A general theorem on error estimates with application to optimal control

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Joint work with

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Outline

- A control problem for a quasilinear elliptic equation
- Nonconvex optimization problem in Banach space
- Main result on error estimates
- Application to the control problem

An elliptic control problem

(P)

$$\min J(u) \; := \int_\Omega L(x, y_u(x), u(x))\, dx,$$
$$\alpha \leq u(x) \leq \beta \quad \text{for a.e. } x \in \Omega,$$

where y_u is the solution of the quasilinear state equation

$$\begin{array}{rclll} -\text{div}\,[a(x,y(x))\,\nabla y(x)] + f(x,y(x)) & = & u(x) & \text{in} & \Omega \\ y(x) & = & 0 & \text{on} & \Gamma. \end{array}$$

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Under associated assumptions, the mapping $u \mapsto y_u$ is of class C^2 from $L^2(\Omega)$ to $C(\bar{\Omega}) \cap H_0^1(\Omega)$. Therefore, the functional J is of class C^2 in $L^\infty(\Omega)$.

Related work

We are interested in sharp error estimates for a FE approximation of this problem. For elliptic control problems, there is an extensive list of references.

Arada, Casas, T.	*COAP 2002*	*semilinear, distributed control*
Casas, Mateos, T.	*COAP 2005*	*semilinear, boundary control*
Meyer, Rösch	*SICON 2004*	*semilinear, superconvergence*
Hinze	*COAP 2005,*	*linear, variational discretization*
Deckelnick, Hinze	*SINUM 2007*	*state constraints, var. discret.*
Casas, T.	*SICON 2009*	*quasilinear, second-order cond.*
Casas, T.	*ESAIM COCV*	*quasilinear, FE estimates*
Deckelnick, Günter, Hinze	*SICON 2009*	*Dirichlet boundary control*
Casas, Dhamo	*COAP*	*quasilinear, boundary control*

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(A1) Lower semicontinuity of J

$$\{u_k\}_{k=1}^\infty \subset \mathcal{K} \text{ and } u_k \rightharpoonup u \text{ in } U_2 \Rightarrow J(u) \le \liminf_{k\to\infty} J(u_k)$$

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Local solution of $(\mathcal{P})$: $\bar{u} \in \mathcal{K}$ such that, with some $\varepsilon > 0$,

$$J(\bar{u}) \le J(u) \quad \forall u \in \mathcal{K} \cap \{u \in U_\infty : \|u - \bar{u}\|_\infty < \varepsilon\}.$$

If analogously $J(\bar{u}) < J(u)$ for $u \neq \bar{u}$, then $\bar{u}$ is a strict local solution.

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Theorem (First order necessary condition)

If $\bar{u}$ is a local solution of $(\mathcal{P})$ and J is differentiable at $\bar{u}$, both in the sense of U_∞, then there holds the variational inequality

$$J'(\bar{u})(u - \bar{u}) \geq 0 \quad \forall u \in \mathcal{K}.$$

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- $J'(u)$ and $J''(u)$ can be continuously extended to U_2:
  $\exists\, r > 0,\ M_i,\ i = 1, 2$, such that $\forall\, v, v_1, v_2 \in U_\infty,\ u \in B_2(\bar{u}, r) \cap \mathcal{K}$

$$|J'(u)v| \le M_1 \|v\|_2 \quad \text{and} \quad |J''(u)(v_1, v_2)| \le M_2 \|v_1\|_2 \|v_2\|_2$$

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- Continuity of $u \mapsto J'(u)$, $u \mapsto J''(u)$:
  $\forall\, \varepsilon > 0\, \exists\, \delta > 0$ such that $\forall\, u_1, u_2 \in B_\infty(\bar{u}, r),\ v \in U_\infty$

$$\|u_1 - u_2\|_\infty < \delta \quad \Rightarrow \quad \begin{cases} |[J'(u_1) - J'(u_2)]v| \le \varepsilon \|v\|_2, \\ |[J''(u_1) - J''(u_2)]v^2| \le \varepsilon \|v\|_2^2. \end{cases}$$

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- The quadratic form $Q : v \mapsto J''(\bar{u})v^2$, $Q : U_2 \longrightarrow \mathbb{R}$ is a Legendre form.

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Cone of feasible directions:

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Cone of critical directions:

$$C_{\bar u} = cl_2\,(S_{\bar u}) \cap \{v \in U_2 : J'(\bar u)v = 0\}.$$

Theorem (Second-order sufficient condition)

Assume (A2)*; If* $\bar{u} \in \mathcal{K}$ *satisfies the first-order necessary condition and*

$$J''(\bar{u})v^2 > 0 \quad \forall v \in C_{\bar{u}} \setminus \{0\}.$$

Then, there exist $\varepsilon > 0$ *and* $\delta > 0$ *such that*

$$J(u) \geq J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|_2{}^2 \quad \forall u \in \mathcal{K} \cap B_\infty(\bar{u}, \varepsilon)$$

(quadratic growth condition).

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(A3) Approximation of $\mathcal{K}$ by $\mathcal{K}_h$

$\mathcal{K}_h \subset \mathcal{K}$ is convex and closed in U_2. For all $u \in \mathcal{K}$ there exist $u_h \in \mathcal{K}_h$ such that

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(A4) Lower semicontinuity of J_h for all fixed $h > 0$

If $\{u_k\}_{k=1}^\infty \subset \mathcal{K}_h$ and $u_k \rightharpoonup u$ in U_2, then $J_h(u) \le \liminf_{k\to\infty} J_h(u_k)$.

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To guarantee that the family $\{(\mathcal{P}_h)\}_h$ well approximates problem $(\mathcal{P})$, we need two further assumptions. We skip them, because they are standard and not needed for our theorem on error estimates.

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Conversely, we are able to show under appropriate conditions that to each strict local solution $\bar{u}$ of $(\mathcal{P})$ there exists a sequence $\{\bar{u}_h\}_{h>0}$ of local solutions to $(\mathcal{P}_h)$ converging strongly in the norm of U_2 to $\bar{u}$.

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$$|[J_h'(u) - J'(u)]v| \le \varepsilon_h \|v\|_2, \quad \forall (u, v) \in \mathcal{K} \times U_2,$$

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$$\text{if } v_k \rightharpoonup 0 \text{ in } U_2 \text{ then } \liminf_{k\to\infty} J''(u_k)v_k^2 \ge \Lambda \liminf_{k\to\infty} \|v_k\|_2^2.$$

General error estimate

Theorem

Assume (A2), (A3) *and* (A5); *let* $\{\bar{u}_h\}_{h>0}$ *be a sequence of local solutions to* $(\mathcal{P}_h)$ *converging strongly to* $\bar{u}$ *in* U_2. *Under the second-order sufficiency condition at* $\bar{u}$ *there exist* $C > 0$ *such that*

$$\|\bar{u} - \bar{u}_h\|_2^2 \le C\left[\varepsilon_h^2 + \|\bar{u} - u_h\|_2^2 + J'(\bar{u})(u_h - \bar{u})\right]$$

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- The discussion of the last term is delicate.

Sketch of the proof

To show: $$\|\bar{u} - \bar{u}_h\|_2^2 \le C \left[\varepsilon_h^2 + \|\bar{u} - u_h\|_2^2 + J'(\bar{u})(u_h - \bar{u})\right].$$

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- If this is false, with $\{h_k\}_{k=1}^{\infty}$ converging to 0 and $u_{h_k} \in \mathcal{K}_{h_k}$

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Sketch of the proof

To show: $$\|\bar u - \bar u_h\|_2^2 \le C\left[\varepsilon_h^2 + \|\bar u - u_h\|_2^2 + J'(\bar u)(u_h - \bar u)\right].$$

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$$[J'(\bar{u}_h) - J'(\bar{u})](\bar{u}_h - \bar{u}) \le [J_h'(\bar{u}_h) - J'(\bar{u}_h)](\bar{u} - \bar{u}_h)$$

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We apply the mean value theorem to the left-hand side and estimate the right-hand side,

$$\begin{aligned} J''(\hat{u}_h)(\bar{u}_h - \bar{u})^2 \quad &\leq \quad \varepsilon_h(\|\bar{u} - \bar{u}_h\|_2 + \|\bar{u} - u_h\|_2) \\ &\quad +M_2\|\bar{u}_h - \bar{u}\|_2\|\bar{u} - u_h\|_2 + J'(\bar{u})(u_h - \bar{u}). \end{aligned}$$

with some $\hat{u}_h$ in $[\bar{u}_h, \bar{u}]$.

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SSC $\Rightarrow$ $v = 0.$

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Case 2: (A5) implies then (we know that $v_{h_k} \rightharpoonup 0$)

$$0 \ge \liminf_{k\to\infty} J''(\hat{u}_{h_k})v_{h_k}^2 \ge \Lambda \underbrace{\liminf_{k\to\infty} \|v_{h_k}\|_2^2}_{=1} = \Lambda > 0.$$

□

The quasilinear control problem

(P)

$$\min J(u) := \int_\Omega L(x, y(x), u(x))\, dx,$$

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Main assumptions:

- $L(x, y, u)$, $a(x, y)$, $f(x, y)$ are measurable with respect to x and twice differentiable with respect to (y, u).
- Certain (local) Lipschitz properties of L, a, f and their derivatives w.r. to (y, u), L Lipschitz w.r. to x.
- $a(x, y) \ge \delta > 0, \quad f_y(x, y) \ge 0, \quad L_{uu}(x, y, u) \ge \Lambda > 0.$

Application of the general result

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- Error estimates:

Variational discretization	$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega_h)} \le C\, h^2$
Piecewise constant controls	$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega_h)} \le C\, h$
	$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Omega_h)} \le C\, h.$

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Thank you