Adaptive Space-Time Finite Elements for Optimal Control of Second Order Hyperbolic Equations

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Control and Optimization of PDEs
Graz, October 13th, 2011

Overview

1. Optimal control problem
2. Discretization
3. A posteriori error estimates
4. Numerical example
5. Behaviour of the energy

References

Dual weighted residual method (DWR) for optimal control

- Becker & Rannacher 2001
- Meidner 2008
- Meidner & Vexler 2007
- Benedix, Günther, Hintermüller, Hinze, Hoppe, Wollner, ...

DWR for wave equations

- Bangerth & Rannacher 1999, 2001
- Bangerth & Geiger & Rannacher 2010
- Rademacher 2009
- ...

DWR for parabolic equations

- Schmich & Vexler 2008
- ...

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Optimal control problem

$$\text{Minimize} \quad J(u,y), \quad u \in U, \quad y \in Y, \quad s.t.$$

$$\begin{aligned} y_{tt} - A(u,y) &= f && \text{in } (0,T)\times\Omega, \\ y(0) &= y_0(u) && \text{in } \Omega, \\ y_t(0) &= y_1(u) && \text{in } \Omega. \end{aligned}$$

	solution	estimator
continuous problem:	(u, y)	
semi-discretization in time:	(u_k, y_k)	η_k
semi-discretization in space:	(u_{kh}, y_{kh})	η_h
discretization of the control:	(u_σ, y_σ)	η_d

Aim

$$J(u,y) - J(u_\sigma, y_\sigma) \approx \eta_k + \eta_h + \eta_d$$

Notation

- V, H Hilbert spaces with $V \hookrightarrow H \hookrightarrow V^*$ being a Gelfand triple (e.g. $V = \left\{ v \in H^1(\Omega) | v|_{\Gamma_D} = 0 \right\}, \quad H = L^2(\Omega)$),
- $I = (0, T)$ for given $T > 0$, $\quad U \subset L^2(I, Q)$ for a Hilbert space Q
- Inner product in H: $(\cdot, \cdot)_H, \quad (\cdot, \cdot)_I = \int_0^T (\cdot, \cdot)_H \, dt$

Spaces

$$X = L^2(I, V) \cap H^1(I, H) \cap H^2(I, V^*),$$
$$\bar{X} = L^2(I, H) \cap H^1(I, V^*),$$
$$Y = X \times \bar{X}$$

- We introduce a semilinear form

$$\bar{a}\colon Q \times V \times V \to \mathbb{R}$$

for a differential operator $A\colon Q \times V \to V^*$ by

$$\bar{a}(u, y)(\xi) = \langle A(u, y), \xi \rangle_{V^*, V}$$

and define the form $a(\cdot, \cdot)(\cdot)$ on $U \times X \times X$ by

$$a(u, y)(\xi) = \int_0^T \bar{a}(u(t), y(t))(\xi(t))\, dt$$

State equation

State equation as a system

The function $(y^1, y^2) \in Y$ is a solution of the state equation, if

$$\begin{aligned} (y_t^2, \xi^1)_I + a(u, y^1)(\xi^1) + (y^2(0) - y_1(u), \xi^1(0))_H &= (f, \xi^1)_I \qquad \forall \xi^1 \in X, \\ (y_t^1, \xi^2)_I - (y^2, \xi^2)_I - (y_0(u) - y^1(0), \xi^2(0))_H &= 0 \qquad \forall \xi^2 \in \bar{X} \end{aligned}$$

with $y_0(u) \in V$, $y_1(u) \in H$, $f \in L^2(I, H)$.

Idea: Set $y^1 = y$ and $y^2 = y_t$

Optimal control problem

- Three times Fréchet differentiable functionals:

$$J_1\colon H \to \mathbb{R}, \quad J_2\colon H \to \mathbb{R}$$

- Cost functional:

$$J(u, y^1) = \int_0^T J_1(y^1(t))\,dt + J_2(y^1(T)) + \frac{\alpha}{2}\|u\|_U^2,$$

$\alpha > 0, u \in U,\ y^1 \in X.$

Control problem

Minimize $\quad J(u, y^1) \quad$ for $(u, y^1) \in U \times X, \quad$ s.t. the state equation

Optimality system

- Define the Lagrangian $\mathcal{L}\colon U \times Y \times Y \to \mathbb{R}$

$$\mathcal{L}(u, y, p) = J(u, y^1) + (f - y_t^2, p^1)_I - a(u, y^1)(p^1) - (y_t^1 - y^2, p^2)_I \\ - (y^2(0) - y_1(u), p^1(0))_H + (y_0(u) - y^1(0), p^2(0))_H$$

for $y = (y^1, y^2)$ and $p = (p^1, p^2)$.

Optimality system

$$\mathcal{L}'_p(u, y, p)(\delta p) = 0 \quad \forall \delta p \in Y \quad \text{(state equation)},$$
$$\mathcal{L}'_y(u, y, p)(\delta y) = 0 \quad \forall \delta y \in Y \quad \text{(adjoint equation)},$$
$$\mathcal{L}'_u(u, y, p)(\delta u) = 0 \quad \forall \delta u \in U \quad \text{(optimality condition)}.$$

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Time discretization

- Petrov-Galerkin scheme
  - continuous ansatz functions
  - discontinuous test functions
- Time points: $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T$
- Partition: $\bar{I} = [0, T] = \{0\} \cup I_1 \cup \ldots \cup I_M$ with $I_m = (t_{m-1}, t_m]$, $k_m = |I_m|$
- Semi-discrete spaces:

$$X_k^r = \left\{ v_k \in C(\bar{I}, H) | v_k|_{I_m} \in \mathcal{P}_r(I_m, V) \right\},$$
$$\widetilde{X}_k^{r-1} = \left\{ v_k \in L^2(I, V) | v_k|_{I_m} \in \mathcal{P}_{r-1}(I_m, V) \text{ and } v_k(0) \in H \right\}$$

Time discretization

Semi-discrete state equation (E_k)

The function $(y_k^1, y_k^2) \in X_k^r \times X_k^r$ is a solution of the (in time) discretized state equation, if

$$\sum_{m=1}^{M} (\partial_t y_k^2, \xi^1)_{I_m} + a(u_k, y_k^1)(\xi^1) + (y_k^2(0) - y_1(u_k), \xi^1(0))_H = (f, \xi^1)_I \qquad \forall \xi^1 \in \widetilde{X}_k^{r-1}$$

$$\sum_{m=1}^{M} (\partial_t y_k^1, \xi^2)_{I_m} - (y_k^2, \xi^2)_I - (y_0(u_k) - y_k^1(0), \xi^2(0))_H = 0 \qquad \forall \xi^2 \in \widetilde{X}_k^{r-1}$$

Semi-discrete control problem

Minimize $\quad J(u_k, y_k^1) \quad$ for $(u_k, y_k^1) \in U \times X_k^r, \quad$ s.t. (E_k)

Space discretization

- Conforming finite elements
- Discrete space:

$$V_h^s = \{v \in V | v|_K \in \mathcal{Q}^s(K) \text{ for } K \in \mathcal{T}_h\}, \quad s \in \mathbb{N}^+$$

- Associate with t_m a mesh $\mathcal{T}_h^m$ and a finite element space $V_h^{s,m}$
- Let $\{\tau_0, \ldots, \tau_r\}$ be a basis of $\mathcal{P}_r(I_m, \mathbb{R})$ with

$$\tau_0(t_{m-1}) = 1, \quad \tau_0(t_m) = 0, \quad \tau_i(t_{m-1}) = 0, \quad i = 1, \ldots, r,$$

then define

$$X_{k,h}^{r,s,m} = \text{span}\left\{\tau_i v_i | v_0 \in V_h^{s,m-1},\ v_i \in V_h^{s,m}, i = 1, \ldots, r\right\} \subset \mathcal{P}_r(I_m, V),$$

$$X_{k,h}^{r,s} = \left\{v_{kh} \in C(\bar{I}, H) | v_{kh}|_{I_m} \in X_{k,h}^{r,s,m}\right\},$$

$$\widetilde{X}_{k,h}^{r-1,s} = \left\{v_{kh} \in L^2(I, V) | v_{kh}|_{I_m} \in \mathcal{P}^{r-1}(I_m, V_h^{s,m}) \text{ and } v_{kh}(0) \in V_h^{s,0}\right\}$$

- We obtain a $cG(r)cG(s)$ discretization

Schmich & Vexler 2008

Space discretization

Space and time discretized state equation (E_{kh})

The function $(y^1_{kh}, y^2_{k,h}) \in X^{r,s}_{k,h} \times X^{r,s}_{k,h}$ is a solution of the (in time and space) discretized state equation, if

$$\sum_{m=1}^{M} (\partial_t y^2_{kh}, \xi^1)_{I_m} + a(u_{kh}, y_{kh})(\xi^1) + (y^2_{kh}(0) - y_1(u_{kh}), \xi^1(0))_H = (f, \xi^1)_I \qquad \forall \xi^1 \in \widetilde{X}^{r-1,s}_{k,h}$$

$$\sum_{m=1}^{M} (\partial_t y^1_{kh}, \xi^2)_{I_m} - (y^2_{kh}, \xi^2)_I - (y_0(u_{kh}) - y^1_{kh}(0), \xi^2(0))_H = 0 \qquad \forall \xi^2 \in \widetilde{X}^{r-1,s}_{k,h}$$

Semi-discrete control problem

Minimize $J(u_{kh}, y^1_{kh}), \quad u_{kh} \in U, \quad y^1_{kh} \in X^{r,s}_{k,h}$ s.t. (E_{kh})

Control discretization

- Control discretization: $U_d \subset U$

Fully-discrete control problem

$$\text{Minimize} \quad J(u_\sigma, y_\sigma^1), \quad u_\sigma \in U, \quad y_\sigma^1 \in X_{k,h}^{r,s},$$

subject to the fully discretized state equation.

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Error in the cost functional

Error

$$J(u, y^1) - J(u_\sigma, y^1_\sigma) = \underbrace{(J(u, y^1) - J(u_k, y^1_k))}_{\approx \eta_k} + \underbrace{(J(u_k, y^1_k) - J(u_{kh}, y^1_{kh}))}_{\approx \eta_h} + \underbrace{(J(u_{kh}, y^1_{kh}) - J(u_\sigma, y^1_\sigma))}_{\approx \eta_d}$$

$$J(u, y^1) - J(u_\sigma, y^1_\sigma) \approx \eta_k + \eta_h + \eta_d$$

Error estimate

For stationary points there holds

$$
\begin{aligned}
\mathcal{L}'(u, y, p)(\delta u, \delta y, \delta p) &= 0 \\
&\qquad \forall (\delta u, \delta y, \delta p) \in U \times Y \times Y, \\
\mathcal{L}'(u_k, y_k, p_k)(\delta u_k, \delta y_k, \delta p_k) &= 0 \\
&\qquad \forall (\delta u_k, \delta y_k, \delta p_k) \in U \times (X_k^r)^2 \times (\widetilde{X}_k^{r-1})^2, \\
\mathcal{L}'(u_{kh}, y_{kh}, p_{kh})(\delta u_{kh}, \delta y_{kh}, \delta p_{kh}) &= 0 \\
&\qquad \forall (\delta u_{kh}, \delta y_{kh}, \delta p_{kh}) \in U \times (X_{k,h}^{r,s})^2 \times (\widetilde{X}_{k,h}^{r-1,s})^2, \\
\mathcal{L}'(u_\sigma, y_\sigma, p_\sigma)(\delta u_\sigma, \delta y_\sigma, \delta p_\sigma) &= 0 \\
&\qquad \forall (\delta u_\sigma, \delta y_\sigma, \delta p_\sigma) \in U_d \times (X_{k,h}^{r,s})^2 \times (\widetilde{X}_{k,h}^{r-1,s})^2.
\end{aligned}
$$

Error estimate

Hence, we have

$$J(u, y^1) - J(u_k, y_k^1) = \frac{1}{2}\mathcal{L}'(u_k, y_k, p_k)(u - \hat{u}_k, y - \hat{y}_k, p - \hat{p}_k) + \mathcal{R}_k,$$

$$J(u_k, y_k^1) - J(u_{kh}, y_{kh}^1) = \frac{1}{2}\mathcal{L}'(u_{kh}, y_{kh}, p_{kh})(u_k - \hat{u}_{kh}, y_k - \hat{y}_{kh}, p_k - \hat{p}_{kh}) + \mathcal{R}_h,$$

$$J(u_{kh}, y_{kh}^1) - J(u_\sigma, y_\sigma^1) = \frac{1}{2}\mathcal{L}'(u_\sigma, y_\sigma, p_\sigma)(u_{kh} - \hat{u}_\sigma, y_{kh} - \hat{y}_\sigma, p_{kh} - \hat{p}_\sigma) + \mathcal{R}_d$$

with

$$(\hat{u}_k, \hat{y}_k, \hat{p}_k) \in U \times (X_k^r)^2 \times (\widetilde{X}_k^{r-1})^2,$$

$$(\hat{u}_{kh}, \hat{y}_{kh}, \hat{p}_{kh}) \in U \times (X_{k,h}^{r,s})^2 \times (\widetilde{X}_{k,h}^{r-1,s})^2,$$

$$(\hat{u}_\sigma, \hat{y}_\sigma, \hat{p}_\sigma) \in U_d \times (X_{k,h}^{r,s})^2 \times (\widetilde{X}_{k,h}^{r-1,s})^2$$

arbitrarily chosen.

Becker & Rannacher 2002, Meidner 2008

Error estimate

The statement can be reduced to

$$
\begin{aligned}
J(u,y^1) - J(u_k,y_k^1) &\approx \frac{1}{2}\left(\mathcal{L}'_y(u_k,y_k,p_k)(y-\hat{y}_k) + \mathcal{L}'_p(u_k,y_k,p_k)(p-\hat{p}_k)\right),\\
J(u_k,y_k^1) - J(u_{kh},y_{kh}^1) &\approx \frac{1}{2}\big(\mathcal{L}'_y(u_{kh},y_{kh},p_{kh})(y_k-\hat{y}_{kh})\\
&\qquad + \mathcal{L}'_p(u_{kh},y_{kh},p_{kh})(p_k-\hat{p}_{kh})\big),\\
J(u_{kh},y_{kh}^1) - J(u_\sigma,y_\sigma^1) &\approx \frac{1}{2}\mathcal{L}'_u(u_\sigma,y_\sigma,p_\sigma)(u_{kh}-\hat{u}_\sigma).
\end{aligned}
$$

Error estimate

We replace the weights by interpolations in higher-order finite element spaces

$$y - \hat{y}_k \approx P_k^y y_k, \qquad p - \hat{p}_k \approx P_k^p p_k, \qquad u_{kh} - \hat{u}_\sigma \approx P_d u_\sigma,$$
$$y_k - \hat{y}_{kh} \approx P_h y_{kh}, \qquad p_k - \hat{p}_{kh} \approx P_h p_{kh}.$$

Further, we replace the continuous and semi-discrete functions by the fully discrete functions

$$J(u, y^1) - J(u_k, y_k^1) \approx \frac{1}{2}\Big(\mathcal{L}'_y(u_\sigma, y_\sigma, p_\sigma)(P_k^{(2)} y_\sigma) + \mathcal{L}'_p(u_\sigma, y_\sigma, p_\sigma)(P_k^{(1)} p_\sigma)\Big),$$
$$J(u_k, y_k^1) - J(u_{kh}, y_{kh}^1) \approx \frac{1}{2}(\mathcal{L}'_y(u_\sigma, y_\sigma, p_\sigma)(P_h^{(2)} y_\sigma) + \mathcal{L}'_p(u_\sigma, y_\sigma, p_\sigma)(P_h^{(2)} p_\sigma)),$$
$$J(u_{kh}, y_{kh}^1) - J(u_\sigma, y_\sigma^1) \approx \frac{1}{2}\mathcal{L}'_u(u_\sigma, y_\sigma, p_\sigma)(P_d u_\sigma).$$

Becker & Rannacher 2001

Meidner & Vexler 2007

Schmich & Vexler 2008

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Dynamical Lamé system

$$
\begin{cases}
y_{tt} - \mathsf{div}(\sigma(y)) = f & \text{in } I \times \Omega, \\
y(0) = y_0 & \text{in } \Omega, \\
y_t(0) = y_1 & \text{in } \Omega, \\
y = 0 & \text{in } I \times \partial\Omega.
\end{cases}
$$

- initial data: $y_0 \in H_0^1(\Omega)^d$, $y_1 \in L^2(\Omega)^d$, $f \in L^2(I, L^2(\Omega)^d)$
- strain tensor: $\varepsilon(y) = \frac{1}{2}\left((\nabla y)^T + \nabla y\right)$
- stress tensor: $\sigma_{ij}(y) = \lambda \delta_{ij} \operatorname{tr}(\varepsilon(y)) + 2\mu\varepsilon_{ij}(y), \quad \lambda, \mu > 0$
- Proof for existence of a solution in X uses Korn's first inequality.

Optimal control of the Lamé system

$$
\begin{cases}
\text{Minimize} \quad J(u,y) = \frac{1}{2}\|y - y_d\|^2_{L^2(I,L^2(\Omega)^d)} + \frac{\alpha}{2}\|u\|^2_U, \\
\quad u \in U = L^2(I,\mathbb{R}^m), \quad y \in X, \quad \text{s.t.} \\
\quad y_{tt} - \operatorname{div}(\sigma(y)) = \mathcal{B}u \quad \text{in } Q, \\
\quad y(0) = y_0 \quad \text{in } \Omega, \\
\quad y_t(0) = y_1 \quad \text{in } \Omega, \\
\quad y = 0 \quad \text{in } \Sigma.
\end{cases}
$$

- spaces: $U = L^2(I,\mathbb{R}^m)$, $m \in \mathbb{N}$
- initial data: $y_0 \in H^1_0(\Omega)^d$, $y_1 \in L^2(\Omega)^d$, $f \in L^2(I, L^2(\Omega)^d)$
- operator: $\mathcal{B}\colon U \to L^2(I, L^2(\Omega)^d)$, $\quad \mathcal{B}u = \sum_{i=1}^m u_i(t) g_i(x)$, $\quad g_i \in L^2(\Omega)^d$

Data

$$y_0(x) = \begin{cases} (\sin(8\pi(x_1 - 0.125))\sin(8\pi(x_2 - 0.125)), 0)^T, & 0.125 < x_1, x_2 < 0.25, \\ (0,0)^T, & \text{else}, \end{cases}$$

$$y_1(x) = (0,0)^T, \quad y_d(t,x) = 0,$$

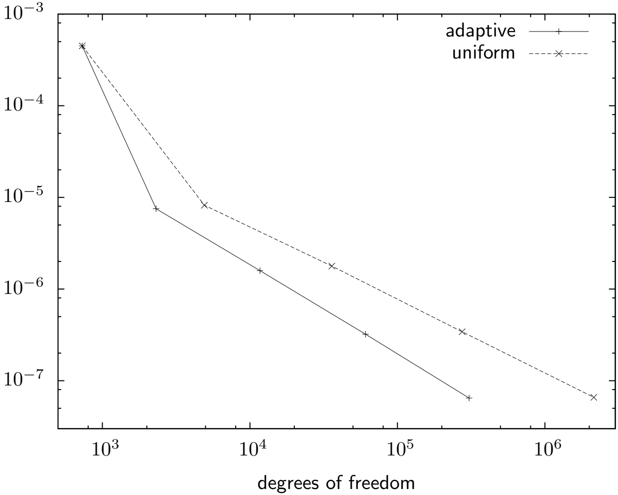
$$g_1(x) = \begin{cases} (1,1)^T, & \text{for } x_1 < 0, \\ (0,0)^T, & \text{else} \end{cases} \qquad g_2(x) = \begin{cases} (1,1)^T, & \text{for } x_1 > 0, \\ (0,0)^T, & \text{else} \end{cases}$$

$$\alpha = 0.001, \quad d = 2, \quad \lambda = 1, \quad \mu = 1, \quad m = 2$$

$$(t,x) = (t, x_1, x_2) \in [0, T] \times \Omega = [0, 0.5] \times [-1, 1]^2.$$

Discretization: $cG(1)cG(1)$, U_d: piecewise constants in time with values in $\mathbb{R}^2$

Numerical example



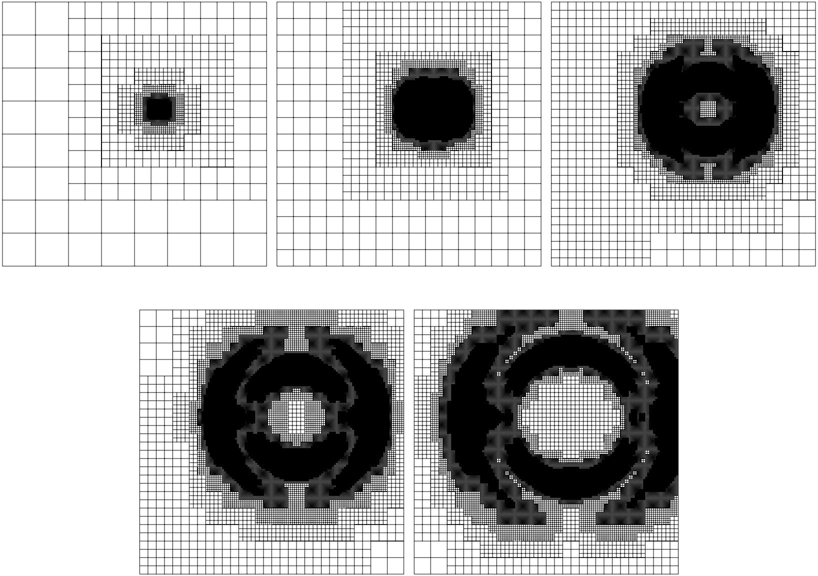
Error for adaptive and uniform refinement

Numerical example

refinement	CPU-time	dof	error
uniform	100%	100%	$6.6 \cdot 10^{-8}$
adaptive	34%	15%	$6.5 \cdot 10^{-8}$

Comparison of the CPU-time for uniform and adaptive refinement

Numerical example



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Energy

- Rademacher 2009, Eriksson et al. 1996
- Wave equation for $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$:

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } Q, \\ y(0) = y_0 & \text{in } \Omega, \\ y_t(0) = y_1 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma \end{cases}$$

- Energy:

$$E(t) = \frac{1}{2}(\|y_t(t)\|^2 + \|\nabla y(t)\|^2) = \frac{1}{2}(\|y_1\|^2 + \|\nabla y_0\|^2) = E(0)$$

Discrete energy

- Discrete equation as a time stepping scheme

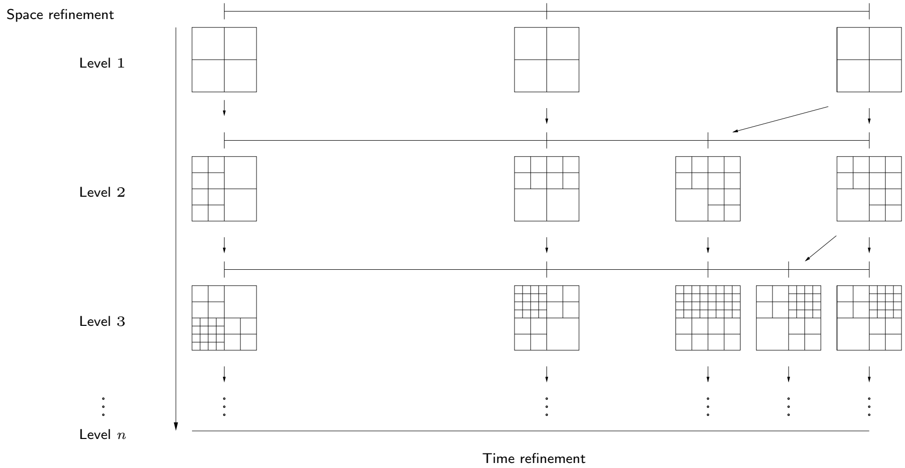
$$
\begin{aligned}
(Y_0^1,\xi) = (y_0,\xi), \quad (Y_0^2,\xi) &= (y_1,\xi) && \forall \xi \in V_h^{1,0},\\
(Y_m^2,\xi^1) + \frac{k_m}{2}(\nabla Y_m^1,\nabla\xi^1) &= (Y_{m-1}^2,\xi^1) - \frac{k_m}{2}(\nabla Y_{m-1}^1,\nabla\xi^1) && \forall \xi^1 \in V_h^{1,m},\\
(Y_m^1,\xi^2) - \frac{k_m}{2}(Y_m^2,\xi^2) &= (Y_{m-1}^1,\xi^2) + \frac{k_m}{2}(Y_{m-1}^2,\xi^2) && \forall \xi^2 \in V_h^{1,m}
\end{aligned}
$$

$Y_m^i = y_\sigma^i(t_m)$ $(i = 1,2)$ and for $m = 1,\ldots,M$.

- Discrete energy

$$
E_{k,h}(t_m) = \frac{1}{2}\left(\|Y_m^2\|^2 + \|\nabla Y_m^1)\|^2\right) \quad m = 0,\ldots,M.
$$

Refinement


Space refinement
Level 1
Level 2
Level 3
Level n
Time refinement

Behaviour of the energy

Theorem

Let $\pi_m\colon\ V_h^{1,m-1} \to V_h^{1,m}$ *for* $m = 1,\dots,M$. *Then*

$$E_{k,h}(t_m) = E_{k,h}(t_{m-1}) - \frac{1}{k_m}(Y^1_{m-1} - \pi_m Y^1_{m-1},\, Y^2_m - Y^2_{m-1})$$
$$- \frac{1}{k_m}(\pi_m Y^2_{m-1} - Y^2_{m-1},\, Y^1_m - Y^1_{m-1}) - \frac{1}{2}(Y^2_{m-1} - \pi_m Y^2_{m-1},\, Y^2_m + Y^2_{m-1})$$
$$- \frac{1}{2}(\nabla Y^1_m + \nabla Y^1_{m-1}, \nabla(Y^1_{m-1} - \pi_m Y^1_{m-1})).$$

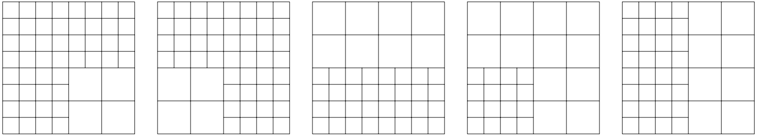
Corollary

On a given discretization level there holds

$$E_{k,h}(t_m) = E_{k,h}(t_{m-1}) \quad \textit{for } m = 1,\dots,M$$

independent of the size of k_m, *if for all steps from* t_m *to* t_{m+1} *the spatial mesh is only refined and not coarsened.*

Spatial meshes



(a) $\mathcal{T}_1$ (b) $\mathcal{T}_2$ (c) $\mathcal{T}_3$ (d) $\mathcal{T}_4$ (e) $\mathcal{T}_5$

Spatial meshes

Time point	t_0	t_1	t_2	t_3	t_4	t_5	t_6
Mesh	$\mathcal{T}_1$	$\mathcal{T}_1$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_3$
Energy	2.5327	2.5327	2.5327	2.5361	2.5361	2.5346	2.5346

Time point	t_7	t_8	t_9	t_{10}
Mesh	$\mathcal{T}_4$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_5$
Energy	2.5441	2.5441	2.5441	2.5441

Energy on a sequence of spatial meshes

Summary

- DWR method for optimal control of second order hyperbolic equations
- Separating the influence of time, space, and control discretization
- Better accuracy of the discrete solution
- Behaviour of the energy

Software

- Software package: RoDoBo, www.rodobo.uni-hd.de

Reference

- A. Kröner,
  Adaptive finite element methods for optimal control of second order hyperbolic equations
  Comput. Methods Appl. Math., 2011,

Thank you for your attention.