}_1}{\mathrm{d}\xi} + \int_{\Omega_1} \text{Bi}'\, u_1 \hat{u}_1$$

- Operators:

$$\mathcal{R}_1(u_{2|\Gamma_2}) = u_{2,1|\Gamma_1} = \frac{1}{|\Gamma_2|} \int_{\Gamma_2} u_2 \quad \text{and} \quad \mathcal{R}_1^*(\lambda_1) = \lambda_{1|\Gamma_2}$$

Condition number Augm. Dirichlet-Neumann method

	2 1D fins	4 1D fins	6 1D fins	8 1D fins
grid 1	3.1095 (4)	3.1615 (4)	3.1799 (6)	3.1994 (5)
grid 2	3.0685 (4)	3.1123 (4)	3.1282 (5)	3.1437 (4)
grid 3	3.0549 (4)	3.0970 (4)	3.1121 (5)	3.1264 (4)
grid 4	3.0506 (4)	3.0923 (4)	3.1072 (5)	3.1211 (4)
grid 5	3.0493 (3)	3.0909 (4)	3.1058 (5)	3.1195 (4)



AN APPLICATION TO HEMODYNAMICS

- Coupling of 3D-0D domains in blood-flow simulations: Navier-Stokes equations with algebraic models.
- Arbitrary number of $\mathbb{S}(0)$-dimensional interfaces with coupling quantities
  - *volumetric flow rate*

$$\boxed{Q_{\mathbb{S}} = \int_{\Gamma_{\mathbb{C}}} \mathbf{u}_f \cdot \mathbf{n}} \qquad (\mu_{\mathbb{S}})$$

  - *coupling stress* (average of the normal component of the traction vector)

$$\boxed{\Sigma_{\mathbb{S}} = \frac{1}{|\Gamma_{\mathbb{C}}|} \int_{\Gamma_{\mathbb{C}}} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{n}} \qquad (\lambda_{\mathbb{S}})$$

- Global nonlinear interface system

$$\boxed{\mathcal{S}_{NS}(Q_{\mathbb{S}}, \Sigma_{\mathbb{S}}) = 0}$$

solved with the Newton method.

[Malossi et al. (2011)]

MODELING FILTRATION OF INCOMPRESSIBLE FLOWS THROUGH POROUS MEDIA

THE DARCY-STOKES PROBLEM

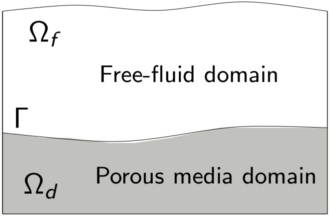
- **Fluid flow:** Stokes equations

$$\begin{aligned} -\mathbf{div}\, \mathrm{T}(\mathbf{u}_f, p_f) &= \mathbf{f} \\ \mathrm{div}\, \mathbf{u}_f &= 0 \end{aligned} \quad \text{in } \Omega_f$$

where $\mathrm{T}(\mathbf{u}_f, p_f) = \nu(\nabla \mathbf{u}_f + \nabla^T \mathbf{u}_f) - p_f \mathbf{I}$ is the Cauchy stress tensor.

- **Fluid through porous medium:** Darcy's equations

$$\begin{aligned} \mathbf{u}_d = -\frac{k}{\nu} \nabla p_d \\ \mathrm{div}\, \mathbf{u}_d = 0 \end{aligned} \quad \text{in } \Omega_d \quad \Rightarrow \quad -\mathrm{div}\,(\frac{k}{\nu} \nabla p_d) = 0 \quad \text{in } \Omega_d$$



COUPLING (INTERFACE) CONDITIONS

The solution must satisfy three regularity conditions across Γ:

- the continuity of the normal velocities

$$\mathbf{u}_f \cdot \mathbf{n} = \mathbf{u}_d \cdot \mathbf{n} \;\Leftrightarrow\; \mathbf{u}_f \cdot \mathbf{n} = -\frac{k}{\nu}\nabla p_d \cdot \mathbf{n}$$

  a consequence of the incompressibility;

- the continuity of the normal stresses

$$-\mathbf{n} \cdot \mathrm{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n} = p_d$$

  (pressures can be discontinuous across Γ);

- a condition on the tangential component of the normal stress: Beavers–Joseph–Saffman equation

$$-\boldsymbol{\tau} \cdot \mathrm{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n} = \frac{\nu\alpha}{\sqrt{k}}\mathbf{u}_f \cdot \boldsymbol{\tau}$$

[Jäger, Mikelić (1996); Miglio, Discacciati, Quarteroni (2002); Layton, Schieweck, Yotov (2003)]

WEAK FORM OF THE COUPLED DARCY-STOKES PROBLEM

Find $\mathbf{u}_f \in H^1(\Omega_f)$, $p_f \in L^2(\Omega_f)$, $p_d \in H^1(\Omega_p)$:

$$\int_{\Omega_f} \nu \nabla \mathbf{u}_f : \nabla \mathbf{v} + \int_{\Gamma} \frac{\nu\alpha}{\sqrt{k}} (\mathbf{u}_f \cdot \boldsymbol{\tau})(\mathbf{v} \cdot \boldsymbol{\tau}) - \int_{\Omega_f} p_f \mathrm{div}\mathbf{v} + \int_{\Gamma} p_d (\mathbf{v} \cdot \mathbf{n}) = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v}$$

$$\int_{\Omega_f} q \mathrm{div}\mathbf{u}_f = 0$$

$$\int_{\Omega_d} \frac{k}{\nu} \nabla p_d \cdot \nabla \psi - \int_{\Gamma} \psi (\mathbf{u}_f \cdot \mathbf{n}) = 0$$

INTERFACE EQUATION FOR THE DARCY-STOKES PROBLEM

We can express the Darcy-Stokes problem in terms of the solution λ (normal velocity across Γ) of the interface problem

$$\boxed{S_s\lambda + S_d\lambda = \chi \quad \text{on } \Gamma}$$

- S_s *fluid* operator:

$$S_s : \lambda \text{ (normal velocities on } \Gamma) \xrightarrow[\textit{Stokes}]{\textit{solve}} -\mathbf{n}\cdot \mathrm{T}(\mathbf{u}_f, p_f)\cdot\mathbf{n} \text{ (normal stresses on } \Gamma).$$

- S_d *porous media* operator:

$$S_d : \lambda \text{ (normal velocities on } \Gamma) \xrightarrow[\textit{Darcy}]{\textit{solve}} p_{d|\Gamma} \text{ (pressure } p_d \text{ on } \Gamma) \ .$$

[Discacciati, Quarteroni (2003)]

ALGEBRAIC FORM

- Considering a suitable conforming finite element approximation of the coupled problem leads to the linear system:

$$\left(\begin{array}{ccc|c} (\mathrm{A}_f + \bar{\alpha}\mathrm{M}_\Gamma)_\tau & 0 & \mathrm{D}_\tau^T & 0 \\ 0 & (\mathrm{A}_f)_n & \mathrm{D}_n^T & \mathrm{C}_\Gamma^T \\ \mathrm{D}_\tau & \mathrm{D}_n & 0 & 0 \\ \hline 0 & -\mathrm{C}_\Gamma & 0 & \mathrm{A}_d \end{array}\right) \begin{pmatrix} (\mathbf{u}_f)_\tau \\ (\mathbf{u}_f)_n \\ p_f \\ p_d \end{pmatrix} = \begin{pmatrix} \mathbf{b}_\tau \\ \mathbf{b}_n \\ \mathbf{b}_p \\ \mathbf{b}_d \end{pmatrix}$$

  with $(\mathbf{u}_f)_n$ vector of the nodal values of $\mathbf{u}_{f,h} \cdot \mathbf{n}$ on Γ

- Discrete interface equation for the normal velocity:

$$\boxed{(\boldsymbol{\Sigma}_s + \boldsymbol{\Sigma}_d)(\mathbf{u}_f)_n = \chi_s + \chi_d}$$

PRECONDITIONING TECHNIQUES (I)

- S_s is *spectrally equivalent* to $S_s + S_d$: there exist two positive constants c and C (independent of η) such that

$$c\langle S_s\eta,\eta\rangle \;\leq\; \langle (S_s+S_d)\eta,\eta\rangle \;\leq\; C\langle S_s\eta,\eta\rangle \qquad \forall\eta$$

- At the discrete level, c and C are also independent of h, so we could use $\boldsymbol{\Sigma}_s$ as an optimal preconditioner for $(\boldsymbol{\Sigma}_s + \boldsymbol{\Sigma}_d)$

PCG iterations with $P = \boldsymbol{\Sigma}_s$

ν	K	h=1/7	h=1/14	h=1/28	h=1/56
1	1	5	5	5	5
10^{-3}	10^{-2}	20	54	73	56
10^{-4}	10^{-3}	20	59	#	#
10^{-6}	10^{-4}	20	59	148	#

[Discacciati, Quarteroni (2003,2004)]

PRECONDITIONING TECHNIQUES (II)

- We introduced a *Robin-Robin* method associated to a preconditioner:

$$P = (\gamma_f I + S_s)(\gamma_d I + S_d)$$

Parameters		Iterations for			
ν	K	h=1/7	h=1/14	h=1/28	h=1/56
10^{-4}	10^{-3}	19	19	19	19
10^{-6}	10^{-4}	20	20	20	20
10^{-6}	10^{-7}	20	20	20	20

[Discacciati, Quarteroni, Valli (2007)]

PRECONDITIONING TECHNIQUES (II)

This preconditioner can be characterized from a purely algebraic point of view following the *generalized Hermitian and skew-Hermitian splitting method* proposed by Golub and Benzi:

$$\boxed{P_{GHSS} = (\boldsymbol{\Sigma}_s + \alpha I)(\boldsymbol{\Sigma}_d + \alpha I)} \qquad \alpha \simeq \sqrt{\nu}$$

These preconditioners

- have a multiplicative structure;
- can be used within GMRES iterations.

[Benzi (2009), Bai et al. (2003), Discacciati (2011, submitted)]

NUMERICAL RESULTS (II)

Comparison between CG iterations without preconditioner and GMRES iterations with preconditioner P_{GHSS}

	$\nu = 10^{-4}$, K $= 10^{-3}$		$\nu = 10^{-6}$, K $= 10^{-5}$		$\nu = 10^{-6}$, K $= 10^{-8}$	
	CG	**GMRES** + P_{GHSS}	CG	**GMRES** + P_{GHSS}	CG	**GMRES** + P_{GHSS}
h_1	9	5 $(\alpha = 10^{-2})$	9	4 $(\alpha = 10^{-3})$	9	4 $(\alpha = 10^{-3})$
h_2	20	7 $(\alpha = 10^{-2})$	20	4 $(\alpha = 10^{-3})$	20	4 $(\alpha = 10^{-3})$
h_3	42	9 $(\alpha = 10^{-3})$	42	4 $(\alpha = 10^{-3})$	42	4 $(\alpha = 10^{-3})$
h_4	64	9 $(\alpha = 10^{-3})$	66	4 $(\alpha = 10^{-3})$	66	4 $(\alpha = 10^{-3})$

PRECONDITIONING TECHNIQUES (III)

Effective preconditioners with additive structure can be characterized as well, considering the following *augmented interface systems*:

- Discrete augmented Dirichlet-Dirichlet (aDD) problem:

$$\begin{pmatrix} \boldsymbol{\Sigma}_s & \mathrm{C}_\Gamma^T \\ -\mathrm{C}_\Gamma & \boldsymbol{\Sigma}_p \end{pmatrix} \begin{pmatrix} (\mathbf{u}_f)_n \\ p_{d|\Gamma} \end{pmatrix} = \begin{pmatrix} \chi_s \\ \chi_p \end{pmatrix}$$

  with $\boldsymbol{\Sigma}_p \approx \boldsymbol{\Sigma}_d^{-1}$

- Discrete augmented Neumann-Neumann (aNN) problem:

$$\begin{pmatrix} \boldsymbol{\Sigma}_d & -\mathrm{C}_\Gamma^T \\ \mathrm{C}_\Gamma & \boldsymbol{\Sigma}_f \end{pmatrix} \begin{pmatrix} (\mathbf{u}_f)_n \\ p_{d|\Gamma} \end{pmatrix} = \begin{pmatrix} \chi_d \\ \chi_f \end{pmatrix}$$

  with $\boldsymbol{\Sigma}_f \approx \boldsymbol{\Sigma}_s^{-1}$

In this case, we can characterize the preconditioners

- for the aDD problem

$$P_{aDD} = \begin{pmatrix} \boldsymbol{\Sigma}_s + \alpha_{aDD} I & 0 \\ 0 & \boldsymbol{\Sigma}_p + \alpha_{aDD} I \end{pmatrix} \begin{pmatrix} \alpha_{aDD} I & \mathrm{C}_\Gamma^T \\ -\mathrm{C}_\Gamma & \alpha_{aDD} I \end{pmatrix}$$

- for the aNN problem

$$P_{aNN} = \begin{pmatrix} \boldsymbol{\Sigma}_d + \alpha_{aNN} I & 0 \\ 0 & \boldsymbol{\Sigma}_f + \alpha_{aNN} I \end{pmatrix} \begin{pmatrix} \alpha_{aNN} I & -\mathrm{C}_\Gamma^T \\ \mathrm{C}_\Gamma & \alpha_{aNN} I \end{pmatrix}$$

These preconditioners

- allow solving the fluid and the porous-media subproblems independently in a parallel fashion;
- can be used within GMRES iterations.

NUMERICAL RESULTS (III)

Comparison between GMRES iterations without preconditioner for the augmented systems:

- with preconditioner P_{aDD} for the aDD problem

	$\nu = 10^{-4}$, K $= 10^{-3}$		$\nu = 10^{-6}$, K $= 10^{-5}$		$\nu = 10^{-6}$, K $= 10^{-8}$	
	GMRES	**GMRES + P_{aDD}**	GMRES	**GMRES + P_{aDD}**	GMRES	**GMRES + P_{aDD}**
h_1	17	14 $(\alpha_{aDD} = 10^{-3})$	17	7 $(\alpha_{aDD} = 10^{-3})$	17	8 $(\alpha_{aDD} = 10^{-3})$
h_2	33	17 $(\alpha_{aDD} = 10^{-3})$	33	8 $(\alpha_{aDD} = 10^{-3})$	33	10 $(\alpha_{aDD} = 10^{-3})$
h_3	63	22 $(\alpha_{aDD} = 5 \cdot 10^{-4})$	65	8 $(\alpha_{aDD} = 5 \cdot 10^{-4})$	65	10 $(\alpha_{aDD} = 5 \cdot 10^{-4})$
h_4	67	23 $(\alpha_{aDD} = 5 \cdot 10^{-4})$	79	9 $(\alpha_{aDD} = 5 \cdot 10^{-4})$	101	11 $(\alpha_{aDD} = 5 \cdot 10^{-4})$

- with preconditioner P_{aNN} for the aNN problem

	$\nu = 10^{-4}$, K $= 10^{-3}$		$\nu = 10^{-6}$, K $= 10^{-5}$		$\nu = 10^{-6}$, K $= 10^{-8}$	
	GMRES	**GMRES + P_{aNN}**	GMRES	**GMRES + P_{aNN}**	GMRES	**GMRES + P_{aNN}**
h_1	17	16 $(\alpha_{aNN} = 0.1)$	16	9 $(\alpha_{aNN} = 0.5)$	9	8 $(\alpha_{aNN} = 1)$
h_2	32	18 $(\alpha_{aNN} = 0.1)$	32	8 $(\alpha_{aNN} = 0.5)$	16	7 $(\alpha_{aNN} = 0.5)$
h_3	59	20 $(\alpha_{aNN} = 5 \cdot 10^{-2})$	58	10 $(\alpha_{aNN} = 0.1)$	30	5 $(\alpha_{aNN} = 0.8)$
h_4	82	27 $(\alpha_{aNN} = 5 \cdot 10^{-2})$	81	8 $(\alpha_{aNN} = 0.1)$	44	5 $(\alpha_{aNN} = 0.8)$

[Discacciati (2011, submitted)]

DIMENSIONLESS FORM OF THE DARCY-STOKES PROBLEM

- We define the dimensionless numbers:

$$\mathrm{Re}_f = \frac{U_f X_f}{\nu} \qquad \text{(Reynolds number)}$$

$$\mathrm{N}_k = \frac{k}{X_f^2}$$

$$\mathrm{E}_f = \frac{\Pi_f}{\rho U_f^2} \qquad \text{(Euler number)}$$

$$N_{BJS} = \alpha(\mathrm{N}_k^{\frac{1}{2}} \mathrm{Re}_f)^{-1}$$

- We characterize the Darcy-Stokes problem using three (independent) dimensionless numbers: Re_f, N_k, E_f and the dimensionless coefficient α.

ALGEBRAIC (DIMENSIONLESS) FORM

- We assume that the physical quantities are constant in the domain.
- Using a conforming finite element approximation we get a linear system with matrix:

$$\begin{pmatrix} (\mathsf{Re}_f\mathsf{E}_f)^{-1}(\mathrm{A}_f + \alpha\mathsf{N}_k^{-\frac{1}{2}}\mathrm{M}_\Gamma)_\tau & 0 & \mathrm{D}_\tau^T & 0 \\ 0 & (\mathsf{Re}_f\mathsf{E}_f)^{-1}(\mathrm{A}_f)_n & \mathrm{D}_n^T & \mathrm{C}_\Gamma^T \\ \mathrm{D}_\tau & \mathrm{D}_n & 0 & 0 \\ 0 & -\mathrm{C}_\Gamma & 0 & (\mathsf{Re}_f\mathsf{E}_f\mathsf{N}_k)\mathrm{A}_d \end{pmatrix} \begin{pmatrix} (\mathbf{u}_f)_\tau \\ (\mathbf{u}_f)_n \\ p_f \\ p_d \end{pmatrix}$$

DIMENSIONLESS SCHUR COMPLEMENT SYSTEM

- We consider now the Schur complement system with respect to the variable $(\mathbf{u}_f)_n$:

$$\boxed{(\boldsymbol{\Sigma}_s + \boldsymbol{\Sigma}_d)(\mathbf{u}_f)_n = \chi_s + \chi_d}$$

where

$$\begin{aligned}
\boldsymbol{\Sigma}_s &= (\mathrm{Re}_f \mathrm{E}_f)^{-1}\left((\mathrm{A}_f)_n + \mathrm{D}_n^T\left(\mathrm{D}_\tau(\mathrm{A}_f + \alpha \mathrm{N}_k^{-\frac{1}{2}} \mathrm{M}_\Gamma)_\tau^{-1} \mathrm{D}_\tau^T\right)^{-1} \mathrm{D}_n\right) \\
\boldsymbol{\Sigma}_d &= (\mathrm{Re}_f \mathrm{E}_f \mathrm{N}_k)^{-1} \mathrm{C}_\Gamma^T \mathrm{A}_d^{-1} \mathrm{C}_\Gamma
\end{aligned}$$

- We introduce the notations:

$$\begin{aligned}
\boldsymbol{\Sigma}_s &= (\mathrm{Re}_f \mathrm{E}_f)^{-1} \hat{\boldsymbol{\Sigma}}_s \\
\boldsymbol{\Sigma}_d &= (\mathrm{Re}_f \mathrm{E}_f \mathrm{N}_k)^{-1} \widetilde{\boldsymbol{\Sigma}}_d
\end{aligned}$$

- The Schur complement system then reads:

$$(\mathrm{Re}_f \mathrm{E}_f)^{-1} \hat{\mathbf{\Sigma}}_s (\mathbf{u}_f)_n + (\mathrm{Re}_f \mathrm{E}_f \mathrm{N}_k)^{-1} \widetilde{\mathbf{\Sigma}}_d (\mathbf{u}_f)_n = \chi_s + \chi_d$$

- If we multiply $(\mathrm{Re}_f \mathrm{E}_f \mathrm{N}_k)$, we obtain:

$$\boxed{(\mathrm{N}_k \hat{\mathbf{\Sigma}}_s + \widetilde{\mathbf{\Sigma}}_d)(\mathbf{u}_f)_n = \widetilde{\chi}_s + \widetilde{\chi}_d}$$

or, equivalently,

$$\boxed{(\widetilde{\mathbf{\Sigma}}_s + \widetilde{\mathbf{\Sigma}}_d)(\mathbf{u}_f)_n = \widetilde{\chi}_s + \widetilde{\chi}_d}$$

- Remark that we have independence of Re_f.

SPECTRAL EQUIVALENCE AND PRECONDITIONERS

- We can prove the following results about the corresponding discrete Steklov-Poincaré operators:

$$c_s \mathsf{N}_k \|\eta_h\|^2_\Lambda \leq \langle \widetilde{S}_{s,h}\eta_h, \eta_h \rangle \leq C_s \mathsf{N}_k \|\eta_h\|^2_\Lambda$$

$$c_d h^2 \|\eta_h\|^2_\Lambda \leq c'_d \|\eta_h\|^2_{\Lambda'} \leq \langle \widetilde{S}_{d,h}\eta_h, \eta_h \rangle \leq C_d \|\eta_h\|^2_\Lambda$$

- Using these estimates, at the algebraic level we can prove that

$$ch(\mathsf{N}_k + c_{ds} h^2)[\boldsymbol{\eta}, \boldsymbol{\eta}] \leq [(\widetilde{\boldsymbol{\Sigma}}_s + \widetilde{\boldsymbol{\Sigma}}_d)\boldsymbol{\eta}, \boldsymbol{\eta}] \leq C(\mathsf{N}_k + C_{ds})[\boldsymbol{\eta}, \boldsymbol{\eta}]$$

so that

$$cond((\widetilde{\boldsymbol{\Sigma}}_s + \widetilde{\boldsymbol{\Sigma}}_d)) \leq C' \frac{\mathsf{N}_k + C_{ds}}{h(\mathsf{N}_k + c_{ds} h^2)}$$

DIRICHLET-NEUMANN TYPE PRECONDITIONER (I)

Consider the operator $\widetilde{\boldsymbol{\Sigma}}_s$.

We can prove the following equivalence:

$$c\left(1+\frac{h^2}{\mathrm{N}_k}\right)[\widetilde{\boldsymbol{\Sigma}}_s\boldsymbol{\eta},\boldsymbol{\eta}] \le [(\widetilde{\boldsymbol{\Sigma}}_s+\widetilde{\boldsymbol{\Sigma}}_d)\boldsymbol{\eta},\boldsymbol{\eta}] \le C\left(1+\frac{1}{\mathrm{N}_k}\right)[\widetilde{\boldsymbol{\Sigma}}_s\boldsymbol{\eta},\boldsymbol{\eta}]$$

so that

$$\boxed{cond(\widetilde{\boldsymbol{\Sigma}}_s^{-1}(\widetilde{\boldsymbol{\Sigma}}_s+\widetilde{\boldsymbol{\Sigma}}_d)) \le \frac{\mathrm{N}_k+C_1}{\mathrm{N}_k+C_2h^2}}$$

We can see that

- if $\mathrm{N}_k \gg h^2$, then $cond(\widetilde{\boldsymbol{\Sigma}}_s^{-1}(\widetilde{\boldsymbol{\Sigma}}_s+\widetilde{\boldsymbol{\Sigma}}_d)) \sim 1$
- if $\mathrm{N}_k \ll h^2$, then $cond(\widetilde{\boldsymbol{\Sigma}}_s^{-1}(\widetilde{\boldsymbol{\Sigma}}_s+\widetilde{\boldsymbol{\Sigma}}_d)) \sim h^{-2}$

DIRICHLET-NEUMANN TYPE PRECONDITIONER (II)

Consider the operator $\widetilde{\boldsymbol{\Sigma}}_d$.

We can prove the following equivalence:

$$c\,(1+\mathrm{N}_k)\,[\widetilde{\boldsymbol{\Sigma}}_d\boldsymbol{\eta},\boldsymbol{\eta}] \leq [(\widetilde{\boldsymbol{\Sigma}}_s+\widetilde{\boldsymbol{\Sigma}}_d)\boldsymbol{\eta},\boldsymbol{\eta}] \leq C\left(1+\frac{\mathrm{N}_k}{h^2}\right)[\widetilde{\boldsymbol{\Sigma}}_d\boldsymbol{\eta},\boldsymbol{\eta}]$$

so that

$$\boxed{cond(\widetilde{\boldsymbol{\Sigma}}_d^{-1}(\widetilde{\boldsymbol{\Sigma}}_s+\widetilde{\boldsymbol{\Sigma}}_d)) \leq C\cdot\frac{1}{h^2}\cdot\frac{\mathrm{N}_k+C_1h^2}{\mathrm{N}_k+C_2}}$$

We can see that

- if $\mathrm{N}_k \gg h^2$, then $cond(\widetilde{\boldsymbol{\Sigma}}_d^{-1}(\widetilde{\boldsymbol{\Sigma}}_s+\widetilde{\boldsymbol{\Sigma}}_d)) \sim h^{-2}$
- if $\mathrm{N}_k \ll h^2$, then $cond(\widetilde{\boldsymbol{\Sigma}}_d^{-1}(\widetilde{\boldsymbol{\Sigma}}_s+\widetilde{\boldsymbol{\Sigma}}_d)) \sim 1$

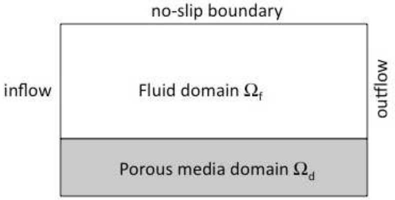
NUMERICAL RESULTS (I)

PCG iterations at $Re_f = 1$, Ω_f and Ω_d unit square domains:

Uf	1		tol=1.e-10								
Xf	1		Nk>h2								
alfa	0										
v	1.00E+00		1.00E-02		1.00E-02		1.00E-03		1.00E-06		
K	1.00E+00		1.00E-03		1.00E-05		1.00E-06		1.00E-09		
Nk	1.00E+00		1.00E-05		1.00E-07		1.00E-09		1.00E-15		
Iterations											
no prec											
0.25	**8**		**9**		**9**		**9**		**9**		0.0625
0.125	**16**	*2.00*	**20**	*2.22*	**20**	*2.22*	**20**	*2.22*	**20**	*2.22*	0.015625
0.0625	**29**	*1.81*	**41**	*2.05*	**47**	*2.35*	**48**	*2.40*	**49**	*2.45*	0.00390625
0.03125	**42**	*1.45*	**42**	*1.02*	**75**	*1.60*	**77**	*1.60*	**77**	*1.57*	0.00097656
DN I											
0.25	**5**		**10**		**10**		**10**		**10**		0.0625
0.125	**5**	*1.00*	**24**	*2.40*	**24**	*2.40*	**24**	*2.40*	**24**	*2.40*	0.015625
0.0625	**5**	*1.00*	**51**	*2.13*	**61**	*2.54*	**63**	*2.63*	**63**	*2.63*	0.00390625
0.03125	**5**	*1.00*	**73**	*1.43*	**120**	*1.97*	**130**	*2.06*	**132**	*2.10*	0.00097656
DN II											
0.25	**10**		**4**		**3**		**2**		**1**		0.0625
0.125	**24**	*2.40*	**6**	*1.50*	**3**	*1.00*	**2**	*1.00*	**2**	*2.00*	0.015625
0.0625	**62**	*2.58*	**9**	*1.50*	**3**	*1.00*	**2**	*1.00*	**2**	*1.00*	0.00390625
0.03125	**160**	*2.58*	**15**	*1.67*	**3**	*1.00*	**2**	*1.00*	**2**	*1.00*	0.00097656

NUMERICAL RESULTS (II)

We consider the following setting:



- $\Omega_f = 0.015 \times 0.005$ m
- $\Omega_d = 0.015 \times 0.0025$ m
- Inflow: parabolic profile with max horizontal velocity 0.1 m/s
- We impose $p_d = 0$ kg/(ms^2) on the bottom of the domain Ω_d
- Permeability k varies between 10^{-6} and 10^{-14} m^2
- Characteristic quantities: $X_f = 0.005$ m, $U_f = 0.1$ m/s

PCG iterations taking the fluid as water:

Uf	0.1	tol=1.e-10									
Xf	0.005	Nk>h2									
alfa	0.5										
	dyn. Visc.	density	kin. Visc.	Re							
Water	1.00E-03	1000	0.000001	500							
permeab. (k)	1.00E-06		1.00E-08		1.00E-10		1.00E-12		1.00E-14		
Nk	4.00E-02		4.00E-04		4.00E-06		4.00E-08		4.00E-10		
Nbjs	5.00E-03		5.00E-02		5.00E-01		5.00E+00		5.00E+01		
Iterations											
no prec											
0.25	**16**		**28**		**32**		**32**		**32**		0.0625
0.125	**23**	*1.44*	**36**	*1.29*	**74**	*2.31*	**76**	*2.38*	**76**	*2.38*	0.015625
0.0625	**34**	*1.48*	**32**	*0.89*	**116**	*1.57*	**150**	*1.97*	**150**	*1.97*	0.00390625
0.03125	**48**	*1.41*	**32**	*1.00*	**122**	*1.05*	**224**	*1.49*	**224**	*1.49*	0.00097656
DN I											
0.25	**12**		**33**		**35**		**36**		**35**		0.0625
0.125	**12**	*1.00*	**45**	*1.36*	**98**	*2.80*	**103**	*2.86*	**104**	*2.97*	0.015625
0.0625	**12**	*1.00*	**49**	*1.09*	**206**	*2.10*	**250**	*2.43*	**244**	*2.35*	0.00390625
0.03125	**12**	*1.00*	**50**	*1.02*	-		-		-		0.00097656
DN II											
0.25	**35**		**13**		**5**		**3**		**2**		0.0625
0.125	**99**	*2.83*	**27**	*2.08*	**6**	*1.20*	**3**	*1.00*	**2**	*1.00*	0.015625
0.0625	**250**	*2.53*	**63**	*2.33*	**10**	*1.67*	**3**	*1.00*	**2**	*1.00*	0.00390625
0.03125	-		**150**	*2.38*	**18**	*1.80*	**3**	*1.00*	**2**	*1.00*	0.00097656

PCG iterations taking the fluid as oil:

Uf	0.1	tol=1.e-10									
Xf	0.005	Nk>h2									
alfa	0.5										
	dyn. Visc.	**density**	**kin. Visc.**	**Re**							
Olive oil	0.1	900	0.000111	4.5							
permeab. (k)	1.00E-06		1.00E-08		1.00E-10		1.00E-12		1.00E-14		
Nk	4.00E-02		4.00E-04		4.00E-06		4.00E-08		4.00E-10		
Nbjs	5.56E-01		5.56E+00		5.56E+01		5.56E+02		5.56E+03		
Iterations no prec											
0.25	**16**		**28**		**32**		**32**		**32**		0.0625
0.125	**23**	*1.44*	**36**	*1.29*	**74**	*2.31*	**76**	*2.38*	**76**	*2.38*	0.015625
0.0625	**34**	*1.48*	**32**	*0.89*	**115**	*1.55*	**146**	*1.92*	**149**	*1.96*	0.00390625
0.03125	**48**	*1.41*	**32**	*1.00*	**120**	*1.04*	**224**	*1.53*	**226**	*1.52*	0.00097656
DN I											
0.25	**12**		**33**		**35**		**35**		**35**		0.0625
0.125	**12**	*1.00*	**44**	*1.33*	**98**	*2.80*	**106**	*3.03*	**103**	*2.94*	0.015625
0.0625	**12**	*1.00*	**50**	*1.14*	**202**	*2.06*	**250**	*2.36*	**250**	*2.43*	0.00390625
0.03125	**12**	*1.00*	**50**	*1.00*	-	-	-		-		0.00097656
DN II											
0.25	**34**		**13**		**5**		**3**		**2**		0.0625
0.125	**94**	*2.76*	**27**	*2.08*	**6**	*1.20*	**3**	*1.00*	**2**	*1.00*	0.015625
0.0625	**250**	*2.66*	**61**	*2.26*	**10**	*1.67*	**3**	*1.00*	**2**	*1.00*	0.00390625
0.03125	-	-	**146**	*2.39*	**19**	*1.90*	**4**	*1.33*	**2**	*1.00*	0.00097656

[Discacciati (2012, in preparation)]

AN EXAMPLE IN HEMODYNAMICS

Take a generic artery with parameters [Zunino (2002)]

- radius = 0.3 cm
- wall thickness = 0.03 cm
- blood density $\rho_b = 1.04$ g/cm^3
- blood viscosity $\nu_b = 0.033$ cm^2/s
- wall permeability $k = 2 \cdot 10^{-14}$ cm^2

This corresponds to

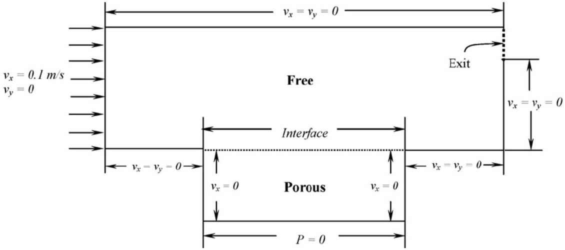
- $\mathrm{Re}_f = 272.73$
- $\mathrm{N}_k = 2.78 \cdot 10^{-14}$

We solve the coupled problem using the preconditioner $\widetilde{\boldsymbol{\Sigma}}_d$ and we obtain

Dofs ($\mathbf{u}_f + p_f + p_d$)	h	No prec.	PCG with $\widetilde{\boldsymbol{\Sigma}}_d$
259 + 73 + 138	0.25	32	1
973 + 259 + 495	0.125	76	1
3769 + 973 + 1869	0.0625	147	2
14833 + 3769 + 7257	0.03125	226	2

APPLICATION TO A TIME DEPENDENT CASE

We consider the setting proposed in Hanspal (2009):



The permeability k varies with time.
We perform time-discretization with a backward Euler method and at each time-step we solve the preconditioned Schur complement system with preconditioner either $\widetilde{\boldsymbol{\Sigma}}_s$ or $\widetilde{\boldsymbol{\Sigma}}_d$.

[Ahmed, master thesis (2012)]

COMMENTS

- When using $\widetilde{\boldsymbol{\Sigma}}_d$, we precondition the Schur complement system by a Laplace problem:

$$\mathbf{y} = \widetilde{\boldsymbol{\Sigma}}_d^{-1}\mathbf{w} \quad \Leftrightarrow \quad y = -\tfrac{k}{\nu}\partial_n\psi \text{ on } \Gamma \quad \text{where}$$

$$\begin{array}{rcll} -\mathrm{div}(\frac{k}{\nu}\nabla\psi) & = & 0 & \text{in } \Omega_d \\ \psi & = & w & \text{on } \Gamma \end{array}$$

- When using $\widetilde{\boldsymbol{\Sigma}}_s$, we precondition the Schur complement system by a Stokes problem:

$$\mathbf{y} = \widetilde{\boldsymbol{\Sigma}}_s^{-1}\mathbf{w} \quad \Leftrightarrow \quad y = \mathbf{u}_f \cdot \mathbf{n} \text{ on } \Gamma \quad \text{where}$$

$$\begin{array}{rcll} Stokes(\mathbf{u}_f, p_f) & = & 0 & \text{in } \Omega_f \\ -\mathbf{n}\cdot \mathrm{T}(\mathbf{u}_f, p_f)\cdot\mathbf{n} & = & \mathbf{w} & \text{on } \Gamma \end{array}$$

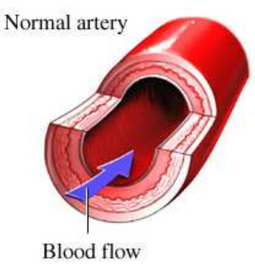
MODELING FSI IN LARGE VESSELS

Joint work with S. Deparis (EPFL), G. Fourestey (CSCS)

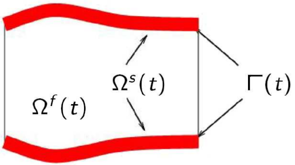
and A. Quarteroni (EPFL and MOX-Milano)

FLUID-VESSEL INTERACTION – PHYSICAL SETTING

Arterial structure



Schematic representation



- Blood flow equations (ALE formulation):

$$\left.\begin{array}{rcl}\rho_f\left(\partial_t \mathbf{u}_f + (\mathbf{u}_f \cdot \nabla)\mathbf{u}_f\right) - \boldsymbol{\nabla} \cdot \mathsf{T}(\mathbf{u}_f, p_f) & = & \mathbf{f} \\ \nabla \cdot \mathbf{u}_f & = & 0\end{array}\right. \quad \text{in } \Omega_f$$

- Vessel equation (Lagrangian formulation):

$$\partial^2_{tt}\mathbf{d} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\mathbf{d}) = f(\mathbf{d}) \quad \text{in } \Omega_s$$

where $\boldsymbol{\sigma}_s(\mathbf{d}) = \mu^l(\nabla \mathbf{d} + \nabla^T \mathbf{d}) + \lambda^l \nabla \cdot (\mathbf{d})\mathsf{I}$.

- Coupling equations:

$$\left.\begin{array}{l}\boldsymbol{\sigma}_s(\mathbf{d}) \cdot \mathbf{n} = \mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n} \\ \mathbf{u}_f = \partial_t \mathbf{d}\end{array}\right. \quad \text{on } \Gamma$$

THE NONLINEAR INTERFACE EQUATION

Consider the coupled problem at a given time $t = t^{n+1}$.
We choose as interface variable the displacement $\lambda(t) = \mathbf{d}_{|\Gamma}(t)$ of the fluid-structure interface, and we define the nonlinear interface operators:

- S_f as the *fluid map*

$$S_f : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \qquad \lambda \to \sigma_f(\lambda)$$

i.e. solve the Navier-Stokes problem with b.c. $\mathbf{u}_{f\,|\Gamma} = (\lambda - \mathbf{d}_{|\Gamma})/\delta t$ on Γ and recover the normal stress $\sigma_f = (\mathrm{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n})_{|\Gamma}$.

- S_s as the *structure map*

$$S_s : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \qquad \lambda \to \sigma_s(\lambda),$$

i.e. solve a structure problem with b.c. $\mathbf{d}_{|\Gamma} = \lambda$ on Γ and recover the normal stress $\sigma_s = \boldsymbol{\sigma}_s(\mathbf{d}) \cdot \mathbf{n}$ on Γ.

- The coupled fluid-structure problem can be expressed in terms of the solution λ of the following nonlinear Steklov-Poincaré interface problem:

$$\boxed{\text{find } \lambda \in H^{1/2}(\Gamma): \quad S_f(\lambda) + S_s(\lambda) = 0.}$$

- Another possible formulation (commonly used for FSI problems):

$$\boxed{\text{find } \lambda: \; S_s^{-1}(-S_f(\lambda)) - \lambda = 0 \text{ on } \Gamma}$$

See also [Badea, Discacciati, Quarteroni (2010)]

ITERATIVE METHODS FOR THE INTERFACE EQUATION

We consider a preconditioned Richardson scheme:

`Given` λ^0, `for` $k \geq 0$,

1.	`compute` $\sigma_s^k = \mathcal{S}_s(\lambda^k)$;	$\Leftrightarrow$	1 structure solve
2.	`compute` $\sigma_f^k = \mathcal{S}_f(\lambda^k)$;	$\Leftrightarrow$	1 fluid solve
3.	$r^k = -(\sigma_f^k + \sigma_p^k)$;		
4.	`solve` $P_k \mu^k = r^k$;	$\Leftrightarrow$	P_k maps the space $H^{1/2}(\Gamma)$ (diplacements) onto $H^{-1/2}(\Gamma)$ (normal stresses)
5.	`update` $\lambda^{k+1} = \lambda^k + \omega^k \mu^k$.		

At each step k, we require to solve *separately* the fluid and the structure problems and then to apply a scaling operator.

We define the generic linear operator:

$$P_k^{-1} = \alpha_f^k \, S_f'(\lambda^k)^{-1} + \alpha_s^k \, S_s'(\lambda^k)^{-1}$$

for two given scalars α_f^k and α_s^k. In particular, we have:

Dirichlet-Neumann:	$P_{DN} = S_s'(\lambda^k)$	(for $\alpha_f^k = 0$, $\alpha_s^k = 1$)
Neumann-Dirichlet:	$P_{ND} = S_f'(\lambda^k)$	(for $\alpha_f^k = 1$, $\alpha_s^k = 0$)
Neumann-Neumann:	$P_{NN}^{-1} = P_k^{-1}$	(if $\alpha_f^k + \alpha_s^k = 1$, $\alpha_f^k, \alpha_s^k \neq 0$)

- Numerical tests show that these preconditioners give convergence with a rate independent of h, but comparable to a fixed point method, usually quite slow!
- Other preconditioners based on Robin-type operators can be found in Nobile, Vergara (2008), Badia, Nobile, Vergara (2009), Yang, Zulehner (2011), Yang (2012).

NEWTON METHOD FOR THE INTERFACE EQUATION

- The Newton method for the interface equation is obtained considering

$$\boxed{P_{Newton} = P_k = S'_f(\lambda^k) + S'_s(\lambda^k)}$$

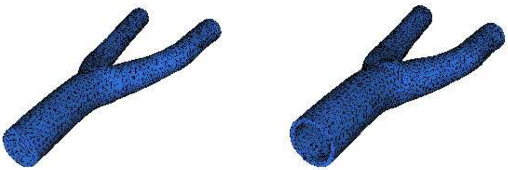
- Algorithm:

$$\begin{aligned}&\text{solve } [\mathcal{S}'_f(\lambda^k) + \mathcal{S}'_s(\lambda^k)]\mu^k = -(\mathcal{S}_f(\lambda^k) + \mathcal{S}_s(\lambda^k))\\ &\text{update } \lambda^{k+1} = \lambda^k + \omega^k \mu^k\end{aligned}$$

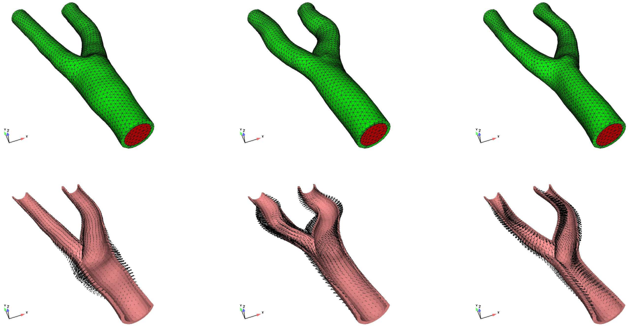
- Remark that this is not equivalent to a Newton method on the problem $S_s^{-1}(-S_f(\lambda)) - \lambda = 0$ on Γ, since now all the steps can be performed in parallel.

NUMERICAL RESULTS

Using this preconditioner we simulate a pressure wave in the carotid bifurcation:
linear structure with thickness 0.5 *mm* and inflow diameter of 0.67 *cm*;
fluid viscosity $\mu = 0.03$ *poise*,
densities $\rho_f = 1\ g/cm^3$ and $\rho_s = 1.2\ g/cm^3$.
Initially at rest and a normal stress of $1.3332 \cdot 10^4\ dynes/cm^2$ is imposed on the inlet for $3 \cdot 10^{-3}\ s$.



[see Karner, Perktold, Hofer, Liepsch (1999)]



$t = 5ms$ $t = 10ms$ $t = 15ms$

	$\delta t = 0.001$			$\delta t = 0.0005$		
Method	FS eval	FS' eval	CPU time	FS eval	FS' eval	CPU time
Newton	3	7.5	8h51'	3	10	19h41'
DD-Newton*	3	7.5	8h12'	3	10	19h33'

* sequential computations [Deparis et al. (2006)]

SUMMARIZING

- DD methods allow us to reduce coupled multiphysics problems to an interface equation
- This approach is suitable for setting up parallel substructuring methods
- The study of the interface operators helps characterizing effective preconditioners
- Preconditioners based on 'ad-hoc' reduced models would reduce the computational cost

- ¿ Effective strategies for nonlinear problems ?