Universität Hamburg

# Pointwise convergence of the feasibility violation for Moreau-Yosida regularized optimal control problems.

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# Outline of the talk

1. Pointwise Convergence
   - Introduction & Previous Results
   - Error Estimates
   - Examples

2. Application to Gradient Constraints on the State
   - Motivation
   - Existence & Optimality Conditions
   - A Priori Error Estimation

# Content

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# Model Problem

Consider $\Omega \subset \mathbb{R}^2$:

$$(P) \quad \begin{aligned} \text{Minimize}_{Q\times V} \; J(q,u) &:= \frac{1}{2}\|u - u^d\|^2 + \frac{1}{2}\|q\|^2 \\ \text{s.t.} \quad -\Delta u &= q \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \\ u &\geq u^c \text{ in } \overline{\Omega} \end{aligned}$$

and its regularization:

$$(P_\gamma) \quad \begin{aligned} \text{Minimize}_{Q\times V} \; J(q,u) &+ \frac{\gamma}{2}\|(u^c - u)^+\|^2 \\ \text{s.t.} \quad -\Delta u &= q \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

(Assume throughout the existence of a Slater point)

# Known Results

Various contributions: Ito, Kunisch, Hintermüller, Hinze, M. Ulbrich, ...

Assuming that any solution $u \in C(\overline{\Omega})$ (in fact $u \in C^{0,\beta}(\overline{\Omega})$)
gives: $(P_\gamma) \to (P)$ as $\gamma \to \infty$

Sometimes even rates for the convergence:
Depends on the feasibility violation to vanish in appropriate norms.

(Hintermüller, Hinze): $\|\overline{q} - \overline{q}_\gamma\|^2 \le C(\gamma^{-1} + \|(u^c - \overline{u}_\gamma)^+\|_\infty) \le c\gamma^{-\frac{\beta}{2+2\beta}}$

(M. Ulbrich): $\|\overline{q} - \overline{q}_\gamma\|^2 \le c\gamma^{\frac{4}{13}+\epsilon}$

Looks independent of $\beta$ but needs $H^2$ regularity

# A Simple Estimate

## Lemma

*Let $(\overline{q}, \overline{u})$ be the solution to $(P)$ and $(\overline{q}_\gamma, \overline{u}_\gamma)$ the solution to $(P_\gamma)$. Assume that it holds*

$$\|(u^c - \overline{u}_\gamma)^+\|_\infty \leq c\gamma^{-\theta}$$

*for some $\theta > 0$ then*

$$0 \leq J(\overline{q}, \overline{u}) - J(\overline{q}_\gamma, \overline{u}_\gamma) \leq c\gamma^{-\theta}.$$

*and*

$$\|\overline{q} - \overline{q}_\gamma\|^2 \leq c\gamma^{-\theta}.$$

*hold.*

## Lemma

*For the solution $(\overline{q}_\gamma, \overline{u}_\gamma)$ of $(P_\gamma)$ it holds*

$$\|(u^c - \overline{u}_\gamma)^+\|^2 \leq c\gamma^{-1}\|(u^c - \overline{u}_\gamma)^+\|_\infty \leq c\gamma^{-1-\theta}$$

# A Bootstrapping Argument

## Lemma

*Let $(\overline{q}_\gamma, \overline{u}_\gamma)$ be a solution of $(P_\gamma)$. Assume that $u^c - \overline{u}_\gamma \in C^{0,\beta}(\overline{\Omega})$ and that for some $\alpha > 0$ it holds*

$$\|(u^c - \overline{u}_\gamma)^+\|^2 \leq c\gamma^{-\alpha}$$

*then we have*

$$\|(u^c - \overline{u}_\gamma)^+\|_\infty \leq c\gamma^{-\alpha\frac{\beta}{2+2\beta}}.$$

## Theorem

*Under the assumptions above it holds*

$$\|(u^c - \overline{u}_\gamma)^+\|^2 \leq c\gamma^{-1-\frac{\beta}{2+\beta}}$$

*then we have*

$$\|(u^c - \overline{u}_\gamma)^+\|_\infty \leq c\gamma^{-\frac{\beta}{2+\beta}}.$$

# Comments

Results can be improved depending on the shape of the active set.

Maximum attained on a line:

$$\|(u^c - \overline{u}_\gamma)^+\|_\infty \le c\gamma^{-\frac{\beta}{1+\beta}}.$$

Maximum attained on a volume:

$$\|(u^c - \overline{u}_\gamma)^+\|_\infty \le c\gamma^{-1}.$$

# Example with a point measure (Cherednichenko, Krumbiegel, Rösch, 2008)

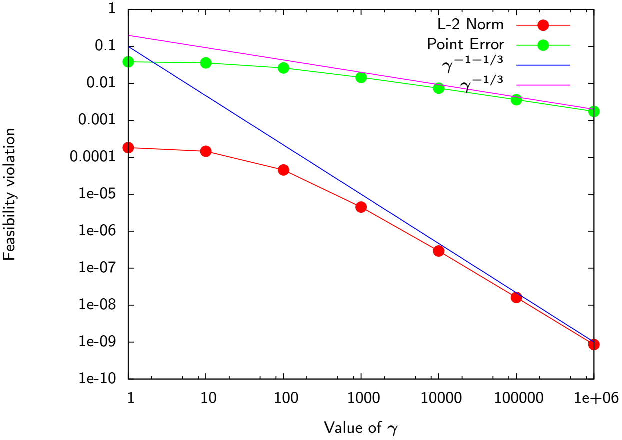


Figure: Convergence of $\|(u^c - \overline{u}_\gamma)^+\|_\infty$

# Example with a line measure (Benedix, Vexler, 2009)

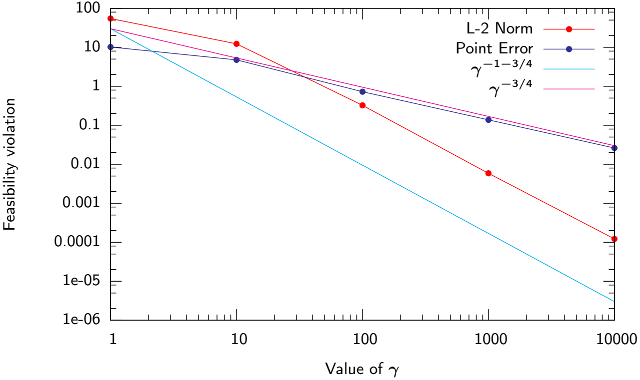


Figure: Convergence of $\|(u^c - \overline{u}_\gamma)^+\|_\infty$

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# Model Problem

Consider:

$$
(P) \quad
\begin{aligned}
\text{Minimize}_{Q\times V} \; & \frac{1}{2}\|u - u^d\|^2 + \frac{\alpha}{r}\|q\|_{L^r}^r \\
\text{s.t.} \quad -\Delta u &= q \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega \\
|\nabla u|^2 &\leq c \text{ in } \overline{\Omega}
\end{aligned}
$$

- Discretize with FEM
- A priori error analysis: Deckelnick, Günther, Hinze; Ortner, Wollner
- $\|q - q_h\|^r = O(h^{1-2/t})$
- Need a 'smooth' domain to have $u \in C^1(\overline{\Omega})$.
- What if $\Omega \subset \mathbb{R}^2$ is not smooth but is a (nonconvex) polygon?

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# Existence

The solution $u$ of the state equation can be rewritten (one corner, a.a. $t$):

$$u = u^r + u^s$$

- $u^r \in W^{2,t}(\Omega) \cap H_0^1(\Omega) \subset C^1(\overline{\Omega})$ $(t > 2)$
- $u^s = \sum_{\substack{j=1 \\ \frac{j\pi}{\omega_c} \neq 1}}^{j < \frac{2\omega_c}{\pi t'}} \alpha_j r^{j\pi/\omega} \varphi(\theta)$
- $u \in W^{1,\infty}(\Omega)$ implies $\alpha_j = 0$

And $-\Delta : W := W^{2,t}(\Omega) \cap H_0^1(\Omega) \mapsto I = \Delta W$ has a continuous inverse.

$\to$ Solution to $(P)$ by standard arguments on the control space $Q = I \cap L^r(\Omega)$.

# Necessary Conditions

For $(\overline{q},\overline{u}) \in I \times W$ there exist $\overline{\mu} \in C(\overline{\Omega})^*$ and a function $\overline{z} \in L^{t'}(\Omega)$ s.t.

$$
\begin{aligned}
(\nabla\overline{u},\nabla\varphi) &= (\overline{q},\varphi) && \forall\ \varphi \in H_0^1(\Omega),\\
\langle -\Delta\varphi,\overline{z}\rangle &= (\overline{u}-u^d,\varphi) + \langle\overline{\mu},\nabla\overline{u}\nabla\varphi\rangle && \forall\ \varphi \in W,\\
(\overline{q}|\overline{q}|^{r-2},\delta q) &= -\langle\delta q,\overline{z}\rangle && \forall\,\delta q \in Q \cap I,\\
\langle\overline{\mu},\varphi\rangle &\le 0 && \forall\ \varphi \in C(\overline{\Omega}), \varphi \le 0,\\
0 &= \langle\overline{\mu},|\nabla\overline{u}|^2 - c\rangle.
\end{aligned}
$$

Note that $\overline{z}$ is determined modulo $I^\perp$ only.

$\rightarrow \overline{q} \in W^{\frac{1-2/t}{r-1},r}(\Omega)$

# Discretization & Simple Error Estimates

$$(P_h^{\perp}) \quad \begin{aligned} &\text{Minimize}_{I\times V_h}\ J(q_h,u_h)\\ &\text{s.t.}\ (\nabla u_h,\nabla\varphi) = (q,\varphi)\,\forall\varphi\in V_h\\ &|\nabla u_h|^2 \le c \text{ in } \overline{\Omega}\end{aligned}$$

We follow the ideas of Ortner, Wollner Numer. Math. (2011)

## Theorem

*Let $(q,u)$ be solutions to $(P)$ and $(q_h,u_h)$ solutions to $(P_h^{\perp})$ then there exists $t>2$ and a constant $C$ such that for any $\epsilon>0$:*

$$|J(q,u) - J(q_h^{\perp},u_h^{\perp})| \le Ch^{1-2/t-\epsilon} =: Ch^{\beta}$$

*and*

$$\|q-q_h\|_r^r \le Ch^{\beta}$$

# Regularization

$$(P_h)\quad \begin{aligned} &\text{Minimize}_{Q\times V_h}\ J(q_h,u_h)\\ &\text{s.t.}\ (\nabla u_h,\nabla\varphi) = (q,\varphi)\,\forall\varphi\in V_h\\ &|\nabla u_h|^2 \le c \text{ in } \overline{\Omega}\end{aligned}$$

Problem: Continuous solution $u^h$ to $\overline{q}_h$ is not in $W^{1,\infty}$.

$$(P_\gamma)\quad \begin{aligned} &\text{Minimize}_{Q\times V}\ J(q,u) = J(q,u) + \frac{\gamma}{2}\|(|\nabla u|^2 - c)^+\|^2\\ &\text{s.t.}\ (\nabla u,\nabla\varphi) = (q,\varphi)\,\forall\varphi\in V\\ &|\nabla u|^2 \le c \text{ in } \overline{\Omega}\end{aligned}$$

Observation:

$$\|(|\nabla u^h|^2 - c)^+\|^2 \le ch^{2\pi/\omega}$$

Allows to estimate the distance between $(P_h)$ and $(P_\gamma)$.

# Error Estimates

## Theorem

*Assume that*

$$\|(|\nabla u_\gamma^r|^2 - c)^+\|_\infty \leq c\gamma^{-\theta}$$

*holds then*

$$|J(\overline{q}_\gamma, \overline{u}_\gamma) - J(\overline{q}, \overline{u})| \leq c\gamma^{-\theta}$$

## Lemma

*It holds*

$$|\alpha_\gamma| \leq c\gamma^{-\frac{1-\pi/\omega}{1+\pi/\omega}}$$

$$\|(|\nabla u_\gamma^r|^2 - c)^+\|_\infty \leq c\gamma^{-\frac{\beta}{1+\beta}\frac{1-\pi/\omega}{1+\pi/\omega}}$$

Note that the rates can be improved depending on the type of the active set.

# Error Estimates - cont.

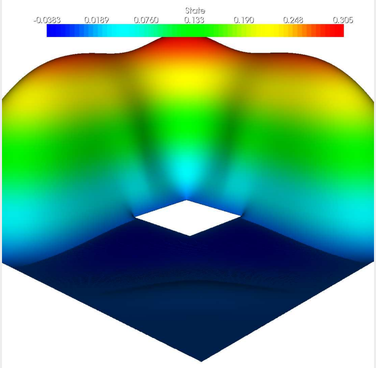
**Theorem**

*It holds for the solution $(\overline{q}_h, \overline{u}_h)$ to $(P_h)$ and $(\overline{q}, \overline{u})$ to $(P)$ that*

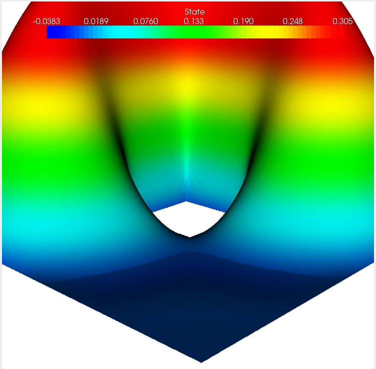
$$|J(\overline{q}, \overline{u}) - J(\overline{q}, \overline{u})| \leq ch^{\beta \frac{2\pi/\omega(1-\pi/\omega)}{1+\pi/\omega+2\beta}}$$

# Example

## Optimal State

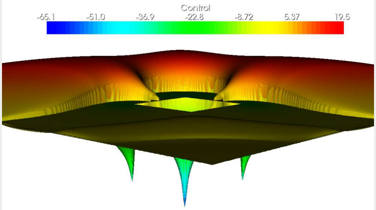



## State Unconstraint

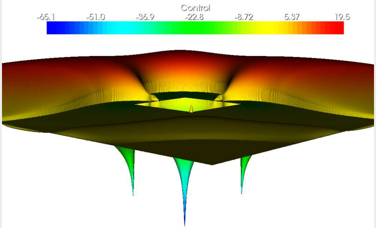


# Example

## Optimal Control on Mesh 5




## Optimal Control on Mesh 6



# Error in the Cost Functional

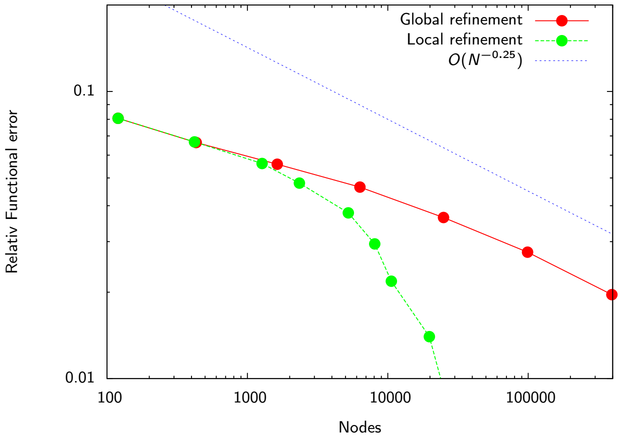



Figure: Error in the cost functional

# Conclusions and Outlook

- New (improved) error estimates for Moreau-Yosida regularized state constraints.
- Shown existence and necessary optimality conditions on non smooth domains.
- Derived error estimates for the discretization error.

Thank you for your attention!

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Thank you for your attention!