Time-optimal control of the wave equation

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Outline

1. **Time-optimal control problem**

2. Penalized problem

3. Parametric problem, numerical results

$$\min \tau$$

subject to the wave equation

$$
\begin{aligned}
y'' - \Delta y &= \chi_\omega u && \text{on } (0,\tau)\times\Omega \\
y &= 0 && \text{on } (0,\tau)\times\Gamma \\
y(0) &= y_1 \\
y'(0) &= y_2
\end{aligned}
$$

the terminal constraints

$$y(\tau) = z_1, \quad y'(\tau) = z_2$$

and the control constraints

$$\|u(t)\|_{L^2(\Omega)} \le \gamma \qquad \text{on } (0,\tau).$$

Data: $\omega \subset \Omega$, $y_1, z_1 \in H^1_0(\Omega)$, $y_2, z_2 \in L^2(\Omega)$, $\gamma > 0$.

Existence of solutions

Theorem: If there is a feasible control, then the time-optimal control problem is solvable.

Important ingredience: control constraints have non-empty interior in $L^\infty(0, T; L^2(\Omega))$.

Box constraints $|u(x,t)| \leq \gamma$*:* Existence of solutions for approximate problem

$$\|y(1) - z\| \leq \epsilon$$

for all $\epsilon > 0$.

References: Fattorini, Lions, Zuazua, Lempio, Leugering, Gugat, ...

Definition: System is *controllable* in time T if there exists for all initial values (y_1, y_2) and terminal values (z_1, z_2) a control, such that the associated state fulfills the initial and terminal conditions.

Geometrical condition: [Bardos, Lebeau, Rauch]

System is controllable in time T if every ray in Ω reflected on Γ hits ω within time T.

Observability: Controllability is equivalent to observability for adjoint equation. [Lions]

The system is controllable in time $T > 0$ if there is a $c > 0$ such that

$$\|p(1)\|_{L^2}^2 + \|p'(1)\|_{H^{-1}}^2 \leq c \int_0^T \int_\omega |p(x,t)|^2 dx\, dt$$

for all solutions p of the adjoint equation $p'' - \Delta p = 0$.

Transformation on fixed interval

Time transformation: $t \to t/\tau$, $I := (0,1)$

Vector notation: $\mathbf{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ \chi_\omega \end{pmatrix}$,

Scaling of velocity component: $\Theta_\tau = \begin{pmatrix} 0 \\ \tau \end{pmatrix}$

Wave equation:

$$\mathbf{y}' = \tau(\mathbf{A}\mathbf{y} + \mathbf{B}u)$$

$$\mathbf{y}(0) = \Theta_\tau \mathbf{y}_0$$

This is a wave equation in the first component of $\mathbf{y}$.

Theorem: Let $\mathbf{y}_0 \in H_0^1(\Omega) \times L^2(\Omega)$, $u \in L^2(0, T; L^2(\omega))$ be given. Then the wave equation admits a unique weak solution $\mathbf{y}$ that satisfies

$$\mathbf{y} \in C(I;\ H_0^1(\Omega) \times L^2(\Omega))$$

with

$$\mathbf{y}_t \in L^2(I;\ L^2(\Omega) \times H^{-1}(\Omega)).$$

$\rightarrow$ *amplitude component one order more regular than velocity component.*

Fixed-interval problem:

$$\min \tau$$

subject to

$$\mathbf{y}' = \tau(\mathbf{A}\mathbf{y} + \mathbf{B}u)$$

$$\mathbf{y}(0) = \Theta_\tau \mathbf{y}_0, \quad \mathbf{y}(1) = \Theta_\tau \mathbf{z},$$

$$\|u(t)\|_{L^2(\Omega)} \leq \gamma \qquad \text{on } (0,1).$$

The resulting problem is clearly **non-linear and non-convex**.

Let $(\tau^*, \mathbf{y}^*, u^*)$ be solutions of the transformed problem. Let the system be controllable for τ^*.

Bang-bang principle: Then there exists $\mathbf{p}^* \neq 0$ that fulfills

$$-\mathbf{p}' = \tau^* \mathbf{A}^* \mathbf{p}$$

and

$$\tau^* (\mathbf{B}^* \mathbf{p},\ u - u^*) \geq 0 \text{ for all admissible } u.$$

[Fattorini]

Note: The adjoint equation is a wave equation in $\mathbf{p}_2$.

Fritz John type condition: Additionally, there is $\lambda_0 \geq 0$ such that

$$\lambda_0 + \langle \mathbf{A}\mathbf{y}^* + \mathbf{B}u^*,\ \mathbf{p}^* \rangle + (y_2, \mathbf{p}_2^*(0))_{L^2} - (z_2, \mathbf{p}_2^*(1))_{L^2} = 0$$

Question: $\lambda_0 > 0$?

1. Time-optimal control problem
2. **Penalized problem**
3. Parametric problem, numerical results

Let $\epsilon > 0$ be given. Consider the penalized problem:

$$\min \tau \left(1 + \frac{\epsilon}{2}\|u\|_{L^2}^2\right) + \frac{1}{2\epsilon}\|\mathbf{y}(1) - \Theta_\tau \mathbf{z}\|_{L^2 \times H^{-1}}^2$$

subject to

$$\begin{aligned} \mathbf{y}' &= \tau(\mathbf{A}\mathbf{y} + \mathbf{B}u), \\ \Theta_\tau \mathbf{y}(0) &= \mathbf{y}_0, \end{aligned}$$

and

$$\|u(t)\|_{L^2(\Omega)} \leq \gamma \qquad \text{on } (0, 1).$$

Let the original time-optimal control problem be solvable with solution $(\tau^*, \mathbf{y}^*, u^*)$.

Convergence: $\tau_\epsilon \to \tau^*$, weak* limits of $(\mathbf{y}_\epsilon, u_\epsilon)$ are solutions of time-optimal control problem.

Sketch of proof:

$$\tau_\epsilon \left(1 + \frac{\epsilon}{2}\|u_\epsilon\|^2_{L^2(I;L^2(\omega))}\right) + \frac{1}{2\epsilon}\|\mathbf{y}_\epsilon(1) - \Theta_{\tau_\epsilon}\mathbf{z}\|^2_{L^2\times H^{-1}} \leq \tau^* \left(1 + \frac{\epsilon}{2}\|u^*\|^2_{L^2(I;L^2(\omega))}\right).$$

$$\limsup \tau_\epsilon \leq \tau^*$$

Necessary optimality condition: There exists $\mathbf{p}_\epsilon$ such that the following system is fulfilled:

$$-\mathbf{p}_{\epsilon,t} = \tau_\epsilon \mathbf{A}^* \mathbf{p}_\epsilon$$

$$\mathbf{p}_\epsilon(1) = \frac{1}{\epsilon} \begin{pmatrix} \mathbf{y}_{\epsilon,1}(1) - z_1 \\ (-\Delta)^{-1}(\mathbf{y}_{\epsilon,2}(1) - \tau_\epsilon z_2) \end{pmatrix}$$

$$\tau_\epsilon(\epsilon u^* + \mathbf{B}^* \mathbf{p}_\epsilon,\ u - u_\epsilon) \geq 0 \text{ for all admissible } u.$$

$$1 + \frac{\epsilon}{2}\|u_\epsilon\|^2_{L^2} + \langle \mathbf{A}\mathbf{y}_\epsilon + \mathbf{B}u_\epsilon,\ \mathbf{p}_\epsilon\rangle + (y_2, \mathbf{p}_{\epsilon,2}(0))_{L^2} - (z_2, \mathbf{p}_{\epsilon,2}(1))_{L^2} = 0$$

Question: Which of these equations remain valid if $\epsilon \to 0$?

Penalization term:

$$\|\mathbf{y}(1)-\Theta_\tau \mathbf{z}\|^2_{L^2\times H^{-1}} = \|y(1)-\mathbf{z}_1\|^2_{L^2}+\Big((-\Delta)^{-1}(y'(1)-\tau\mathbf{z}_2),\, y'(1)-\tau\mathbf{z}_2\Big)_{L^2}$$

$\rightarrow$ velocity component is smoothed

Terminal value of adjoint equation:

$$\mathbf{p}_\epsilon(1) = \frac{1}{\epsilon}\begin{pmatrix} \mathbf{y}_{\epsilon,1}(1)-z_1 \\ (-\Delta)^{-1}(\mathbf{y}_{\epsilon,2}(1)-\tau_\epsilon z_2) \end{pmatrix}$$

$\rightarrow$ amplitude component $\mathbf{p}_2$ is one order smoother than velocity component $\mathbf{p}_1$.

Transversality conditions

Integrated transversality condition:

$$1 + \frac{\epsilon}{2}\|u_\epsilon\|_{L^2}^2 + \langle \mathbf{A}\mathbf{y}_\epsilon + \mathbf{B}u_\epsilon,\ \mathbf{p}_\epsilon\rangle + (y_2, \mathbf{p}_{\epsilon,2}(0))_{L^2} - (z_2, \mathbf{p}_{\epsilon,2}(1))_{L^2} = 0$$

Time derivative of integrand: If $\mathbf{y}_0 \in H^2 \times H_0^1$ then we have

$$\frac{d}{dt}\left(\frac{\epsilon}{2}\|u_\epsilon(t)\|_{L^2}^2 + \langle \mathbf{A}\mathbf{y}_\epsilon(t) + \mathbf{B}u_\epsilon(t),\ \mathbf{p}_\epsilon(t)\rangle\right) = 0$$

Point-wise transversality condition: If $\mathbf{y}_0 \in H^2 \times H_0^1$ then it holds

$$\begin{aligned} 1 + \frac{\epsilon}{2}\|u_\epsilon(t)\|_{L^2}^2 &+ \langle \mathbf{A}\mathbf{y}_\epsilon(t) + \mathbf{B}u_\epsilon(t),\ \mathbf{p}_\epsilon(t)\rangle \\ &+ (y_2, \mathbf{p}_{\epsilon,2}(0))_{L^2} - (z_2, \mathbf{p}_{\epsilon,2}(1))_{L^2} = 0 \quad \forall t \in [0,1]. \end{aligned}$$

Important relation: If $\mathbf{z} \in H^2 \times H_0^1$ then it holds

$$(\mathbf{A}\mathbf{y}_\epsilon(1),\, \mathbf{p}_\epsilon(1))_{L^2(\Omega)^2} = \langle \mathbf{A}\Theta_{\tau_\epsilon}\mathbf{z},\, \mathbf{p}_\epsilon(1)\rangle.$$

$\to$ we can replace $\mathbf{y}_\epsilon(1)$ by $\Theta_{\tau_\epsilon}\mathbf{z}$ here, although in general $\mathbf{y}_\epsilon(1) \neq \Theta_{\tau_\epsilon}\mathbf{z}$!

End-time transversality condition: If $\mathbf{y}_0, \mathbf{z} \in H^2 \times H_0^1$ then we have

$$1 + \frac{\epsilon}{2}\|u_\epsilon(1)\|_{L^2}^2 + \langle \mathbf{A}\Theta_{\tau_\epsilon}\mathbf{z} + \mathbf{B}u_\epsilon(1),\, \mathbf{p}_\epsilon(1)\rangle \\ + (y_2, \mathbf{p}_{\epsilon,2}(0))_{L^2} - (z_2, \mathbf{p}_{\epsilon,2}(1))_{L^2} = 0.$$

$\to$ this condition has the lowest regularity requirements on the solution.

Lemma 1:

$$(u_\epsilon(t), \mathbf{B}^*\mathbf{p}_\epsilon(t))_{L^2(\omega)} = -\epsilon\|u_\epsilon(t)\|^2_{L^2(\omega)} - \gamma\|\epsilon u_\epsilon(t) + \mathbf{B}^*\mathbf{p}_\epsilon(t)\|_{L^2(\omega)}$$

Lemma 2: Let $\mathbf{z} \in H^2 \times H^1_0$. Then there exists $\bar{\epsilon} > 0$ and $\delta > 0$ such that

$$\|\mathbf{p}_\epsilon(1)\|_{L^2 \times H^{-1}} \geq \delta$$

for all $\epsilon \in (0, \bar{\epsilon}]$,

Passing to limit

Let $\tau_\epsilon \to \tau^*$, $(\mathbf{y}_\epsilon, u_\epsilon) \rightharpoonup^* (\tilde{\mathbf{y}}, \tilde{u})$ in $L^\infty(I; H_0^1 \times L^2) \times L^\infty(L^2)$.

Let the system be controllable for some $\tau' < \tau^*$.

Theorem: Assume that $\tilde{u}$ is bang-bang or the sequence $\{\frac{\mathrm{p}_\epsilon(1)}{\|\mathrm{p}_\epsilon(1)\|_{H^{-1}\times L^2}}\}$ is bounded in $L^2 \times H_0^1$. Then the sequence $\{\frac{\mathrm{p}_\epsilon}{\|\mathrm{p}_\epsilon(1)\|_{H^{-1}\times L^2}}\}$ has a subsequence converging weakly* in $L^\infty(I; L^2 \times H^{-1})$ to $\tilde{\mathbf{p}} \neq 0$ that satisfies

$$-\tilde{\mathbf{p}}' = \mathbf{A}^*\tilde{\mathbf{p}}$$

and

$$(\mathbf{B}^*\tilde{\mathbf{p}},\, u - \tilde{u}) \geq 0 \quad \text{for all admissible } u.$$

Controllability is essential to prove $\tilde{p} \neq 0$.

Theorem (ctd): Let in addition $\tilde{u}$ be continuous at $t = 1$.

If $\{\mathbf{p}_\epsilon(1)\}$ is bounded in $L^2 \times H_0^1$ then

$$1 + \langle \mathbf{A}\Theta_{T^*}\mathbf{z} + \mathbf{B}\tilde{u}(1), \tilde{\mathbf{p}}(1)\rangle + (y_2, \tilde{\mathbf{p}}_2(0))_{L^2} - (z_2, \tilde{\mathbf{p}}_2(1))_{L^2} = 0.$$

Otherwise if $\{\frac{p_\epsilon(1)}{\|p_\epsilon(1)\|_{H^{-1}\times L^2}}\}$ is bounded in $L^2 \times H_0^1$ then

$$0 + \langle \mathbf{A}\Theta_{T^*}\mathbf{z} + \mathbf{B}\tilde{u}(1), \tilde{\mathbf{p}}(1)\rangle + (y_2, \tilde{\mathbf{p}}_2(0))_{L^2} - (z_2, \tilde{\mathbf{p}}_2(1))_{L^2} = 0.$$

Missing case: What if the norm $\|\mathbf{p}_\epsilon(1)\|_{L^2\times H_0^1}$ tends faster to infinity than $\|\mathbf{p}_\epsilon(1)\|_{H^{-1}\times L^2}$?

Barbu's technique

Technique from Barbu '84 for *parabolic equations*.

Consider the case $\omega = \Omega$, $\mathbf{z} = 0$, $y_2 = 0$ (from rest to origin). Then $\mathbf{B}^*\mathbf{p} = \mathbf{p}_2$.

The transversality condition reads

$$1 + \frac{\epsilon}{2}\|u_\epsilon(1)\|^2_{L^2(\omega)} + (u_\epsilon(1), \mathbf{p}_\epsilon(1))_{L^2(\Omega)} = 0.$$

By Lemma 1 we obtain

$$\begin{aligned} 1 + \frac{\epsilon}{2}\|u_\epsilon(1)\|^2_{L^2(\Omega)} &= -(u_\epsilon(1), \mathbf{p}_{\epsilon,2}(1))_{L^2(\Omega)} \\ &= \epsilon\|u_\epsilon(1)\|^2_{L^2(\Omega)} + \gamma\|\epsilon u_\epsilon(1) + \mathbf{p}_{\epsilon,2}(1)\|_{L^2(\Omega)} \\ &\geq \epsilon\|u_\epsilon(1)\|^2_{L^2(\Omega)} + \gamma\|\mathbf{p}_{\epsilon,2}(1)\|_{L^2(\Omega)} - \gamma\|\epsilon u_\epsilon(1)\|_{L^2(\Omega)}, \end{aligned}$$

$\rightarrow \{\mathbf{p}_{\epsilon,2}(1)\}$ bounded in $L^2(\Omega)$. What about $\mathbf{p}_{\epsilon,1}$?

Ito & Kunisch '10

Consider time-optimal control of the ODE system

$$y' = Ay + Bu.$$

They proved fulfillment of Zowe-Kurcyusz condition directly for ODE-system (A, B) if

- one component i^* is inactive for a small intervall
- system (A, B_{i^*}) is controllable

Then KKT-System is necessary, which includes transversality condition.

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$$\min \tau(1 + \frac{\epsilon}{2}\|u\|_{L^2}^2) + \frac{1}{2\epsilon}\|\mathbf{y}(1) - \Theta_\tau \mathbf{z}\|_{L^2 \times H^{-1}}^2$$

subject to

$$\begin{aligned} \mathbf{y}' &= \tau(\mathbf{A}\mathbf{y} + \mathbf{B}u), \\ \Theta_\tau \mathbf{y}(0) &= \mathbf{y}_0, \end{aligned}$$

and

$$\|u(t)\|_{L^2(\Omega)} \leq \gamma \qquad \text{on } (0,1).$$

Existence: Problem is strictly convex w.r.t. u, unique solution $u_{\tau,\epsilon}$.

Define

$$\mathcal{V}(\tau) := \tau\left(1 + \frac{\epsilon}{2}\|u_{\tau,\epsilon}\|^2_{L^2}\right) + \frac{1}{2\epsilon}\|\mathbf{y}_{\tau,\epsilon}(1) - \Theta_\tau \mathbf{z}\|^2_{L^2\times H^{-1}}$$

Then

$$\frac{d}{d\tau}\mathcal{V}(\tau) = 1 + \frac{\epsilon}{2}\|u_{\tau,\epsilon}\|^2_{L^2} + \langle \mathbf{A}\mathbf{y}_{\tau,\epsilon} + \mathbf{B}u_{\tau,\epsilon},\ \mathbf{p}_{\tau,\epsilon}\rangle$$
$$+ (y_2, \mathbf{p}_{\tau,\epsilon,2}(0))_{L^2} - (z_2, \mathbf{p}_{\tau,\epsilon,2}(1))_{L^2}$$

Let us assume that there are positive constants τ_0, C_0 such that for all $\tau > \tau_0$ there exists a control $u_{0,\tau}$ that is admissible for the original time-optimal control problem and satisfies

$$\|u_{0,\tau}\|_{L^\infty(I;L^2(\omega))} \leq C_0 \tau^{-1}.$$

Then it holds

$$|\mathcal{V}(\tau) - \tau)| \leq c\ \epsilon\ \tau^{-1},$$

$$\left|\frac{d}{d\tau}\mathcal{V}(\tau) - 1\right| \leq c\ \tau^{-1/2}.$$

Algorithm: Three nested iterations.

- Outer-most loop: adapt ϵ (refine discretization if necessary).
- Middle loop: solve for τ, gradient algorithm for $\mathcal{V}$.
- Inner loop: solve the parametric problem for $u_{\tau,\epsilon}$ for fixed τ, ϵ.

Observations:

- Semi-smooth Newton method applied to penalized problem did not converge (system matrix is non-symmetric)
- Sometimes $\tau = 0$ is a local minimum
- SSN converges faster for pointwise box-constraints then for the constraints used in the talk

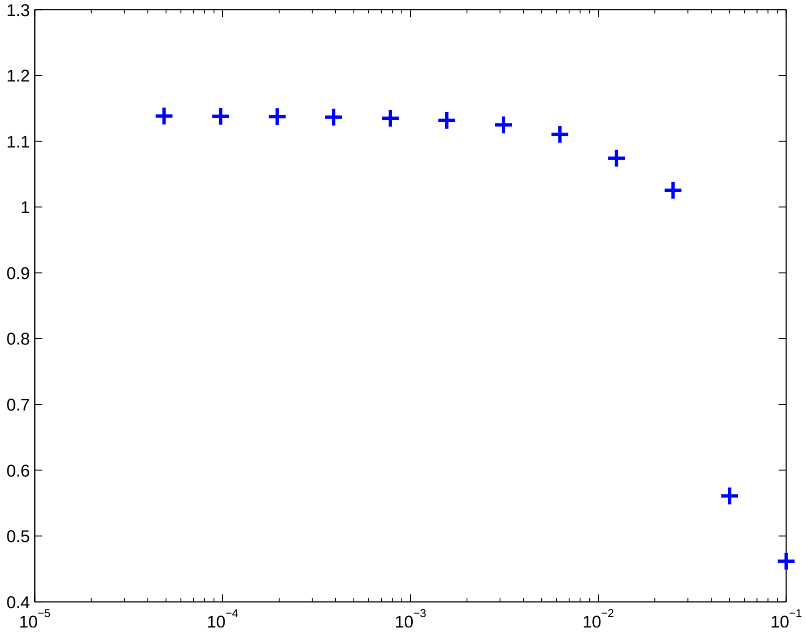
Challenges:

- The function $\mathcal{V}(\tau)$ has many local minima.
- Some of these minima do not disappear for $\epsilon \to 0$.
- When do mesh refinement?

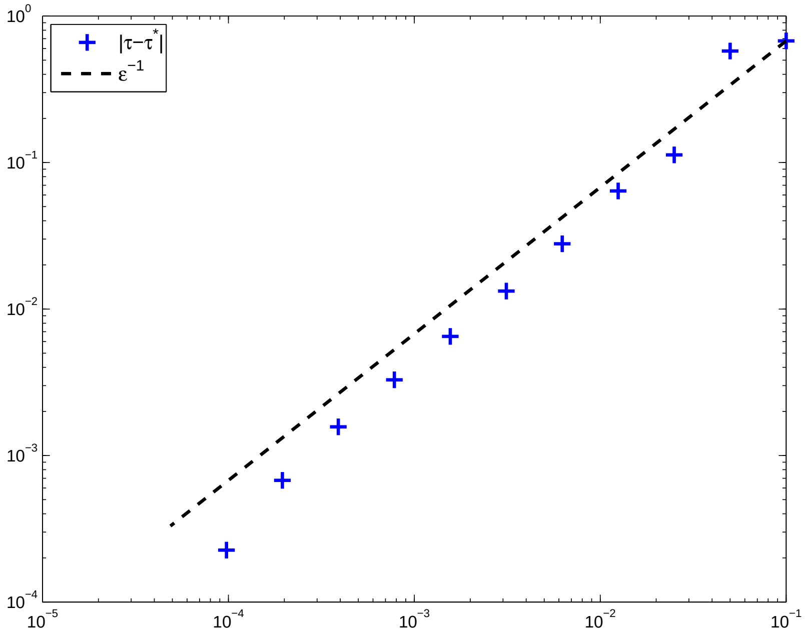
Results

Data: $\Omega = (0,1)^2$, $y_1 = 4xy(1-x)(1-y)$, $y_2 = z_1 = z_2 = 0$, $\gamma = 1$.

Discretization: FEM P1/P0/P0 for amplitude/velocity/controls; C-N



τ_ϵ vs. ϵ



$|\tau_\epsilon - \tau^*|$, ϵ^{-1} vs. ϵ

Observed rates: $|\tau_\epsilon - \tau^*|$, $\|\mathbf{y}_\epsilon(1) - \Theta_{\tau_\epsilon}\mathbf{z}\|_{H^{-1}\times L^2} \sim \epsilon^{-1}$

Boundedness of $\|\mathbf{p}_\epsilon(1)\|_{L^2\times H^1}$

Summary:

- Strict transversality for time-optimal control is an open problem.
- If strict transversality is not fulfilled, the first-order system is under-determined.

Future work:

- Convergence of semi-smooth Newton for the parametric and the penalized problem
- Investigation of structure of parametric problem
- Analysis of convergence rates

Summary:

- Strict transversality for time-optimal control is an open problem.
- If strict transversality is not fulfilled, the first-order system is under-determined.

Future work:

- Convergence of semi-smooth Newton for the parametric and the penalized problem
- Investigation of structure of parametric problem
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Thank you!