Bang Bang control of elliptic PDEs

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Adaptivity and Model Order Reduction in PDE Constrained Optimization

Organisers
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Model problem

$$(\mathbb{P})^\alpha \quad \begin{cases} \min\limits_{u \in U_{ad}} J(u) = \frac{1}{2}\int_\Omega |y - y_0|^2 + \frac{\alpha}{2}\|u\|_{L^2}^2 \\ \text{subject to } y = \mathcal{G}(u). \end{cases}$$

Here, $\alpha \geq 0$ and we are interested in the solution for $\alpha = 0$.

$$U_{ad} := \{v \in L^2(\Omega); a \leq u \leq b\} \subseteq L^2(\Omega)$$

with $a < b$ constants, and $y = \mathcal{G}(u)$ iff

$$-\Delta y = u \text{ in } \Omega, \text{ and } y = 0 \text{ on } \partial\Omega.$$

More general elliptic operators may be considered, and also control operators which map abstract controls to feasible right-hand sides of the elliptic equation.

Existence and uniqueness, optimality conditions

The optimal control problem admits a unique solution.

The function $u \in U_{ad}$ is a solution of the optimal control problem iff there exists an adjoint state p such that $y = \mathcal{G}(u)$, $p = \mathcal{G}(y - y_0)$ and

$$(\alpha u + p, v - u) \geq 0 \text{ for all } v \in U_{ad}.$$

There holds $u = P_{U_{\text{ad}}}\left(-\frac{1}{\alpha}p\right)$ for $\alpha > 0$,

$$u = \begin{cases} a, & \alpha u + p > 0, \\ -\frac{1}{\alpha}p, & \alpha u + p = 0, \\ b, & \alpha u + p < 0, \end{cases} \quad \text{if } \alpha > 0, \text{ and } u \begin{cases} = a, & p > 0, \\ \in [a, b] & p = 0, \\ = b, & p < 0, \end{cases} \quad \text{if } \alpha = 0.$$

Variational discretization

Discrete optimal control problem:

$$(\mathbb{P})_h^\alpha \quad \begin{cases} \min\limits_{u \in U_{ad}} J(u) = \frac{1}{2}\int_\Omega |y_h - y_0|^2 + \frac{\alpha}{2}\|u\|_{L^2}^2 \\ \text{subject to } y_h = \mathcal{G}_h(u). \end{cases}$$

Here, $\mathcal{G}_h(u)$ denotes the piecewise linear and continuous finite element approximation to $y(u)$, i.e.

$$a(y_h, v_h) := (\nabla y_h, \nabla v_h) = (u, v_h) \text{ for all } v_h \in X_h,$$

where with the triangulation $\mathcal{T}_h$

$$X_h := \{w \in C^0(\bar{\Omega}); w_{|\partial\Omega} = 0, w_{|T} \text{ linear for all } T \in \mathcal{T}_h\}.$$

This problem is still $\infty-$dimensional.

Ritz projection $R_h : H_0^1(\Omega) \to X_h$,

$$a(R_h w, v_h) = a(w, v_h) \text{ for all } v_h \in X_h$$

Existence and uniqueness, optimality conditions for discrete problem

The variational-discrete optimal control problems admits a solution $u_h \in U_{ad}$, which is unique in the case $\alpha > 0$. The state y_h is unique (also in the case $\alpha = 0$).

Let $u_h \in U_{ad}$ be a solution of the optimal control problem. Then there exists a unique adjoint state p_h such that $y_h = \mathcal{G}_h(u_h)$, $p_h = \mathcal{G}_h(y_h - y_0)$ and

$$(\alpha u_h + p_h, v - u_h) \geq 0 \text{ for all } v \in U_{ad}.$$

There holds $u_h = P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha} p_h\right)$ for $\alpha > 0$,

$$u_h = \begin{cases} a, & \alpha u_h + p_h > 0, \\ -\frac{1}{\alpha} p_h, & \alpha u_h + p_h = 0, \\ b, & \alpha u_h + p_h < 0, \end{cases} \quad \text{if } \alpha > 0, \text{ and } u_h \begin{cases} = a, & p_h > 0, \\ \in [a, b] & p_h = 0, \\ = b, & p_h < 0, \end{cases} \quad \text{if } \alpha = 0.$$

Error estimates

It is well known that

$$\|y - y_h\| + \alpha\|u - u_h\| \sim \|y - y_h(u)\| + \|p - p_h(y(u))\|$$

So one expects estimates for $y - y_h$ also in the case $\alpha = 0$.

Estimates for $\|u - u_h\|$?

Estimate for the states ($S := \overline{\{x \in \Omega \mid p(x) \neq 0\}} \subset \bar{\Omega}$)

$$\begin{aligned} \|y - y_h\| &\leq C\left(h^2 + (b-a)\|p - R_h p\|_{L^1(\Omega\setminus S)} + \|p - R_h p\|_{L^\infty}\|u - u_h\|_{L^1(S)}\right), \\ \|p - p_h\|_{L^\infty} &\leq C\|y - y_h\| + \|p - R_h p\|_{L^\infty}, \end{aligned}$$

follow from

- $0 \leq (p - p_h, u_h - u) = (R_h p - p_h, u_h - u) + (p - R_h p, u_h - u) \equiv I + II.$
- $I \leq -\frac{1}{2}\|y - y_h\|^2 + \frac{1}{2}\|y - R_h y\|^2$
- $II = \int_{\Omega\setminus S}(p - R_h p)(u_h - u) + \int_S (p - R_h p)(u_h - u).$

Error estimates

Structural assumption

$$\exists C > 0 \forall \epsilon > 0 : \mathcal{L}(\{x \in \bar{\Omega}; |p(x)| \leq \epsilon\}) \leq C\epsilon^{\beta}$$

for the solution u at $\alpha = 0$ with some $\beta \in (0, 1]$ yields

$$\begin{aligned} \|y - y_h\| + \|p - p_h\|_{L^\infty} &\leq C\left(h^2 + \|p - R_h p\|_{L^\infty}^{\frac{1}{2-\beta}}\right); \\ \|u - u_h\|_{L^1} &\leq C\left(h^{2\beta} + \|p - R_h p\|_{L^\infty}^{\frac{\beta}{2-\beta}}\right). \end{aligned}$$

Sketch of proof for $\beta = 1$

$$\|u - u_h\|_{L^1}, \|y - y_h\|, \|p - p_h\|_{L^\infty} \le C\left\{h^2 + \|p - R_h p\|_{L^\infty}\right\}$$

Sketch of proof:

- $0 \le (p - p_h, u_h - u) = (R_h p - p_h, u_h - u) + (p - R_h p, u_h - u) \equiv I + II.$
- $I \le -\frac{1}{2}\|y - y_h\|^2 + \frac{1}{2}\|y - R_h y\|^2$
- $II = \int_S (p - R_h p)(u_h - u)$. Combine now
  - $\|u - u_h\|_{L^1} \le (b - a)\mathcal{L}(\{p > 0, p_h \le 0\} \cup \{p < 0, p_h \ge 0\})$
  - $\{p > 0, p_h \le 0\} \cup \{p < 0, p_h \ge 0\} \subseteq \{|p(x)| \le \|p - p_h\|_\infty\} \Rightarrow$
  - $\mathcal{L}(\{|p(x)| \le \|p - p_h\|_\infty\}) \le C\|p - p_h\|_\infty$
  - $\|u - u_h\|_{L^1} \le C\|p - p_h\|_\infty$
  - $\|p - p_h\|_\infty \le \|p - R_h p\|_\infty + \|R_h p - p_h\|_\infty$
  - $\|R_h p - p_h\|_\infty \le C\|y - y_h\|$

  to estimate II.

Special cases

1. $u_0 \in U_{ad}$ exists such that $y_0 = \mathcal{G}(u_0)$. Then
$$\|y - y_h\| + \|p - p_h\|_{L^\infty} \leq Ch^2.$$

2. If $p \in C^1(\bar{\Omega})$ satisfies
$$\min_{x \in K} |\nabla p(x)| > 0, \qquad \text{where } K = \{x \in \bar{\Omega} \,|\, p(x) = 0\}.$$
Then, the structural assumption is satisfied with $\beta = 1$.

3. If $p \in W^{2,\infty}(\Omega)$ and satisfies the structural assumption, then
$$\|y - y_h\| + \|p - p_h\|_{L^\infty} + \|u - u_h\|_{L^1} \leq Ch^2 |\log h|^{\gamma(d)}.$$

Algorithms for $\mathbb{P}_h^\alpha$

Define

$$G_h(u) = u - P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha} p_h(y_h(u))\right).$$

The optimality condition reads $G_h(u) = 0$ and motivates the fix–point iteration

- u given, do until convergence

$$u^+ = P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha} p_h(y_h(u))\right), \quad u = u^+.$$

1. Is this algorithm numerically implementable?

Yes, whenever for given u it is possible to numerically evaluate the expression

$$P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha} p_h(y_h(u))\right)$$

in the $i - th$ iteration, with an numerical overhead which is *independent* of the iteration counter of the algorithm.

Semi–smooth Newton algorithm for $\alpha > 0$

2. **Does the fix–point algorithm converge?**

Yes, if $\alpha > \|RB^*S_h^*S_hB\|_{\mathcal{L}(U)}$, since $P_{U_{ad}}$ is non–expansive.

Condition too restrictive for our purpose $\to$ semi–smooth Newton method applied to $G_h(u) = 0$:

- u given, solve until convergence

$$G_h'(u)u^+ = -G_h(u) + G_h'(u)u, \quad u = u^+.$$

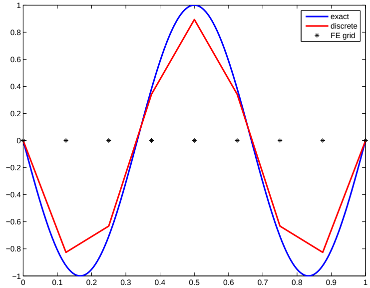
1. This algorithm is implementable whenever the fix–point iteration is, since

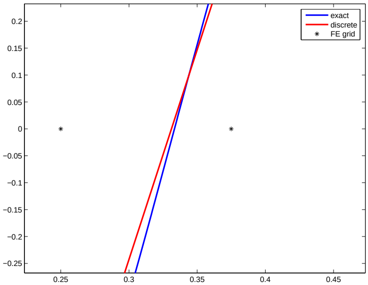
$$-G_h(u) + G_h'(u)u =$$

$$= -P_{U_{ad}}\left(-\frac{1}{\alpha}p_h(u)\right) - \frac{1}{\alpha}P_{U_{ad}}'\left(-\frac{1}{\alpha}p_h(u)\right)S_h^*S_hu.$$

2. For every $\alpha > 0$ this algorithm is locally fast convergent (H. (COAP 2005), Vierling).

Numerical example with 2 switching points, fix-point iteration





Experimental order of convergence:

- $||u - u_h||_{L^1}$: 3.00077834
- Function values 1.99966106
- $||p - p_h||_{L^\infty}$: 1.99979367
- $||y - y_h||_{L^\infty}$: 1.9997965
- $||p - p_h||_{L^2}$: 1.99945711

Homotopy in α with semi–smooth Newton, Tröltzsch checkerboard

D. & G. Wachsmuth (ESAIM: COCV 2011 (Preprint 2009)), von Daniels (Diploma Thesis 2010):

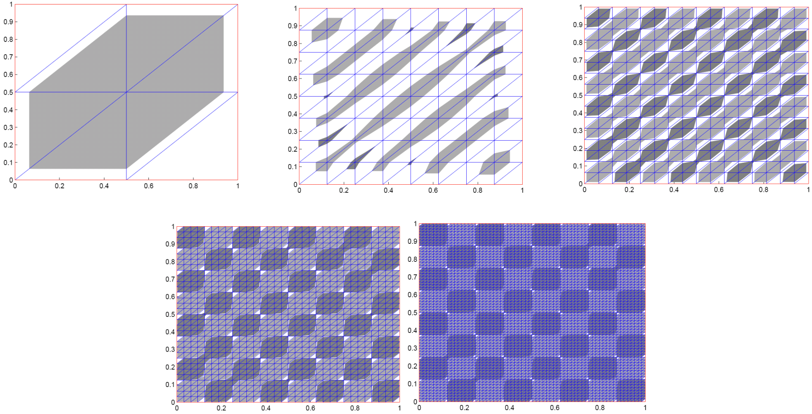
- $\|u_0 - u_\alpha\| \sim \sqrt{\alpha}$,
- $\|u_\alpha - u_{\alpha,h}\| \sim h^2\alpha^{-1}$, thus
- $\|u_0 - u_{\alpha,h}\| \sim h^{\frac{2}{3}}$

$$u(x) = -\text{sign}\, p(x),\ p(x) = -\frac{1}{128\pi^2}\sin(8\pi x_1)\sin(8\pi x_2),\ y(x) = \sin(\pi x_1)\sin(\pi x_2).$$

Loop i	$\|u-u_h\|_{L^1}$	$\|u-u_h\|_{L^2}$	$EOC_{L^1}(u)$	$EOC_{L^2}(u)$	Nit
3	2.5008e-001	4.7416e-001	1.10	0.61	4
4	1.2045e-001	3.4864e-001	1.05	0.44	5
5	3.6487e-002	1.9368e-001	1.72	0.85	4
6	5.8124e-003	6.2070e-002	1.33	0.82	3
7	2.1287e-003	3.7590e-002	1.45	0.72	3
mean			1.33	0.69	

Numerical example by Nicolaus von Daniels

Checkerboard example, plots



Related approaches, next steps

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- **In a recent talk Walter Alt for linear–quadratic optimal control problems with ODEs proposed to use the zeros of the discrete switching function to define the control $\rightarrow$**
- **This relates to post–processing of Meyer/Rösch combined with piecewise constant control approximations in the present situation. Structural assumptions on p imply the required *regularity* of the discrete active set.**

Next steps:

- **Parabolic problems**

Thank you very much for your attention!

Related approaches, next steps

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