Internal exponential stabilization to a nonstationary solution for 3D Navier–Stokes equations

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Outline

The equations

The Navier-Stokes system in a 3D bounded domain $\Omega \subseteq \mathbb{R}^3$ with boundary Γ reads:

$$
\begin{aligned}
u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = h + \zeta \quad \text{in} \quad \Omega;\\
\nabla \cdot u = 0 \quad \text{in} \quad \Omega;\\
u\big|_\Gamma = 0;\\
u(0) = u_0.
\end{aligned}
$$

Functional spaces/regularity

- To rewrite the equations as an evolutionary equation in H:

$$H := \{u \in L^2(T\Omega) \mid \nabla \cdot u = 0 \;\;\&\;\; u \cdot \mathbf{n} = 0 \,\mathrm{on}\, \Gamma\};$$
$$V := \{u \in H^1(T\Omega) \mid \nabla \cdot u = 0 \;\;\&\;\; u = 0 \,\mathrm{on}\, \Gamma\};$$
$$U := D(L) = H^2(T\Omega) \cap V.$$

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- Scalar products and norms: let Π be the orthogonal projection in $L^2(T\Omega)$ onto H and let $L = -\nu\Pi\Delta$ be the Stokes operator;

$$(u,\, v)_H := (u,\, v)_{L^2(T\Omega)}, \quad (u,\, v)_V := \langle Lu,\, v\rangle_{V',V},$$
$$(u,\, v)_{\mathrm{D}(L)} := (Lu,\, Lv)_{L^2(T\Omega)}.$$

Fix a function $h \in L^2(\mathbb{R}_+, H)$ and write the system

$$u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = h + \zeta, \qquad \nabla \cdot u = 0 \quad \text{in} \quad \Omega;$$

as an evolutionary equation in the space H of divergence free vector fields H:

$$u_t + Lu + Bu = h + \Pi(\zeta).$$

Goal

Fix $\hat{u} \in L^2(\mathbb{R}_+,\ V) \cap \mathcal{W}$ solving the (non-controlled) Navier–Stokes system

$$\hat{u}_t + L\hat{u} + B\hat{u} = h, \quad t > 0; \quad \hat{u}(0) = \hat{u}_0$$

with $\mathcal{W} := W^{1,\infty}(\mathbb{R}_+,\ W^{1,\infty}(T\Omega))$; an element $u_0 \in H$ and a sub-domain $\omega \subseteq \Omega$.

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Goal: find a finite-dimensional subspace $\mathcal{E} \subset L^2(T\omega)$ and a control $\zeta \in L^2_{\mathrm{loc}}(\mathbb{R}_+,\ \mathcal{E})$ such that the solution of the problem

$$u_t + Lu + Bu = h + \Pi\zeta, \quad u(0) = u_0$$

is defined for all $t > 0$ and converges exponentially to $\hat{u}$, i.e.,

$$|u(t) - \hat{u}(t)|_H \leq \kappa_1 e^{-\kappa_2 t} \quad \text{for } t \geq 0,$$

where κ_1 and κ_2 are non-negative constants; in this case, we say that u converges κ_2-exponentially to $\hat{u}$.

Linearization

The difference $v = u - \hat{u}$ solves

$$v_t + Lv + \mathbb{B}(\hat{u})v + Bv = \Pi\zeta, \quad t > 0; \quad v(0) = v_0 := u_0 - \hat{u}_0;$$

with $\mathbb{B}(\hat{u})v = B(\hat{u}, v) + B(v, \hat{u})$ so, our goal is to find a finite-dimensional subspace $\mathcal{E} \subset L^2(T\omega)$ and a control $\zeta \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathcal{E})$ such that

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We start by considering the linear system

$$v_t + Lv + \mathbb{B}(\hat{u})v = \Pi\zeta, \quad t > 0; \quad v(0) = v_0 := u_0 - \hat{u}_0;$$

with the same goal.

The finite-dimensional space $\mathcal{E}$

- Let $\{\phi_i \mid i \in \mathbb{N}_0\}$ be an orthonormal basis in $L^2(T\Omega)$ formed by the eigenfunctions of the Dirichlet Laplacian and let $0 < \beta_1 \leq \beta_2 \leq \dots$ be the corresponding eigenvalues;

THE FINITE-DIMENSIONAL SPACE $\mathcal{E}$

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- let $\{e_i \mid i \in \mathbb{N}_0\}$ be the orthonormal basis in H formed by the eigenfunctions of the Stokes operator and let $0 < \alpha_1 \leq \alpha_2 \leq \ldots$ be the corresponding eigenvalues.

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- For each $N \in \mathbb{N}_0$, we introduce the N-dimensional subspaces

$$E_N := span\{\phi_i \mid i \leq N\} \subset L^2(T\Omega), \quad F_N := span\{e_i \mid i \leq N\} \subset H$$

and denote by $P_N : L^2(T\Omega) \to E_N$ and $\Pi_N : L^2(T\Omega) \to F_N$ the corresponding orthogonal projections.

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- The required control space can be chosen in the form $\mathcal{E}_M = \chi E_M$, where $\chi \in C_0^\infty(\Omega)$ is a given function not identically equal to zero, and the integer M is sufficiently large. In particular, $\chi E_M \subset C_0^\infty(T\omega)$ for any sub-domain $\omega \subseteq \Omega$ containing $\mathrm{supp}(\chi)$.

Taking controls in $\mathcal{E}_M$ we may rewrite the problem as

$$v_t + Lv + \mathbb{B}(\hat{u})v = \Pi(\chi P_M \eta), \quad v(0) = v_0;$$

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Main result for linear problem

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$$v_t + Lv + \mathbb{B}(\hat{u})v = \Pi(\chi P_M \eta), \quad v(0) = v_0;$$

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Theorem

For each $v_0 \in H$ and $\lambda > 0$, there is an integer $M = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} \geq 1$ and a control $\eta^{\hat{u},\lambda}(v_0) \in L^2(\mathbb{R}_+, E_M)$ such that the solution v of the system satisfies the inequality $|v(t)|_H^2 \leq \kappa |v_0|_H^2 e^{-\lambda t}, \quad t \geq 0$ for some $\kappa = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} > 0$. Moreover, the mapping $v_0 \mapsto e^{(\tilde{\lambda}/2)t} \eta^{\hat{u},\lambda}(v_0)$ is linear and continuous from H to $L^2(\mathbb{R}_+, E_M)$ for all $0 \leq \tilde{\lambda} < \lambda$. Finally, if $v_0 \in V$, then $|v(t)|_V^2 \leq \bar{\kappa} |v_0|_V^2 e^{-\lambda t}, \quad t \geq 0$ for some $\kappa = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} > 0$. The constants κ and $\bar{\kappa}$ do not depend on v_0.

$\overline{C}_{[a_1,\ldots,a_k]}$ denotes a function of non-negative variables a_j that increases in each of its arguments.

Auxiliary lemmas

Let us fix $\tau > 0$ and denote by $S_{\hat{u},\tau}(w_0, \eta)$ the operator that takes the pair (w_0, η) to the solution of

$$v_t + Lv + \mathbb{B}(\hat{u})v = \Pi(\chi P_M \eta),\; t \in I_\tau = (\tau,\, 1+\tau), \quad v(0) = w_0;$$

Lemma

For each $N \in \mathbb{N}$ there is an integer $M = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} \geq 1$ such that, for every $w_0 \in H$ and an appropriate control $\eta \in L^2(I_\tau, E_M)$ we have

$$\Pi_N S_{\hat{u},\tau}(w_0, \eta)(\tau + 1) = 0.$$

Moreover, there is a constant C_χ depending only on $|\hat{u}|_{\mathcal{W}}$ (but not on N and τ) such that

$$|\eta|^2_{L^2(I_\tau,\, E_M)} \leq C_\chi |w_0|^2_H.$$

For the proof: for $\epsilon > 0$ consider the minimization problem.

Problem

Given $M, N \in \mathbb{N}$ and $w_0 \in H$, find the minimum of the quadratic functional $J_\epsilon(v, \eta) := |\eta|^2_{L^2(I_\tau, L^2(T\Omega))} + \frac{1}{\epsilon}|\Pi_N S_{\hat{u},M,\tau}(w_0, \eta)(\tau + 1)|^2_H$ on the set of functions $(v, \eta) \in W(I_\tau, V, V') \times L^2(I_\tau, L^2(T\Omega))$ that solve the system.

- The unique minimizer $(\bar{v}_\epsilon, \bar{\eta}_\epsilon)$ depends linearly on $w_0 \in H$.

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- The unique minimizer $(\bar{v}_\epsilon, \bar{\eta}_\epsilon)$ depends linearly on $w_0 \in H$.
- Using the Karush–Kuhn–Tucker theorem, and making some direct computations, we have that there is a Lagrange multiplier $q^\epsilon \in L^2(I_\tau, V)$ satisfying the time-backward system

$$
\begin{aligned}
q^\epsilon_t - Lq^\epsilon - \mathbb{B}^*(\hat{u})q^\epsilon = 0, \quad t \in I_\tau; \\
q^\epsilon(\tau+1) = -2\epsilon^{-1}\Pi_N \bar{v}^\epsilon(\tau+1);
\end{aligned}
$$

with $2\bar{\eta}_\epsilon = P_M(\chi q^\epsilon)$ and...

$$\int_{I_\tau} |P_M(\chi q^\epsilon)|^2_{L^2(T\Omega)}\, \mathrm{d}t + \epsilon|q^\epsilon(\tau+1)|^2_H = -2(q^\epsilon(\tau),\, \bar{v}^\epsilon(\tau))_H$$

$$\leq \alpha|q^\epsilon(\tau)|^2_H + \alpha^{-1}|\bar{v}^\epsilon(\tau)|^2_H$$

From the truncated observability inequality:

Proposition

For any integer $N \geq 1$ there is $M = \overline{C}_{[N, |\hat{u}|_{\mathcal{W}_\tau}]} \in \mathcal{N}$ such that any solution q for time-backward system
$q_t - Lq - \mathbb{B}^*(\hat{u})q = 0, \quad t \in I_\tau,\ q(\tau+1) = q_1$, with $q_1 \in F_N = \Pi_N H$ satisfies the inequality $|q(\tau)|^2_H \leq D_\chi \int_{I_\tau} |P_M(\chi q)|^2_{L^2(T\Omega)}\, \mathrm{d}t$ for a suitable constant D_χ depending only on χ.

we obtain, setting $\alpha = (2D_\chi)^{-1}$,

$$\int_{I_\tau} |P_M(\chi q^\epsilon)|^2_{L^2(T\Omega)}\, \mathrm{d}t + 2\epsilon|q^\epsilon(\tau+1)|^2_H \leq 4D_\chi|w_0|^2_H.$$

Remark: to proof the proposition: use the finite-dimensionality of F_N and well known obs. ineq. $|q(\tau)|^2_H \leq C_\omega \int_{I_\tau} |q|^2_{L^2(T\omega)}\, \mathrm{d}t$ (Imanuvilov, 2001).

- In particular, we derive that the families $\{\bar{\eta}^\epsilon = \frac{1}{2}P_M(\chi q^\epsilon) \mid \epsilon > 0\}$, $\{\bar{v}^\epsilon \mid \epsilon > 0\}$ and $\{\bar{v}_t^\epsilon \mid \epsilon > 0\}$ are bounded in appropriate spaces.

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- A standard limiting argument shows that there is a limit pair (v^0, η^0) solving
  $v_t^0 + Lv^0 + \mathbb{B}(\hat{u})v^0 = \Pi(\chi P_M \eta^0)$, $t \in I_\tau = (\tau, 1+\tau)$, $\quad v^0(0) = w_0$.
  Furthermore, it follows from above equations that

$$|\Pi_N \bar{v}^\epsilon(\tau+1)|_H^2 = \frac{\epsilon^2}{4}|q^\epsilon(\tau+1)|_H^2 \leq \frac{\epsilon D_\chi}{2}|w_0|_H^2 \to 0 \quad \text{as} \quad \epsilon \to 0.$$

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- We also easily find that

$$|\eta^0|^2_{L^2(I_\tau,\, E_M)} \leq 4D_\chi |w_0|_H^2$$

  and D_χ may be taken independent of τ and N. This ends the proof of the lemma.

In view of latter lemma, it makes sense to consider:

Problem

Given integers $M, N \geq 1$ *and a function* $w_0 \in H$, *find the minimum of the quadratic functional* $J(\eta) := |\eta|^2_{L^2(I_\tau,\, L^2(T\Omega))}$ *on the set of functions* $(v,\, \eta) \in W(I_\tau,\, V,\, V') \times L^2(I_\tau,\, E_M)$ *satisfying*
$v_t + Lv + \mathbb{B}(\hat{u})v = \Pi(\chi P_M \eta)$, $t \in I_\tau$, $v(0) = w_0$ *and* $\Pi_N v(\tau + 1) = 0$.

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Lemma

For any $N \in \mathbb{N}$ *there is an integer* $M = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} \geq 1$ *such that for any* $w_0 \in H$ *the problem has a unique minimizer* $(\bar{v}^{\hat{u},\tau},\, \bar{\eta}^{\hat{u},\tau})$*. Moreover, the mapping* $w_0 \mapsto (\bar{v}^{\hat{u},\tau},\, \bar{\eta}^{\hat{u},\tau})$ *is linear and continuous in the corresponding spaces, and there is a constant* C_χ *depending only on* $|\hat{u}|_{\mathcal{W}}$ *(but not on* N *and* τ*) such that*

$$|\bar{\eta}^{\hat{u},\tau}|^2_{\mathcal{L}(H,\, L^2(I_\tau,\, E_M))} \leq C_\chi.$$

A STABILIZING CONTROL FOR LINEARIZED SYSTEM

- Fix an initial function $v_0 \in H$ and an integer $N = N(\lambda) \geq 1$, and set

$$\eta^{\hat{u},\lambda}(t) = \bar{\eta}^{\hat{u},0}(v_0)(t) \quad \text{for} \quad t \in I_0.$$

Assuming that $\eta^{\hat{u},\lambda}$ is constructed on the interval $(0, n)$ and denoting by $v(t)$ the corresponding solution on $[0, n]$, we define

$$\eta^{\hat{u},\lambda}(t) = \bar{\eta}^{\hat{u},n}(v(n))(t) \quad \text{for} \quad t \in I_n.$$

- By construction, $\eta^{\hat{u},\lambda}$ is an E_M-valued function square integrable on every bounded interval.

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- By construction, $\eta^{\hat{u},\lambda}$ is an E_M-valued function square integrable on every bounded interval.
- The linearity of $\bar{\eta}^{\hat{u},\tau}$ implies that $\eta^{\hat{u},\lambda}$ linearly depends on v_0.
- We claim that, if $N \in \mathbb{N}$ is sufficiently large, then the solution v of $v_t + Lv + \mathbb{B}(\hat{u})v = \Pi(\chi P_M \eta^{\hat{u},\lambda})$, $t \in \mathbb{R}^+$, $v(0) = w_0$, goes λ-exponentially to 0 as $t \to +\infty$.
  Indeed...

- From standard computations we have

$$
\begin{aligned}
|v(1)|_V^2 &\leq \overline{C}_{[|\hat{u}|_{\mathcal{W}}]}(|v_0|_H^2 + 3|\chi|_{L^\infty(\Omega)}^2 |\bar{\eta}^{\hat{u},0}(v_0)|_{L^2(I_0,\, E_M)}^2) \\
&\leq \overline{C}_{[|\hat{u}|_{\mathcal{W}}]}(|v_0|_H^2 + 3|\chi|_{L^\infty(\Omega)}^2 C_\chi |v_0|_H^2).
\end{aligned}
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$$\leq \overline{C}_{[|\hat{u}|_{\mathcal{W}}]}(|v_0|_H^2 + 3|\chi|_{L^\infty(\Omega)}^2 C_\chi |v_0|_H^2).$$

- Since $\Pi_N v(1) = 0$, we obtain $\alpha_N |v(1)|_H^2 \leq |v(1)|_V^2 \leq \overline{C}_{[|\hat{u}|_{\mathcal{W}}]}(\chi)|v_0|^2$. Taking N so large that $\alpha_N \geq e^\lambda \overline{C}_{[|\hat{u}|_{\mathcal{W}}]}(\chi)$, we obtain $|v(1)|_H^2 \leq e^{-\lambda}|v_0|_H^2$. Similarly $|v(n+1)|_H^2 \leq e^{-\lambda}|v(n)|_H^2$. By induction, we see that the solution v corresponding to control $\eta = \eta^{\hat{u},\lambda}$ satisfies the inequality $|v(n)|_H^2 \leq e^{-\lambda n}|v_0|_H^2$:

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$$\le \overline{C}_{[|\hat{u}|_{\mathcal{W}}]}(|v_0|_H^2 + 3|\chi|^2_{L^\infty(\Omega)}C_\chi|v_0|_H^2).$$

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- From this, using some more standard estimates, it is not difficult to derive that $|v(t)|_H^2 \le \overline{C}_{[\lambda,|\hat{u}|_{\mathcal{W}}]}e^{-\lambda t}|v_0|_H^2$ and, if $v_0 \in V$, that $|v(t)|_V^2 \le \overline{C}_{[\lambda,|\hat{u}|_{\mathcal{W}}]}e^{-\lambda t}|v_0|_V^2$.

Finally for any $\tilde{\lambda} < \lambda$, the the continuity of the map $v_0 \mapsto e^{(\tilde{\lambda}/2)t}\eta^{\hat{u},\lambda}$ follows from a simple and direct computation:

$$\begin{aligned}
|e^{(\tilde{\lambda}/2)t}\eta^{\hat{u},\lambda}|^2_{L^2(\mathbb{R}_+,\,E_M)} &= \sum_{n\in\mathbb{N}} |e^{(\tilde{\lambda}/2)t}\bar{\eta}^{\hat{u},n}(v(n))|^2_{L^2(I_n,\,E_M)} \\
&\leq C'_\chi \sum_{n\in\mathbb{N}} e^{\tilde{\lambda}(n+1)}|v(n)|^2_H \\
&\leq C'_\chi e^{\tilde{\lambda}} \sum_{n\in\mathbb{N}} e^{(\tilde{\lambda}-\lambda)n}|v_0|^2_H \leq C_{\chi,\lambda}|v_0|^2_H.
\end{aligned}$$

Theorem (Feedback control)

For any $\hat{u} \in \mathcal{W}$ and $\lambda > 0$ there is an integer $M = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} \in \mathbb{N}$, a family of continuous operators $K_{\hat{u}}^{\lambda}(t) : H \to \mathcal{E}_M$, and a constant $\kappa = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]}$ such that the following properties hold.

(i) *The function $t \mapsto K_{\hat{u}}^{\lambda}(t)$ is continuous in the weak operator topology, and its operator norm is bounded by κ.*

(ii) *For any $s \geq 0$ and $v_0 \in H$, the solution of the problem*

$$v_t + Lv + \mathbb{B}(\hat{u})v = \Pi K_{\hat{u}}^{\lambda}(t)v, \quad v(s) = v_0$$

exists on the time interval $(s, +\infty)$ and satisfies the inequality

$$e^{\lambda(t-s)}|v(t)|_H^2 + \int_s^t e^{\lambda(\tau-s)}\left(|v(\tau)|_V^2 + |v_t(\tau)|_{V'}^2\right)d\tau \leq \kappa |v_0|_H^2, \quad t \geq s$$

. Moreover, if $v_0 \in V$, then

$$e^{\lambda(t-s)}|v(t)|_V^2 + \int_s^t e^{\lambda(\tau-s)}\left(|v(\tau)|_{\mathrm{D}(L)}^2 + |v_t(\tau)|_H^2\right)d\tau \leq \kappa |v_0|_V^2, \quad t \geq s$$

Problem

Put $E^\lambda(X) := \{f \in X \mid e^{\lambda t} f \in X\}$. *Given* $s \geq 0$, $\lambda > 0$, $M \in \mathbb{N}$ *and* $w_0 \in H$, *find the minimum of the functional*

$$M_s^\lambda(v,\, \eta) := \int_{(s,\,+\infty)} e^{\lambda t}(|v|_V^2 + |\eta|_{L^2(T\Omega)}^2)\,\mathrm{d}t$$

on the set of functions (v, η) *that satisfy*
$v_t + Lv + \mathbb{B}(\hat{u})v = \Pi(\chi P_M \eta)$, $t \in I_\tau$, $v(s) = w_0$ *and*
$(v, \eta) \in E^\lambda(W([s,\, +\infty),\, V,\, V')) \times E^\lambda(L^2([s,\, +\infty),\, L^2(T\Omega)))$

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$(v, \eta) \in E^{\lambda}(W([s,\,+\infty),\, V,\, V')) \times E^{\lambda}(L^2([s,\,+\infty),\, L^2(T\Omega)))$

Lemma

For any $\hat{u} \in \mathcal{W}$ *and* $\lambda > 0$ *there is an integer* $M = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} \geq 1$ *such that the problem has a unique minimizer* (v_s^*, η_s^*). *Moreover, there is a continuous operator* $Q_{\hat{u}}^{s,\lambda} : H \to H$ *such that*

$$M_s^{\lambda}(v_s^*,\, \eta_s^*) = \left(Q_{\hat{u}}^{s,\lambda} w_0, w_0\right), \qquad |Q_{\hat{u}}^{s,\lambda}|_{\mathcal{L}(H)} \leq \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} e^{\lambda s},$$

where $C = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} > 0$ *is a constant. Finally,* $Q_{\hat{u}}^{s,\lambda}$ *continuously depends on* s *in the weak operator topology.*

Problem

Given $\lambda > 0$ *and* $v_0 \in H$, *find the minimum of the functional*

$$N_s^\lambda(v,\,\eta) := \int_{(0,\,s)} e^{\lambda t}(|v|_V^2 + |\eta|_{L^2(T\Omega)}^2)\,\mathrm{d}t + (Q_{\hat{u}}^{s,\lambda} v(s),\, v(s))$$

on the set of functions $(v,\,\eta) \in W([0,\,s],\,V,\,V') \times L^2((0,\,s),\,L^2(T\Omega))$ *that satisfy* $v_t + Lv + \mathbb{B}(\hat{u})v = \Pi(\chi P_M \eta)$, $t \in (0,\,s)$, $v(0) = v_0$ *and* M *is the integer constructed in preceding lemma.*

Problem

Given $\lambda > 0$ and $v_0 \in H$, find the minimum of the functional

$$N_s^\lambda(v,\,\eta) := \int_{(0,\,s)} e^{\lambda t}(|v|_V^2 + |\eta|_{L^2(T\Omega)}^2)\,\mathrm{d}t + (Q_{\hat{u}}^{s,\lambda} v(s),\, v(s))$$

on the set of functions $(v,\,\eta) \in W([0,\,s],\,V,\,V') \times L^2((0,\,s),\,L^2(T\Omega))$ that satisfy $v_t + Lv + \mathbb{B}(\hat{u})v = \Pi(\chi P_M \eta)$, $t \in (0,\,s)$, $v(0) = v_0$ and M is the integer constructed in preceding lemma.

This problem has a unique minimizer $(v_s^\bullet,\,\eta_s^\bullet)$, which is a linear function of $v_0 \in H$.

Lemma

Under the hypotheses of preceding lemma, the restriction of (v_0^*, η_0^*) to the interval $(0, s)$ coincides with $(v_s^\bullet,\,\eta_s^\bullet)$ and the restriction of (v_0^*, η_0^*) to the interval $(s, +\infty)$ coincides with $(v_s^*, \eta_s^*)(v_0(s))$.

Using the Karush–Kuhn–Tucker theorem we find some equations that must be satisfied by the optimal control and trajectory of the last problem. It turns out that at time s we must have

$$\eta_s^\bullet(s) = -e^{-\lambda s} P_M \chi Q_{\hat{u}}^{s,\lambda} v_s^\bullet(s).$$

Since s is arbitrary we may conclude that the optimal trajectory v_0^* solves

$$v_t + Lv + B(\hat{u}, v) + B(v, \hat{u}) = \Pi(K_{\hat{u}}^\lambda v), \quad t \in \mathbb{R}_+, \quad v(0) = v_0,$$

where we set

$$K_{\hat{u}}^\lambda(t) := -e^{-\lambda t} \chi P_M \chi Q_{\hat{u}}^{t,\lambda}.$$

Let us consider the nonlinear problem

$$v_t + Lv + Bv + \mathbb{B}(\hat{u})v = K_{\hat{u}}^{\lambda}(t)v, \quad t \in \mathbb{R}_+; \quad v(0) = v_0.$$

Theorem

Let $\hat{u} \in \mathcal{W}$ be an arbitrary function, let $\lambda > 0$, and let $M = \overline{C}_{[|\hat{u}|_{\mathcal{W}},\lambda]}$ the integer constructed in feedback theorem for the linear case. Then there are positive constants ϑ and ϵ depending only on $|\hat{u}|_{\mathcal{W}}$ and λ such that for $|v_0|_V \le \epsilon$ the solution v of the system above is well defined for all $t \ge 0$ and satisfies the inequality

$$|v(t)|_V^2 \le \vartheta e^{-\lambda t}|v_0|_V^2 \quad \textit{for } t \ge 0.$$

Denote by $\mathcal{Z}^{\lambda}$ the space of functions $z \in C(\mathbb{R}_+,\, V) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+,\, U)$ such that

$$|z|_{\mathcal{Z}^{\lambda}} := \sup_{t \ge 0} \left(e^{\lambda t}|z(t)|_V^2 + \int_{(t,\, t+1)} e^{\lambda \tau}|z(\tau)|^2_{\mathrm{D}(L)}\, \mathrm{d}\tau \right)^{1/2} < \infty.$$

For the proof we use the contraction mapping principle. Fix a constant $\vartheta > 0$ and a function $v_0 \in V$ and introduce the following subset of $\mathcal{Z}^\lambda$:

$$\mathcal{Z}^\lambda_\vartheta := \{z \in \mathcal{Z}^\lambda \mid z(0) = v_0, |z|^2_{\mathcal{Z}^\lambda} \le \vartheta |v_0|^2_V\}.$$

We define a mapping $\Xi : \mathcal{Z}^\lambda_\vartheta \to C(\mathbb{R}_+, V) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ that takes a function $a \in \mathcal{Z}^\lambda$ to the solution of the problem

$$b_t + Lb + \mathbb{B}(\hat{u})b = K^\lambda_{\hat{u}} b - Ba, \quad t \in \mathbb{R}_+; \qquad b(0) = v_0. \tag{1}$$

The theorem follows from the following proposition, which proof follows by some technical computations we do not present here.

Proposition

Under the hypotheses of theorem, there exists $\vartheta > 0$ such that for any $\gamma \in (0, 1)$ and an appropriate constant $\epsilon = \epsilon_\gamma > 0$ the mapping Ξ takes the set Z^λ_ϑ into itself and satisfies the inequality

$$|\Xi(a_1) - \Xi(a_2)|_{\mathcal{Z}^\lambda} \le \gamma |a_1 - a_2|_{\mathcal{Z}^\lambda} \quad \text{for all} \quad a_1, a_2 \in \mathcal{Z}^\lambda_\vartheta,$$

provided that $|v_0|_V \le \epsilon$.

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Thanks for your attention