A Priori Error Analysis of the Petrov Galerkin Crank Nicolson Scheme for Parabolic Optimal Control Problems

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October 10 - 14, 2011

Optimal control problem

Cost functional

$$\text{Minimize} \quad J(q,u) = \frac{1}{2}\int_0^T \int_\Omega (u(t,x) - \hat{u}(t,x))^2 \, dx \, dt + \frac{\alpha}{2}\int_0^T |q(t)|^2 \, dt$$

State equation

$$\begin{aligned} \partial_t u - \Delta u &= f + Gq && \text{in } (0,T)\times\Omega, \\ u &= 0 && \text{on } (0,T)\times\partial\Omega, \\ u &= u_0 && \text{in } \{0\}\times\Omega \end{aligned}$$

with $G\colon Q = L^2(0,T;\mathbb{R}^D) \to L^2(0,T;H^1(\Omega))$ and

$(Gq)(t,x) = \sum_{i=1}^D q_i(t) g_i(x), \quad g_i \in V = H_0^1(\Omega).$

Control constraints

$$q_a \leq q(t) \leq q_b \quad \text{a.e. in } (0,T).$$

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A priori error analysis

- Temporal discretization of the state
  → discretization parameter k
- Spatial discretization of the state
  → discretization parameter h
- Treatment of the control?

→ Goal:

Error estimate

$$\|\bar{q} - \tilde{q}_{kh}\|_Q = \mathcal{O}(k^2 + h^2).$$

→ optimal control $\bar{q}$ is not smooth (due to control constraints)

→ Crank Nicolson scheme is of second order in k

→ Adjoint scheme?

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Existing literature

- Error estimates for parabolic problems without constraints:
  - Gunzburger, Hou, Hinze, Deckelnick, Chrysafinos, Winther, ...
  - Apel and Flaig 2011 (Crank-Nicolson)
- Error estimates for parabolic problems with control constraints
  - Lasiecka and Malanowski 1977/78, Malanowski 1981
  - Rösch 2004
  - Meidner and Vexler 2008
  - Neitzel and Vexler 2011 (semilinear)
  - Meidner and Vexler 2011 (Petrov-Galerkin-Crank-Nicolson) $\rightarrow \mathcal{O}(k^2 + h^2)$
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Optimality system and regularity

Optimality system

$$\partial_t \bar{u} - \Delta \bar{u} = f + G\bar{q}, \qquad u(0) = u_0$$
$$-\partial_t \bar{z} - \Delta \bar{z} = \bar{u} - \hat{u}, \qquad z(T) = 0$$
$$(\alpha \bar{q} + G^* \bar{z}, \delta q - \bar{q}) \geq 0 \qquad \forall \delta q \in Q_{\text{ad}}$$

→ $\bar{q} = P_{Q_{\text{ad}}}\left(-\frac{1}{\alpha} G^* \bar{z}\right)$ with the pointwise projection $P_{Q_{\text{ad}}} : Q \to Q_{\text{ad}}$

Assumption 1

Ω is polygonal and convex

$$f, \hat{u} \in H^1(0,T;L^2(\Omega)), \quad f(0), \hat{u}(T) \in H_0^1(\Omega) \quad u_0, \Delta u_0 \in H_0^1(\Omega)$$

Regularity

$$\bar{q} \in W^{1,\infty}(0,T;\mathbb{R}^D), \quad \bar{u}, \bar{z} \in H^1(0,T;H^2(\Omega)) \cap H^2(0,T;L^2(\Omega))$$

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Temporal discretization

→ cG(1) Petrov-Galerkin method

- Partitioning of the time interval $\bar{I} = [0, T]$:

$$\bar{I} = \{0\} \cup I_1 \cup I_2 \cup \cdots \cup I_M$$

with subintervals $I_m = (t_{m-1}, t_m]$ of size k_m and time points

$$0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T$$

- Ansatz space (continuous)

$$X_k^1 = \left\{ v_k \in C(\bar{I}, V) \;\middle|\; v_k\big|_{I_m} \in \mathcal{P}_1(I_m, V),\; m = 1, 2, \ldots, M \right\},$$

- Test space (discontinuous)

$$\tilde{X}_k^0 = \left\{ v_k \in L^2(I, V) \;\middle|\; v_k\big|_{I_m} \in \mathcal{P}_0(I_m, V),\; m = 1, 2, \ldots, M,\; v_k(0) \in V \right\}$$

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- Bilinear form:

$$B(u_k,\phi) := (\partial_t u_k,\phi)_{I\times\Omega} + (\nabla u_k,\nabla\phi)_{I\times\Omega} + (u_{k,0},\phi_0^-).$$

- Temporal discretization of the state:

$$u_k \in X_k^1 \;:\; B(u_k,\phi) = (f+Gq,\phi)_{I\times\Omega} + (u_0,\phi_0^-) \quad \forall\phi\in\tilde{X}_k^0.$$

- Temporal discretization of the adjoint state:

$$z_k \in \tilde{X}_k^0 \;:\; B(\phi,z_k) = (u_k-\hat{u},\phi)_{I\times\Omega} \quad \forall\phi\in X_k^1.$$

→ Discrete adjoint state is piecewise constant.

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Temporal discretization

- cG(1) discretization of the state is a variant of the Crank Nicolson scheme
  $\rightarrow$ second order convergence
- Consistent discretization of the adjoint state $\rightarrow$ piecewise constants
  $\rightarrow$ only first order convergence expected

$\rightarrow$ Variational discretization (no control discretization)
  $\rightarrow$ only first order convergence

$$\|\bar{q} - \bar{q}_k\|_Q = \mathcal{O}(k).$$

$\rightarrow$ How to achieve $\mathcal{O}(k^2)$?

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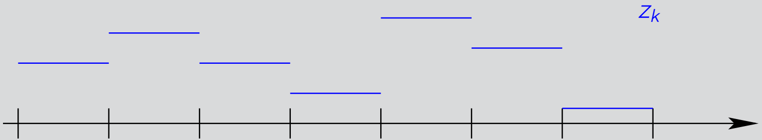
Adjoint discretization

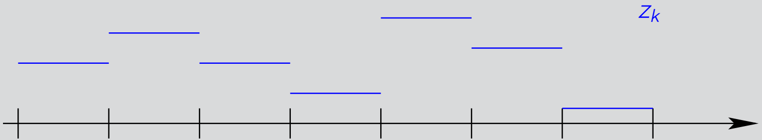


Piecewise linear interpolation at the midpoints

→ Superconvergence result (Meidner & Vexler 2011)

$$\|z - \pi_k z_k\|_{L^2(I\times\Omega)} = \mathcal{O}(k^2)$$

Adjoint discretization



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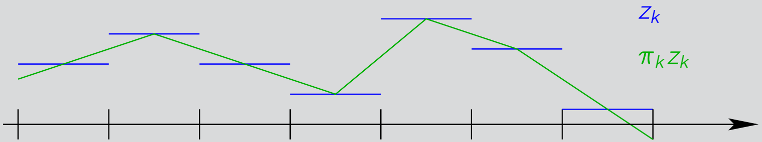

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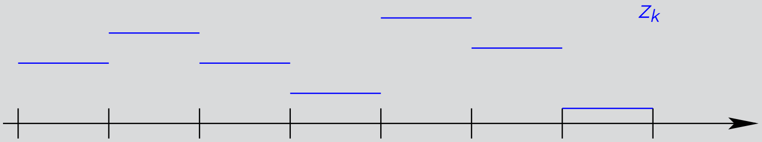

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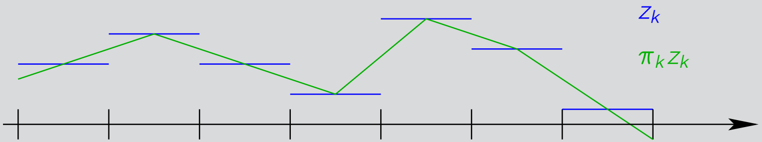

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Theorem (Meidner & Vexler 2011)

Let z be solution of

$$-\partial_t z - \Delta z = g, \quad z(T) = 0$$

with $g \in H^1(0,T;L^2(\Omega))$, $g(T) \in H^1_0(\Omega)$ and $z_k \in \tilde{X}^0_k$ be defined by

$$B(\phi, z_k) = (g, \phi)_{I\times\Omega} \quad \forall \phi \in X^1_k.$$

Then there holds

$$\|z - \pi_k z_k\|_{L^2(I\times\Omega)} \le c\, k^2 \left(\|\partial_t^2 z\|_{L^2(I\times\Omega)} + \|\partial_t \Delta z\|_{L^2(I\times\Omega)} \right)$$

Step 1: Semidiscrete stability estimates

→ diagonal testing is not possible!

$$\|\partial_t v_k\|_{I\times\Omega} + \|P_k \Delta v_k\|_{I\times\Omega} + \|\nabla v_k\|_{I\times\Omega} \le c\|f\|_{I\times\Omega}$$

with L^2-Projection $P_k \colon L^2(0,T;L^2(\Omega)) \to \tilde{X}_k^0$.

Step 2: Supercloseness for the midpoint interpolation

→ duality arguments & stability estimates (step 1)

$$\|\Pi_k z - z_k\|_{L^2(I\times\Omega)} \le c\, k^2 \left(\|\partial_t^2 z\|_{L^2(I\times\Omega)} + \|\partial_t \Delta z\|_{L^2(I\times\Omega)}\right)$$

Step 3: Stability of π_k in $L^2(I\times\Omega)$ for semidiscrete functions

→ π_k is in general not stable w.r.t. $L^2(I\times\Omega)$, but

$$\|\pi_k y_k\|_{I\times\Omega} \le c\|y_k\|_{I\times\Omega} \quad \forall y_k \in \tilde{X}_k^0$$

Superconvergence result: Proof

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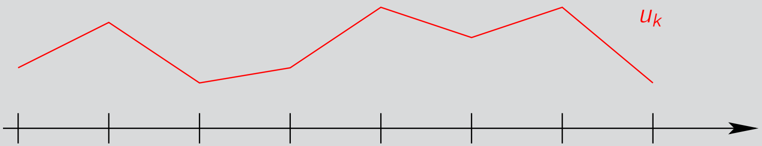

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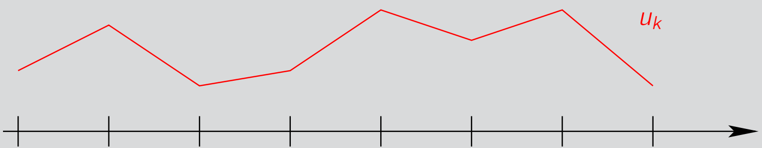
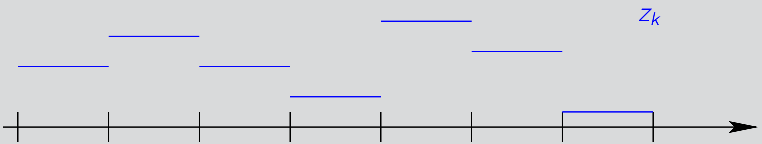
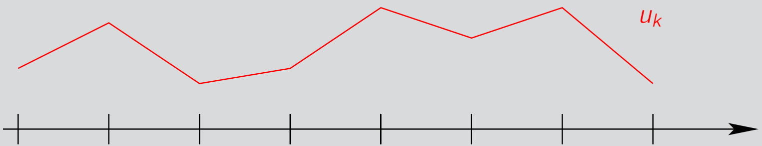


Adjoint and control discretization

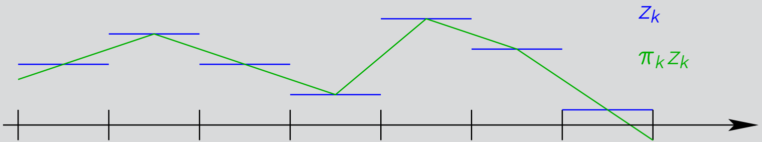


Postprocessing strategy

State discretization

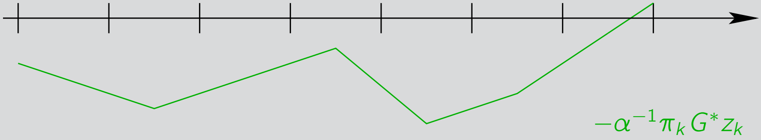


Adjoint discretization and interpolation



Postprocessing strategy

Intermediate step: $-\frac{1}{\alpha}\pi_k G^* z_k$



Postprocessed $\tilde{q}_k = P_{Q_{\text{ad}}}(-\frac{1}{\alpha}\pi_k G^* z_k)$

→ cf. C. Meyer & A. Rösch 2004.

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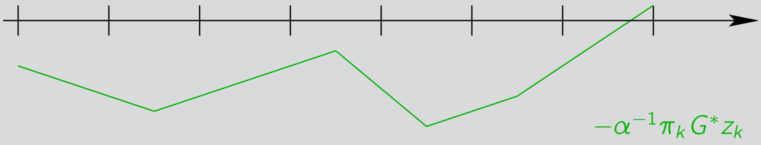

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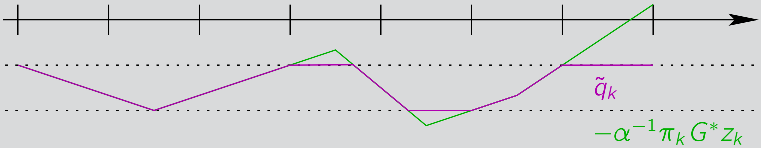

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Postprocessing strategy

Assumption 2

The boundaries of the active sets

$$A_i = \{\, t \in [0, T] \mid q_i(t) = q_{a,i} \;\text{ or }\; q_i(t) = q_{b,i} \,\}, \quad i = 1, 2, \ldots, D$$

consist of a finite number of points.

Theorem

Let Assumptions 1 and 2 be fulfilled. Let $\tilde{q}_k$ be defined as

$$\tilde{q}_k = P_{Q_{\text{ad}}}\left(-\frac{1}{\alpha}\pi_k G^* \bar{z}_k\right).$$

Then there holds:

$$\|\bar{q} - \tilde{q}_k\|_Q = \mathcal{O}(k^2).$$

- D. Meidner and B. Vexler.
  A priori error analysis of the Petrov Galerkin Crank Nicolson scheme for parabolic optimal control problems.
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→ Linear or bilinear finite elements for both state and the adjoint state

- Ansatz space (continuous in time)

$$X_{kh}^1 = \left\{ v_{kh} \in C(\bar{I}, V_h) \;\middle|\; v_k\big|_{I_m} \in \mathcal{P}_1(I_m, V_h),\ m = 1, 2, \dots, M \right\},$$

- Test space (discontinuous in time)

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- Space-time discretization of the state:

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- Space-time discretization of the adjoint state:

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Spatial discretization

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Problem with known exact solution

$$\bar{q}(t) := P_{Q_{ad}}\left(-\frac{\pi^4}{4}\left\{\exp(a\pi^2 t) - \exp(a\pi^2 T)\right\}\right),$$

$$\bar{u}(t,x_1,x_2) := \frac{-1}{2+a}\pi^2 w_a(t,x_1,x_2),$$

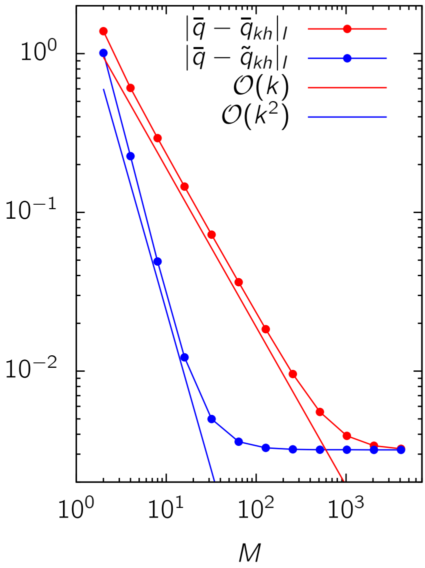
$$\bar{z}(t,x_1,x_2) := w_a(t,x_1,x_2) - w_a(T,x_1,x_2).$$

with

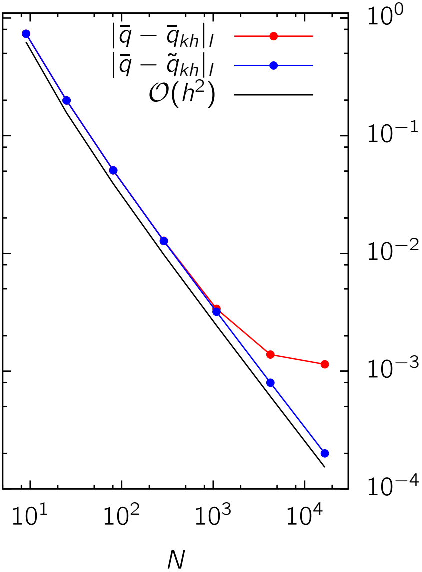
$$w_a(t,x_1,x_2) := \exp(a\pi^2 t)\sin(\pi x_1)\sin(\pi x_2),$$

$$q_a = -70,\ q_b = -1,\ a = -\sqrt{5},\ T = 0.1$$

Numerical example: temporal error



Numerical example: spatial error



Conclusions

→ cG(1) Petrov-Galerkin discretization → Crank Nicolson scheme

→ consistent discretization of the adjoint state

→ second order convergence for the postprocessed control

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