# A Priori Error Estimates for Finite Element Discretizations of Parabolic Optimization Problems with Pointwise State Constraints in Time

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# Model Problem

Minimize

$$J(q,u) := \frac{1}{2}\int_I\int_\Omega (u(t,x)-\hat{u}(t,x))^2\,dx\,dt + \frac{\alpha}{2}\int_I\int_\Omega q(t,x)^2\,dx\,dt$$

with $I := (0,T)$ and $\Omega \subset \mathbb{R}^n$ convex and polygonal, subject to

$$\begin{aligned} \partial_t u - \Delta u &= q && \text{in } I \times \Omega \\ u &= 0 && \text{on } I \times \partial\Omega \\ u &= u_0 && \text{in } \{0\} \times \Omega \end{aligned}$$

with control constraints

$$q_a \le q(t,x) \le q_b \quad \text{a. e. in } I \times \Omega$$

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with control constraints

$$q_a \le q(t,x) \le q_b \quad \text{a. e. in } I \times \Omega$$

and pointwise state constraints in time

$$\int_\Omega u(t,x)\omega(x)\,dx \le b \quad \text{in } \bar{I}.$$

# Outline

# Existing Literature

- Error estimates for elliptic problems with state constraints:
  - Casas 2002: finitely many state constraints
  - Deckelnick and Hinze 2007: pointwise state constraints
  - Deckelnick and Hinze 2008: pointwise control and state constraints
  - Meyer 2008: pointwise control and state constraints
  - De los Reyes, Meyer, and Vexler 2008: pointwise control and state constraints for Stokes
  - Rösch and Steinig 2010: pointwise control and state constraints
  - Deckelnick, Günther, and Hinze 2009: pointwise control and gradient constraints
  - Ortner and Wollner 2011: pointwise control and gradient constraints
- Error estimates for parabolic problems with state constraints:
  - Deckelnick and Hinze 2011: pointwise state constraints with time dependent control

# Model Problem

- Control space:

$$Q := L^2(I, L^2(\Omega)), \quad Q_{ad} := \{\, q \in Q \mid q_a \le q(t,x) \le q_b \text{ a. e. in } I \times \Omega \,\}$$

- State space:

$$X := L^2(I, H_0^1(\Omega)) \cap H^1(I, H^{-1}(\Omega))$$

- State constraint:

$$G(u)(t) := \int_\Omega u(t,x)\omega(x)\,dx, \quad \omega \in L^2(\Omega)$$

**Optimization problem**

$$\min_{(q,u)\in Q_{ad}\times X} J(q,u) \text{ subject to } \begin{cases} (\partial_t u, \varphi)_I + (\nabla u, \nabla \varphi)_I = (q, \varphi)_I \quad \forall \varphi \in X \\ u(0) = u_0 \\ G(u) \le b \end{cases}$$

- Notation:

$$(u,v) = \int_\Omega u(x)v(x)\,dx \qquad (u,v)_I = \int_I \int_\Omega u(x,t)v(x,t)\,dx\,dt$$

# Regularity of the Control

- $u \in X \hookrightarrow C(\bar{I}, L^2(\Omega))$ implies $G(u)\colon t \mapsto (u(t,\cdot),\omega) \in C(\bar{I})$
- Lagrange multiplier for the state constraint $\mu \in C^*(\bar{I})$

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**Adjoint equation**

$$(\partial_t \varphi, z)_I + (\nabla \varphi, \nabla z)_I = (\varphi, u - \hat{u})_I + \langle G(\varphi), \mu \rangle \quad \forall \varphi \in X,\ \varphi(0) = 0$$

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- Solution operator $S\colon L^2(I, H^{-1}(\Omega)) \to X$
- Adjoint solution operator $S^*\colon X^* \to L^2(I, H_0^1(\Omega))$
- $\langle G(\cdot), \mu \rangle \in X^*$ implies $z \in L^2(I, H_0^1(\Omega))$

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**Variational inequality**

$$(\alpha q + z, p - q)_I \geq 0 \quad \forall p \in Q_{\text{ad}} \qquad \Longrightarrow$$

$$q = P_{[q_a, q_b]}(-\alpha^{-1} z) \in L^2(I, H^1(\Omega)) \cap L^\infty(I \times \Omega)$$

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$\Rightarrow$ No information about the temporal regularity of $q$! Error estimates?

# Concept of Discretization

1. Time discretization of the states
   by discontinuous Galerkin methods: dG(0)
   $\Rightarrow$ optimal solution with semidiscretized state $(q_k, u_k) \in Q_{\mathrm{ad}} \times X_k^0$
   - No discretization of $q$
   - No regularity of $q$ required
   - Error estimate for $\|q - q_k\|_I$

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   - Error estimate for $\|q - q_k\|_I$
2. Space discretization of the time-discretized states
   by continuous Galerkin methods: cG(1)dG(0)
   $\Rightarrow$ optimal solution with discrete state $(q_{kh}, u_{kh}) \in Q_{\text{ad}} \times X_{k,h}^{0,1}$
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   - No regularity of $q$ required
   - Error estimate for $\|q_k - q_{kh}\|_I$

# Concept of Discretization

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   - No regularity of $q$ required
   - Error estimate for $\|q - q_k\|_I$
2. Space discretization of the time-discretized states
   by continuous Galerkin methods: cG(1)dG(0)
   $\Rightarrow$ optimal solution with discrete state $(q_{kh}, u_{kh}) \in Q_{\text{ad}} \times X_{k,h}^{0,1}$
   - No discretization of $q$
   - No regularity of $q$ required
   - Error estimate for $\|q_k - q_{kh}\|_I$
3. Space discretization of the controls
   by discontinuous Galerkin methods: dG(0)
   $\Rightarrow$ optimal solution with discrete state and control $(q_\sigma, u_\sigma) \in Q_{\text{ad},d} \times X_{k,h}^{0,1}$
   - Only spatial discretization of $q$
   - Only spatial regularity of $q$ required
   - Error estimate for $\|q_{kh} - q_\sigma\|_I$

# Time Discretization of the State

## The semidiscrete problem

### dG(0) bilinear form

$$B(u_k, \varphi) := (\nabla u_k, \nabla \varphi)_I + \sum_{m=2}^{M} ([u_k]_{m-1}, \varphi_{m-1}^+) + (u_{k,0}^+, \varphi_0^+)$$

### Semidiscrete optimization problem

$$\min_{Q_{\mathrm{ad}} \times X_k^0} J(q_k, u_k) \text{ subject to } \begin{cases} B(u_k, \varphi) = (q_k, \varphi)_I + (u_0, \varphi_0^+) \quad \forall \varphi \in X_k^0 \\ G(u_k)\big|_{I_m} \leq b, \quad m = 1, 2, \ldots, M \end{cases}$$

$$X_k^0 = \left\{ v_k \in L^2(I, H_0^1(\Omega)) \;\middle|\; v_k\big|_{I_m} \in \mathcal{P}^0(I_m, H_0^1(\Omega)) \right\}, \qquad I_m = (t_{m-1}, t_m]$$

# Time Discretization of the State

## The semidiscrete problem

**Semidiscrete adjoint equation for $z_k \in X_k^0$**

$$B(\varphi, z_k) = (\varphi, u_k - \hat{u})_I + \langle G(\varphi), \mu_k \rangle \quad \forall \varphi \in X_k^0$$

$$\langle v, \mu_k \rangle := \sum_{l=1}^{M} \frac{\mu_{k,l}}{k_l} \int_{I_l} v(t)\, dt \qquad \text{with} \qquad \mu_{k,l} \in \mathbb{R}_{\geq 0},\ l = 1, 2, \dots, M$$

**Semidiscrete variational inequality**

$$(\alpha q_k + z_k, p - q_k)_I \geq 0 \quad \forall p \in Q_{ad} \qquad \Longleftrightarrow \qquad q_k = P_{[q_a, q_b]}(-\alpha^{-1} z_k)$$

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$\Rightarrow$ Without discretizing $q_k$ explicitly,
it is already piecewise constant in time!

# Time Discretization of the State
## Error estimate

**Theorem**

$$\alpha\|\bar{q}-\bar{q}_k\|_I^2 \leq C\Big\{\|u(\bar{q})-u_k(\bar{q})\|_I + \|u(\bar{q}_k)-u_k(\bar{q}_k)\|_I \\ + \|u(\bar{q})-u_k(\bar{q})\|_{L^\infty(I,L^2(\Omega))} + \|u(\bar{q}_k)-u_k(\bar{q}_k)\|_{L^\infty(I,L^2(\Omega))}\Big\}$$

# Time Discretization of the State

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- Available estimate:

$$\|u(p) - u_k(p)\|_I \leq Ck\big\{ \|p\|_I + \|\nabla u_0\| \big\}$$

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- We proved the estimate

$$\|u(p)-u_k(p)\|_{L^\infty(I,L^2(\Omega))} \le Ck\left(\ln\frac{T}{k}\right)^{\frac12}\Big\{\|p\|_{L^\infty(I,L^2(\Omega))}+\|\Delta u_0\|\Big\}$$

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$$\|u(p)-u_k(p)\|_{L^\infty(I,L^2(\Omega))} \le Ck\left(\ln\frac{T}{k}\right)^{\frac{1}{2}}\Big\{\|p\|_{L^\infty(I,L^2(\Omega))}+\|\Delta u_0\|\Big\}$$

**Corollary**

$$\|\bar q-\bar q_k\|_I \le \frac{C}{\sqrt{\alpha}}\sqrt{k}\left(\ln\frac{T}{k}\right)^{\frac{1}{4}}$$

# Time Discretization of the State

## Error estimate

### $L^\infty(L^2)$ error estimate

$$\|u(p) - u_k(p)\|_{L^\infty(I,L^2(\Omega))} \leq Ck\left(\ln \frac{T}{k}\right)^{\frac{1}{2}} \left\{\|p\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\|\right\}$$

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Idea of the proof:

$$\begin{aligned} &\|u(p) - u_k(p)\|_{L^\infty(I_m,L^2(\Omega))} \\ &\qquad \leq \|u(p) - u(p)(t_m)\|_{L^\infty(I_m,L^2(\Omega))} + \|u(p)(t_m) - u_k(p)(t_m)\| \end{aligned}$$

# Time Discretization of the State

## Error estimate

**$L^\infty(L^2)$ error estimate**

$$\|u(p) - u_k(p)\|_{L^\infty(I,L^2(\Omega))} \leq Ck\left(\ln \frac{T}{k}\right)^{\frac{1}{2}} \left\{ \|p\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\| \right\}$$

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With suitable dual solution $y$ and semidiscrete counterpart $y_k$:

$$\begin{aligned} &\|u(p)(t_m) - u_k(p)(t_m)\|^2 \\ &\qquad \leq \left\{ \|p\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\| \right\} \left\{ \|y - y_k\|_{L^1(I,L^2(\Omega))} + \|y(0) - y_{k,1}\|_{H^{-2}(\Omega)} \right\} \end{aligned}$$

# Time Discretization of the State

## Error estimate

**$L^\infty(L^2)$ error estimate**

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Idea of the proof:

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With suitable dual solution $y$ and semidiscrete counterpart $y_k$:

$$\|u(p)(t_m) - u_k(p)(t_m)\|^2 \leq \left\{\|p\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\|\right\}\left\{\|y - y_k\|_{L^1(I,L^2(\Omega))} + \|y(0) - y_{k,1}\|_{H^{-2}(\Omega)}\right\}$$

The hard part (weighted norms):

$$\|y - y_k\|_{L^1(I,L^2(\Omega))} + \|y(0) - y_{k,1}\|_{H^{-2}(\Omega)} \leq Ck\left(\ln \frac{T}{k}\right)^{\frac{1}{2}} \|u(p)(t_m) - u_k(p)(t_m)\|$$

# Space Discretization of the State

## The semidiscrete problem

**Semidiscrete optimization problem**

$$\min_{Q_{\text{ad}} \times X_{k,h}^{0,1}} J(q_{kh}, u_{kh}) \text{ subject to } \begin{cases} B(u_{kh}, \varphi) = (q_{kh}, \varphi)_I + (u_0, \varphi_0^+) \quad \forall \varphi \in X_{k,h}^{0,1} \\ G(u_{kh})\big|_{I_m} \leq b, \quad m = 1, 2, \ldots, M \end{cases}$$

$$X_{k,h}^{0,1} = \left\{ v_{kh} \in L^2(I, H_0^1(\Omega)) \;\middle|\; v_{kh}\big|_{I_m} \in \mathcal{P}^0(I_m, V_h^1) \right\}, \; V_h^1 = \left\{ v \in H_0^1 \;\middle|\; v\big|_K \in \mathcal{Q}_1(K) \right\}$$

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### Semidiscrete adjoint equation for $z_{kh} \in X_{k,h}^{0,1}$

$$B(\varphi, z_{kh}) = (\varphi, u_{kh} - \hat{u})_I + \langle G(\varphi), \mu_{kh} \rangle \quad \forall \varphi \in X_{k,h}^{0,1}$$

### Semidiscrete variational inequality

$$(\alpha q_{kh} + z_{kh}, p - q_{kh})_I \geq 0 \quad \forall p \in Q_{\text{ad}} \quad \Longleftrightarrow \quad q_{kh} = P_{[q_a, q_b]}(-\alpha^{-1} z_{kh})$$

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**Semidiscrete adjoint equation for $z_{kh} \in X_{k,h}^{0,1}$**

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**Semidiscrete variational inequality**

$$(\alpha q_{kh} + z_{kh}, p - q_{kh})_I \geq 0 \quad \forall p \in Q_{\text{ad}} \quad \Longleftrightarrow \quad q_{kh} = P_{[q_a, q_b]}(-\alpha^{-1} z_{kh})$$

$\Rightarrow$ $q_{kh}$ is piecewise constant in time but not cellwise polynomial in space!

# Space Discretization of the State
## Error estimate

**Theorem**

$$\alpha\|\bar{q}_k - \bar{q}_{kh}\|_I^2 \leq C\Big\{\|u_k(\bar{q}_k) - u_{kh}(\bar{q}_k)\|_I + \|u_k(\bar{q}_\sigma) - u_{kh}(\bar{q}_\sigma)\|_I + \|u_k(\bar{q}_k) - u_{kh}(\bar{q}_k)\|_{L^\infty(I,L^2(\Omega))} + \|u_k(\bar{q}_\sigma) - u_{kh}(\bar{q}_\sigma)\|_{L^\infty(I,L^2(\Omega))}\Big\}$$

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- Available estimate:

$$\|u_k(p) - u_{kh}(p)\|_I \leq Ch^2\left\{\|p\|_I + \|\nabla u_0\|\right\}$$

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$$+ \|u_k(\bar{q}_k) - u_{kh}(\bar{q}_k)\|_{L^\infty(I,L^2(\Omega))} + \|u_k(\bar{q}_\sigma) - u_{kh}(\bar{q}_\sigma)\|_{L^\infty(I,L^2(\Omega))} \Big\}$$

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- We proved the estimate

$$\|u_k(p) - u_{kh}(p)\|_{L^\infty(I,L^2(\Omega))} \leq Ch^2 \ln \frac{T}{k} \big\{ \|p\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\| \big\}$$

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**Corollary**

$$\|\bar{q}_k - \bar{q}_{kh}\|_I \leq \frac{C}{\sqrt{\alpha}}h\left(\ln\frac{T}{k}\right)^{\frac{1}{2}}$$

# Space Discretization of the State

## Error estimate

**$L^\infty(L^2)$ error estimate**

$$\|u_k(p) - u_{kh}(p)\|_{L^\infty(I,L^2(\Omega))} \leq Ch^2 \ln \frac{T}{k} \left\{ \|p\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\| \right\}$$

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$$\|u_k(p) - u_{kh}(p)\|_{L^\infty(I,L^2(\Omega))} \leq Ch^2 \ln \frac{T}{k} \left\{ \|p\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\| \right\}$$

Idea of the proof:

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$$\|u_k(p)(t_m) - u_{kh}(p)(t_m)\|^2$$
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The hard part (weighted norms):

$$\|y_k - y_{kh}\|_{L^1(I,L^2(\Omega))} + \|y_{k,1} - y_{kh,1}\|_{H^{-2}(\Omega)} \le Ch^2 \ln\frac{T}{k}\|u_k(p)(t_m) - u_{kh}(p)(t_m)\|$$

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- No need to discretize $q_{kh}$ in time since $q_{kh}$ is already piecewise constant in time
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$$\|z_{kh}(\bar{q}_\sigma) - \Pi_d z_{kh}(\bar{q}_\sigma)\|_I \leq Ch\{1 + \|\bar{q}_\sigma\|_I + \|\nabla u_0\|\}$$

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**Corollary**

$$\|q_{kh} - q_\sigma\|_I \leq \frac{C}{\alpha}h$$

# Main Result

Error estimate for cG(1)dG(0) discretization of the state and dG(0)dG(0) discretization of the control:

**Theorem**

*For the error between the continuous optimal control $\bar{q}$ and its discrete approximation $\bar{q}_\sigma$ there holds*

$$\|\bar{q} - \bar{q}_\sigma\|_I \leq \frac{C}{\sqrt{\alpha}} \left\{ \sqrt{k} \left( \ln \frac{T}{k} \right)^{\frac{1}{4}} + h \left( \ln \frac{T}{k} \right)^{\frac{1}{2}} + \frac{1}{\sqrt{\alpha}} h \right\} \approx \mathcal{O}(\sqrt{k} + h).$$

Assumptions on data: $\hat{u} \in L^2(I, L^2(\Omega))$, $f \in L^\infty(I, L^2(\Omega))$, $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$.

- D. Meidner, R. Rannacher, and B. Vexler.
  A priori error estimates for finite element discretizations of parabolic optimization problems with pointwise state constraints in time.
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- We know

$$\bar{q} \in L^2(I, H^1(\Omega)) \cap L^\infty(I \times \Omega)$$

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**Corollary**

*There holds*

$$\bar{q} \in H^s(I, L^2(\Omega)) \quad \textit{for all} \;\; 0 \leq s < \frac{1}{2}.$$

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$$\|v_k\|_{H^s(I,L^2(\Omega))} \leq Ck^{-s}\|v_k\|_I,$$

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- For $0 \le s < \frac{1}{2}$ it holds

$$\sum_{n=1}^{\infty} 2^{-n\left(\frac{1}{2}-s\right)}(n+1)^{\frac{1}{4}} < \infty \quad \Longrightarrow \quad \bar{q} \in H^s(I, L^2(\Omega)).$$

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- Construct example with known continuous solution
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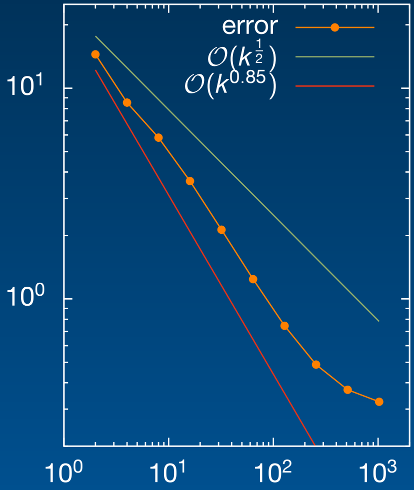


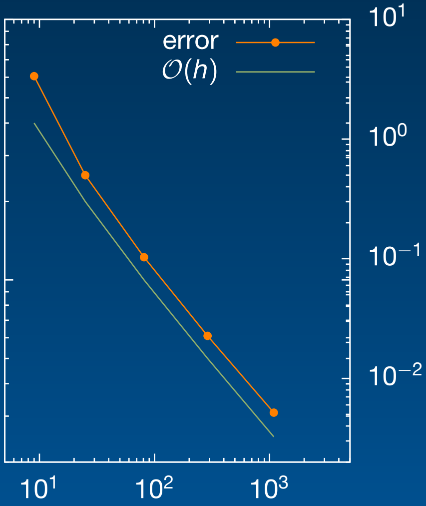

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- A priori error analysis for problems with state constraints in time
- No information about temporal regularity of the control available
- Anyway, derivation of error estimates is possible
- Crucial step: estimates of the discretization errors in terms of $L^\infty(I, L^2(\Omega))$ norms
- Derivation of regularity result for continuous optimal control based on error estimate

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**Thank you for your attention!**

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