Adaptive Space-Time Finite Elements for Optimal Control of Second Order Hyperbolic Equations

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Control and Optimization of PDEs
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Overview

1. Optimal control problem
2. Discretization
3. A posteriori error estimates
4. Numerical example
5. Behaviour of the energy
Dual weighted residual method (DWR) for optimal control

- Becker & Rannacher 2001
- Meidner 2008
- Meidner & Vexler 2007
- Benedix, Günther, Hintermüller, Hinze, Hoppe, Wollner, ...

DWR for wave equations

- Bangerth & Rannacher 1999, 2001
- Bangerth & Geiger & Rannacher 2010
- Rademacher 2009
- ...

DWR for parabolic equations

- Schmich & Vexler 2008
- ...

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Optimal control problem

Minimize \( J(u, y), \quad u \in U, \quad y \in Y, \quad s.t. \)

\[
y_{tt} - A(u, y) = f \quad \text{in} \quad (0, T) \times \Omega,
\]

\[
y(0) = y_0(u) \quad \text{in} \quad \Omega,
\]

\[
y_t(0) = y_1(u) \quad \text{in} \quad \Omega.
\]

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\[
J(u, y) - J(u_\sigma, y_\sigma) \approx \eta_k + \eta_h + \eta_d
\]
V, H Hilbert spaces with $V \hookrightarrow H \hookrightarrow V^*$ being a Gelfand triple
(e.g. $V = \{ v \in H^1(\Omega) | v|_{\Gamma_D} = 0 \}, \quad H = L^2(\Omega)$),

$I = (0, T)$ for given $T > 0$, $U \subset L^2(I, Q)$ for a Hilbert space $Q$

Inner product in $H$: $(\cdot, \cdot)_H$, $(\cdot, \cdot)_I = \int_0^T (\cdot, \cdot)_H dt$
We introduce a semilinear form

\[ \bar{a} : Q \times V \times V \to \mathbb{R} \]

for a differential operator \( A : Q \times V \to V^* \) by

\[ \bar{a}(u, y)(\xi) = \langle A(u, y), \xi \rangle_{V^*, V} \]

and define the form \( a(\cdot, \cdot)(\cdot) \) on \( U \times X \times X \) by

\[ a(u, y)(\xi) = \int_0^T \bar{a}(u(t), y(t))(\xi(t)) \, dt \]
State equation as a system

The function \((y^1, y^2) \in Y\) is a solution of the state equation, if

\[
(y_t^2, \xi^1)_I + a(u, y^1)(\xi^1) + (y^2(0) - y_1(u), \xi^1(0))_H = (f, \xi^1)_I \quad \forall \xi^1 \in X,
\]

\[
(y_t^1, \xi^2)_I - (y^2, \xi^2)_I - (y_0(u) - y^1(0), \xi^2(0))_H = 0 \quad \forall \xi^2 \in \bar{X}
\]

with \(y_0(u) \in V, y_1(u) \in H, f \in L^2(I, H)\).

Idea: Set \(y^1 = y\) and \(y^2 = y_t\)
Three times Fréchet differentiable functionals:

\[ J_1 : H \to \mathbb{R}, \quad J_2 : H \to \mathbb{R} \]

Cost functional:

\[
J(u, y^1) = \int_0^T J_1(y^1(t)) \, dt + J_2(y^1(T)) + \frac{\alpha}{2} \| u \|^2_U,
\]

\[ \alpha > 0, \; u \in U, \; y^1 \in X. \]

Control problem

Minimize \( J(u, y^1) \) for \((u, y^1) \in U \times X\), s.t. the state equation
Define the Lagrangian $\mathcal{L}: U \times Y \times Y \to \mathbb{R}$

$$
\mathcal{L}(u, y, p) = J(u, y^1) + (f - y_t^2, p^1)_I - a(u, y^1)(p^1) - (y_t^1 - y^2, p^2)_I \\
- (y^2(0) - y_1(u), p^1(0))_H + (y_0(u) - y^1(0), p^2(0))_H
$$

for $y = (y^1, y^2)$ and $p = (p^1, p^2)$.

\[ \begin{align*}
\mathcal{L}'_p(u, y, p)(\delta p) &= 0 \quad \forall \delta p \in Y \quad \text{(state equation)}, \\
\mathcal{L}'_y(u, y, p)(\delta y) &= 0 \quad \forall \delta y \in Y \quad \text{(adjoint equation)}, \\
\mathcal{L}'_u(u, y, p)(\delta u) &= 0 \quad \forall \delta u \in U \quad \text{(optimality condition)}. 
\end{align*} \]
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Time discretization

- Petrov-Galerkin scheme
  - continuous ansatz functions
  - discontinuous test functions

- Time points: \( 0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T \)

- Partition: \( \bar{I} = [0, T] = \{0\} \cup I_1 \cup \ldots \cup I_M \) with \( I_m = (t_{m-1}, t_m], k_m = |I_m| \)

- Semi-discrete spaces:

\[
\begin{align*}
X^r_k &= \{ v_k \in C(\bar{I}, H) | v_k|_{I_m} \in \mathcal{P}_r(I_m, V) \}, \\
\tilde{X}^{r-1}_k &= \{ v_k \in L^2(I, V) | v_k|_{I_m} \in \mathcal{P}_{r-1}(I_m, V) \text{ and } v_k(0) \in H \}
\end{align*}
\]
Time discretization

Semi-discrete state equation \((E_k)\)

The function \((y^1_k, y^2_k) \in X^r_k \times X^r_k\) is a solution of the (in time) discretized state equation, if

\[
\sum_{m=1}^{M} (\partial_t y^2_k, \xi^1) I_m + a(u_k, y^1_k)(\xi^1) + (y^2_k(0) - y^1(u_k), \xi^1(0)) H = (f, \xi^1) I
\]

\[
\forall \xi^1 \in \tilde{X}^{r-1}_k
\]

\[
\sum_{m=1}^{M} (\partial_t y^1_k, \xi^2) I_m - (y^2_k, \xi^2) I - (y_0(u_k) - y^1_k(0), \xi^2(0)) H = 0
\]

\[
\forall \xi^2 \in \tilde{X}^{r-1}_k
\]

Semi-discrete control problem

Minimize \(J(u_k, y^1_k)\) for \((u_k, y^1_k) \in U \times X^r_k\), s.t. \((E_k)\)
Space discretization

- Conforming finite elements
- Discrete space:
  \[ V_h^s = \{ v \in V \mid v|_K \in Q^s(K) \text{ for } K \in \mathcal{T}_h \}, \quad s \in \mathbb{N}^+ \]
- Associate with \( t_m \) a mesh \( \mathcal{T}_h^m \) and a finite element space \( V_h^{s,m} \)
- Let \( \{\tau_0, \ldots, \tau_r\} \) be a basis of \( \mathcal{P}_r(I_m, \mathbb{R}) \) with
  \[ \tau_0(t_{m-1}) = 1, \quad \tau_0(t_m) = 0, \quad \tau_i(t_{m-1}) = 0, \quad i = 1, \ldots, r, \]
  then define

\[
X_{k,h}^{r,s,m} = \text{span} \left\{ \tau_i v_i \mid v_0 \in V_{h,m-1}^s, \quad v_i \in V_{h,m}^s, \quad i = 1, \ldots, r \right\} \subset \mathcal{P}_r(I_m, V),
\]
\[
X_{k,h}^{r,s} = \left\{ v_{kh} \in C(\overline{I}, H) \mid v_{kh}|_{I_m} \in X_{k,h}^{r,s,m} \right\},
\]
\[
\tilde{X}_{k,h}^{r-1,s} = \left\{ v_{kh} \in L^2(I, V) \mid v_{kh}|_{I_m} \in \mathcal{P}_{r-1}(I_m, V_{h,m}^s) \text{ and } v_{kh}(0) \in V_{h,0}^s \right\}
\]

- We obtain a \( cG(r)cG(s) \) discretization

Schmich & Vexler 2008
Space discretization

Space and time discretized state equation \((E_{kh})\)

The function \((y_{1, kh}, y_{2, kh}) \in X_{k, h}^{r, s} \times X_{k, h}^{r, s}\) is a solution of the (in time and space) discretized state equation, if

\[
\sum_{m=1}^{M} (\partial_t y_{kh}^2, \xi^1)_m + a(u_{kh}, y_{kh})(\xi^1) + (y_{kh}^2(0) - y_1(u_{kh}), \xi^1(0))_H = (f, \xi^1)_I
\]

\[
\forall \xi^1 \in \tilde{X}_{k, h}^{r-1, s}
\]

\[
\sum_{m=1}^{M} (\partial_t y_{kh}^1, \xi^2)_m - (y_{kh}^2, \xi^2)_I - (y_0(u_{kh}) - y_{kh}^1(0), \xi^2(0))_H = 0
\]

\[
\forall \xi^2 \in \tilde{X}_{k, h}^{r-1, s}
\]

Semi-discrete control problem

Minimize \(J(u_{kh}, y_{kh}^1), \quad u_{kh} \in U, \quad y_{kh}^1 \in X_{k, h}^{r, s}\) s.t. \((E_{kh})\)
Control discretization: \( U_d \subset U \)

**Fully-discrete control problem**

Minimize \( J(u_\sigma, y_{\sigma}^1) \), \( u_\sigma \in U \), \( y_{\sigma}^1 \in X_{k,h}^{r,s} \), subject to the fully discretized state equation.
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Error in the cost functional

\[ J(u, y^1) - J(u_\sigma, y_\sigma^1) = (J(u, y^1) - J(u_k, y_k^1)) + (J(u_k, y_k^1) - J(u_{kh}, y_{kh}^1)) + (J(u_{kh}, y_{kh}^1) - J(u_\sigma, y_\sigma^1)) \]

\[ \approx \eta_k + \eta_h + \eta_d \]
For stationary points there holds

\[ \mathcal{L}'(u, y, p)(\delta u, \delta y, \delta p) = 0 \]
\[ \forall (\delta u, \delta y, \delta p) \in U \times Y \times Y, \]

\[ \mathcal{L}'(u_k, y_k, p_k)(\delta u_k, \delta y_k, \delta p_k) = 0 \]
\[ \forall (\delta u_k, \delta y_k, \delta p_k) \in U \times (X^r_k)^2 \times (\tilde{X}^{r-1}_k)^2, \]

\[ \mathcal{L}'(u_{kh}, y_{kh}, p_{kh})(\delta u_{kh}, \delta y_{kh}, \delta p_{kh}) = 0 \]
\[ \forall (\delta u_{kh}, \delta y_{kh}, \delta p_{kh}) \in U \times (X^r_{k,h})^2 \times (\tilde{X}^{r-1}_{k,h})^2, \]

\[ \mathcal{L}'(u_\sigma, y_\sigma, p_\sigma)(\delta u_\sigma, \delta y_\sigma, \delta p_\sigma) = 0 \]
\[ \forall (\delta u_\sigma, \delta y_\sigma, \delta p_\sigma) \in U_d \times (X^r_{k,h})^2 \times (\tilde{X}^{r-1}_{k,h})^2. \]
Hence, we have

\[
J(u, y^1) - J(u_k, y_k^1) = \frac{1}{2} \mathcal{L}'(u_k, y_k, p_k)(u - \hat{u}_k, y - \hat{y}_k, p - \hat{p}_k) + \mathcal{R}_k,
\]

\[
J(u_k, y_k^1) - J(u_{kh}, y_{kh}^1) = \frac{1}{2} \mathcal{L}'(u_{kh}, y_{kh}, p_{kh})(u_k - \hat{u}_{kh}, y_k - \hat{y}_{kh}, p_k - \hat{p}_{kh}) + \mathcal{R}_h,
\]

\[
J(u_{kh}, y_{kh}^1) - J(u_\sigma, y_\sigma^1) = \frac{1}{2} \mathcal{L}'(u_\sigma, y_\sigma, p_\sigma)(u_{kh} - \hat{u}_\sigma, y_{kh} - \hat{y}_\sigma, p_{kh} - \hat{p}_\sigma) + \mathcal{R}_d
\]

with

\[
(\hat{u}_k, \hat{y}_k, \hat{p}_k) \in U \times (X^r_k)^2 \times (\tilde{X}^{r-1}_k)^2,
\]

\[
(\hat{u}_{kh}, \hat{y}_{kh}, \hat{p}_{kh}) \in U \times (X^r_{k,h})^2 \times (\tilde{X}^{r-1,s}_{k,h})^2,
\]

\[
(\hat{u}_\sigma, \hat{y}_\sigma, \hat{p}_\sigma) \in U_d \times (X^r_{k,h})^2 \times (\tilde{X}^{r-1,s}_{k,h})^2
\]

arbitrarily chosen.

Becker & Rannacher 2002, Meidner 2008
The statement can be reduced to

\[
J(u, y_1) - J(u_k, y_k^1) \approx \frac{1}{2} \left( \mathcal{L}_y'(u_k, y_k, p_k)(y - \hat{y}_k) + \mathcal{L}_p'(u_k, y_k, p_k)(p - \hat{p}_k) \right),
\]

\[
J(u_k, y_k^1) - J(u_{kh}, y_{kh}^1) \approx \frac{1}{2} \left( \mathcal{L}_y'(u_{kh}, y_{kh}, p_{kh})(y_k - \hat{y}_{kh}) + \mathcal{L}_p'(u_{kh}, y_{kh}, p_{kh})(p_k - \hat{p}_{kh}) \right),
\]

\[
J(u_{kh}, y_{kh}^1) - J(u_{\sigma}, y_{\sigma}^1) \approx \frac{1}{2} \mathcal{L}_u'(u_{\sigma}, y_{\sigma}, p_{\sigma})(u_{kh} - \hat{u}_{\sigma}).
\]
We replace the weights by interpolations in higher-order finite element spaces

\[
\begin{align*}
    y - \hat{y}_k & \approx P^y_k y_k, \\
    p - \hat{p}_k & \approx P^p_k p_k, \\
    u_{kh} - \hat{u}_\sigma & \approx P_d u_\sigma, \\
    y_k - \hat{y}_{kh} & \approx P_h y_{kh}, \\
    p_k - \hat{p}_{kh} & \approx P_h p_{kh}.
\end{align*}
\]

Further, we replace the continuous and semi-discrete functions by the fully discrete functions

\[
\begin{align*}
    J(u, y^1) - J(u_k, y^1_k) & \approx \frac{1}{2} \left( \mathcal{L}'_y(u_\sigma, y_\sigma, p_\sigma)(P_{k}^{(2)} y_\sigma) + \mathcal{L}'_p(u_\sigma, y_\sigma, p_\sigma)(P_{k}^{(1)} p_\sigma) \right), \\
    J(u_k, y^1_k) - J(u_{kh}, y^1_{kh}) & \approx \frac{1}{2} \left( \mathcal{L}'_y(u_\sigma, y_\sigma, p_\sigma)(P_{h}^{(2)} y_\sigma) + \mathcal{L}'_p(u_\sigma, y_\sigma, p_\sigma)(P_{h}^{(2)} p_\sigma) \right), \\
    J(u_{kh}, y^1_{kh}) - J(u_\sigma, y^1_\sigma) & \approx \frac{1}{2} \mathcal{L}'_u(u_\sigma, y_\sigma, p_\sigma)(P_d u_\sigma).
\end{align*}
\]

Becker & Rannacher 2001

Meidner & Vexler 2007

Schmich & Vexler 2008
Overview

1. Optimal control problem
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Dynamical Lamé system

\[
\begin{cases}
  y_{tt} - \text{div}(\sigma(y)) = f & \text{in } I \times \Omega, \\
  y(0) = y_0 & \text{in } \Omega, \\
  y_t(0) = y_1 & \text{in } \Omega, \\
  y = 0 & \text{in } I \times \partial\Omega.
\end{cases}
\]

- initial data: \( y_0 \in H^1_0(\Omega)^d, \ y_1 \in L^2(\Omega)^d, \ f \in L^2(I, L^2(\Omega)^d) \)
- strain tensor: \( \varepsilon(y) = \frac{1}{2} (\nabla y)^T + \nabla y \)
- stress tensor: \( \sigma_{ij}(y) = \lambda \delta_{ij} \text{tr}(\varepsilon(y)) + 2\mu\varepsilon_{ij}(y), \ \lambda, \mu > 0 \)
- Proof for existence of a solution in \( X \) uses Korn’s first inequality.
Optimal control of the Lamé system

Minimize

$$J(u, y) = \frac{1}{2} \|y - y_d\|^2_{L^2(I, L^2(\Omega)^d)} + \frac{\alpha}{2} \|u\|^2_U,$$

$$u \in U = L^2(I, \mathbb{R}^m), \quad y \in X, \quad \text{s.t.}$$

$$y_{tt} - \text{div}(\sigma(y)) = Bu \quad \text{in } Q,$$

$$y(0) = y_0 \quad \text{in } \Omega,$$

$$y_t(0) = y_1 \quad \text{in } \Omega,$$

$$y = 0 \quad \text{in } \Sigma.$$ 

- spaces: $U = L^2(I, \mathbb{R}^m)$, $m \in \mathbb{N}$
- initial data: $y_0 \in H^1_0(\Omega)^d$, $y_1 \in L^2(\Omega)^d$, $f \in L^2(I, L^2(\Omega)^d)$
- operator: $B: U \rightarrow L^2(I, L^2(\Omega)^d)$, $Bu = \sum_{i=1}^m u_i(t)g_i(x)$, $g_i \in L^2(\Omega)^d$
Data

\[ y_0(x) = \begin{cases} 
(\sin(8\pi(x_1 - 0.125)) \sin(8\pi(x_2 - 0.125)), 0)^T, & 0.125 < x_1, x_2 < 0.25, \\
(0, 0)^T, & \text{else},
\end{cases} \]

\[ y_1(x) = (0, 0)^T, \quad y_d(t, x) = 0, \]

\[ g_1(x) = \begin{cases} 
(1, 1)^T, & \text{for } x_1 < 0, \\
(0, 0)^T, & \text{else}
\end{cases}, \quad g_2(x) = \begin{cases} 
(1, 1)^T, & \text{for } x_1 > 0, \\
(0, 0)^T, & \text{else}
\end{cases} \]

\( \alpha = 0.001, \quad d = 2, \quad \lambda = 1, \quad \mu = 1, \quad m = 2 \)

\( (t, x) = (t, x_1, x_2) \in [0, T] \times \Omega = [0, 0.5] \times [-1, 1]^2. \)

Discretization: \( cG(1)cG(1), U_d: \) piecewise constants in time with values in \( \mathbb{R}^2 \)
Numerical example

Error for adaptive and uniform refinement
### Numerical Example

Comparison of the CPU-time for uniform and adaptive refinement

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<th>Refinement</th>
<th>CPU-time</th>
<th>dof</th>
<th>Error</th>
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<tr>
<td>Uniform</td>
<td>100%</td>
<td>100%</td>
<td>$6.6 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>Adaptive</td>
<td>34%</td>
<td>15%</td>
<td>$6.5 \cdot 10^{-8}$</td>
</tr>
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Energy

- Rademacher 2009, Eriksson et al. 1996

- Wave equation for $y_0 \in H^1_0(\Omega)$ and $y_1 \in L^2(\Omega)$:

$$
\begin{cases}
  y_{tt} - \Delta y = 0 & \text{in } Q, \\
  y(0) = y_0 & \text{in } \Omega, \\
  y_t(0) = y_1 & \text{in } \Omega, \\
  y = 0 & \text{on } \Sigma
\end{cases}
$$

- Energy:

$$
E(t) = \frac{1}{2}(\|y_t(t)\|^2 + \|\nabla y(t)\|^2) = \frac{1}{2}(\|y_1\|^2 + \|\nabla y_0\|^2) = E(0)
$$
Discrete energy

- Discrete equation as a time stepping scheme

\[
(Y^1_0, \xi) = (y_0, \xi), \quad (Y^2_0, \xi) = (y_1, \xi) \quad \forall \xi \in V^{1,0}_h,
\]

\[
(Y^2_m, \xi^1) + \frac{k_m}{2} (\nabla Y^1_m, \nabla \xi^1) = (Y^2_{m-1}, \xi^1) - \frac{k_m}{2} (\nabla Y^1_{m-1}, \nabla \xi^1) \quad \forall \xi^1 \in V^{1,m}_h,
\]

\[
(Y^1_m, \xi^2) - \frac{k_m}{2} (Y^2_m, \xi^2) = (Y^1_{m-1}, \xi^2) + \frac{k_m}{2} (Y^2_{m-1}, \xi^2) \quad \forall \xi^2 \in V^{1,m}_h.
\]

\[Y^i_m = y^i_\sigma(t_m) \quad (i = 1, 2) \text{ and for } m = 1, \ldots, M.\]

- Discrete energy

\[
E_{k,h}(t_m) = \frac{1}{2} \left( \| Y^2_m \|^2 + \| \nabla Y^1_m \|^2 \right) \quad m = 0, \ldots, M.
\]
Refinement

Space refinement

Level 1

Level 2

Level 3

Level \( n \)

Time refinement
Behaviour of the energy

**Theorem**

Let \( \pi_m : V_h^{1,m-1} \to V_h^{1,m} \) for \( m = 1, \ldots, M \). Then

\[
E_{k,h}(t_m) = E_{k,h}(t_{m-1}) - \frac{1}{k_m} (Y_{m-1}^1 - \pi_m Y_{m-1}^1, Y_m - Y_{m-1}^2)
\]

\[
- \frac{1}{k_m} (\pi_m Y_{m-1}^2 - Y_{m-1}^2, Y_m^1 - Y_{m-1}^1) - \frac{1}{2} (Y_{m-1}^2 - \pi_m Y_{m-1}^2, Y_m^2 + Y_{m-1}^2)
\]

\[
- \frac{1}{2} (\nabla Y_m^1 + \nabla Y_{m-1}^1, \nabla(Y_{m-1}^1 - \pi_m Y_{m-1}^1)).
\]

**Corollary**

On a given discretization level there holds

\[
E_{k,h}(t_m) = E_{k,h}(t_{m-1}) \quad \text{for } m = 1, \ldots, M
\]

independent of the size of \( k_m \), if for all steps from \( t_m \) to \( t_{m+1} \) the spatial mesh is only refined and not coarsened.
Spatial meshes

(a) $\mathcal{T}_1$  (b) $\mathcal{T}_2$  (c) $\mathcal{T}_3$  (d) $\mathcal{T}_4$  (e) $\mathcal{T}_5$

<table>
<thead>
<tr>
<th>Time point</th>
<th>$t_0$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
<th>$t_4$</th>
<th>$t_5$</th>
<th>$t_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh</td>
<td>$\mathcal{T}_1$</td>
<td>$\mathcal{T}_1$</td>
<td>$\mathcal{T}_1$</td>
<td>$\mathcal{T}_2$</td>
<td>$\mathcal{T}_2$</td>
<td>$\mathcal{T}_3$</td>
<td>$\mathcal{T}_3$</td>
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<tr>
<td>Energy</td>
<td>2.5327</td>
<td>2.5327</td>
<td>2.5327</td>
<td>2.5361</td>
<td>2.5361</td>
<td>2.5346</td>
<td>2.5346</td>
</tr>
</tbody>
</table>

Time point

<table>
<thead>
<tr>
<th>$t_7$</th>
<th>$t_8$</th>
<th>$t_9$</th>
<th>$t_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh</td>
<td>$\mathcal{T}_4$</td>
<td>$\mathcal{T}_4$</td>
<td>$\mathcal{T}_5$</td>
</tr>
<tr>
<td>Energy</td>
<td>2.5441</td>
<td>2.5441</td>
<td>2.5441</td>
</tr>
</tbody>
</table>

Energy on a sequence of spatial meshes
Summary

- DWR method for optimal control of second order hyperbolic equations
- Separating the influence of time, space, and control discretization
- Better accuracy of the discrete solution
- Behaviour of the energy

Software

- Software package: RoDoBo, www.rodobo.uni-hd.de

Reference

- A. Kröner,
  *Adaptive finite element methods for optimal control of second order hyperbolic equations*
Thank you for your attention.