Optimality Conditions in Optimal Control of Elastoplasticity

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Gerd Wachsmuth — Numerical Mathematics

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Workshop on Control and Optimization of PDEs

Graz, October 10, 2011

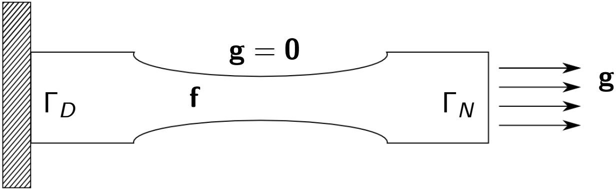
DFG SPP 1253

Outline

Outline

This talk: static (incremental) setting
See talk by Gerd Wachsmuth (Tue, 9:30) for quasi-static setting and numerics

Typical Configuration in Linear Elasticity



Material laws and boundary conditions

$\mathbb{C}^{-1}\boldsymbol{\sigma} = \varepsilon(\mathbf{u})$	in Ω	Hooke's Law
$\nabla \cdot \boldsymbol{\sigma} = -\mathbf{f}$	in Ω	equilibrium condition
$\mathbf{u} = \mathbf{0}$	on Γ_D	displacement b/c
$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{g}$	on Γ_N	stress b/c

$$C_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, (\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk})$$

Variables

$\boldsymbol{\sigma}$ stress tensor

$\mathbf{u}$ displacement vector

$\varepsilon(\mathbf{u})$ lin. strain tensor

$$\varepsilon(\mathbf{u}) = \tfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$$

Linear elasticity

$$\begin{aligned} \text{Minimize} \quad & \tfrac{1}{2} a(\boldsymbol{\sigma}, \boldsymbol{\sigma}) \\ \text{s.t.} \quad & b(\boldsymbol{\sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V \end{aligned}$$

Elasticity & Plasticity: Energy Minimization

Linear elasticity

$$\text{Minimize} \quad \tfrac{1}{2} a(\boldsymbol{\sigma}, \boldsymbol{\sigma})$$
$$\text{s.t.} \quad b(\boldsymbol{\sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V$$

Bilinear and linear forms

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} : \mathbb{C}^{-1} \boldsymbol{\tau} \, \mathrm{d}x$$

$$b(\boldsymbol{\sigma}, \mathbf{v}) = -\int_\Omega \boldsymbol{\sigma} : \varepsilon(\mathbf{v}) \, \mathrm{d}x, \quad \varepsilon(\mathbf{u}) = \tfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$$

$$\langle \ell, \mathbf{v} \rangle = -\int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x - \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v} \, \mathrm{d}s$$

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$$\boldsymbol{\sigma} \in S = L^2(\Omega; \mathbb{R}^{d \times d}_{\text{sym}}), \qquad \mathbf{u} \in V = H^1_{\Gamma_D}(\Omega; \mathbb{R}^d), \quad [\mathbf{u} = 0 \text{ on } \Gamma_D]$$

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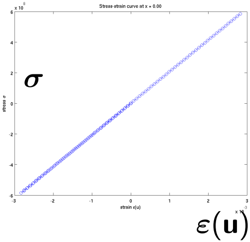

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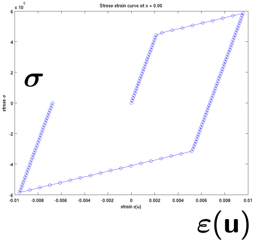

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Elasticity & Plasticity: Energy Minimization

Static plasticity (linear kinematic hardening)

$$
\begin{aligned}
\text{Minimize} \quad & \tfrac{1}{2} a(\boldsymbol{\Sigma}, \boldsymbol{\Sigma}), \qquad \boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi}) \\
\text{s.t.} \quad & b(\boldsymbol{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V \\
\text{and} \quad & \boldsymbol{\Sigma} \in \mathcal{K} \quad \text{(convex)}
\end{aligned}
$$

Bilinear and linear forms

$$a(\boldsymbol{\Sigma}, \mathbf{T}) = \int_\Omega \boldsymbol{\sigma} : \mathbb{C}^{-1} \boldsymbol{\tau} \, \mathrm{d}x + \int_\Omega \boldsymbol{\chi} : \mathbb{H}^{-1} \boldsymbol{\mu} \, \mathrm{d}x$$

$$b(\boldsymbol{\Sigma}, \mathbf{v}) = - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v}) \, \mathrm{d}x, \qquad \boldsymbol{\varepsilon}(\mathbf{u}) = \tfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$$

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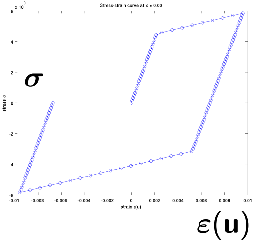


$$\boldsymbol{\chi}, \boldsymbol{\sigma} \in S = L^2(\Omega; \mathbb{R}^{d \times d}_{\text{sym}}), \qquad \mathbf{u} \in V = H^1_{\Gamma_D}(\Omega; \mathbb{R}^d), \qquad [\mathbf{u} = 0 \text{ on } \Gamma_D]$$

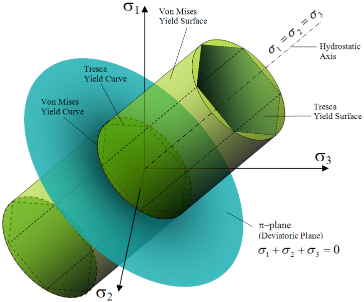
Kinematic Hardening Model: Yield Condition

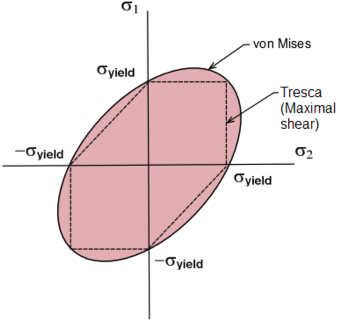
Von Mises yield condition (linear kinematic hardening)

$$K = \Big\{(\boldsymbol{\sigma}, \boldsymbol{\chi}) \in \mathbb{R}^{d\times d}_{\text{sym}} : |\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D|_{\text{Frob}} \le \widetilde{\sigma}_0 := \sqrt{2/3}\,\sigma_0\Big\}$$

$$\mathcal{K} = \{\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi}) \in S \times S : (\boldsymbol{\sigma}(x), \boldsymbol{\chi}(x)) \in K \text{ a.e. in } \Omega\}$$

$$\mathbf{A}^D = \mathbf{A} - \tfrac{1}{d}(\operatorname{trace}\mathbf{A})\,\mathbf{I}$$





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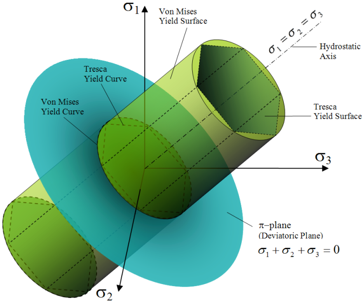
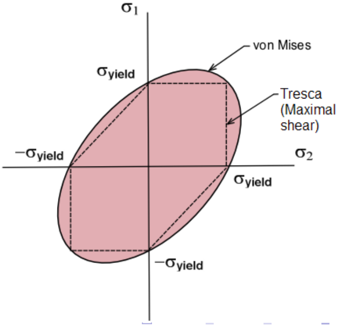
$$\mathcal{D}\boldsymbol{\Sigma} = \boldsymbol{\sigma}^D + \boldsymbol{\chi}^D$$





Two Ways to Write the Forward Problem

The unique minimizer $(\boldsymbol{\sigma}, \boldsymbol{\chi}, \mathbf{u}) \in S \times S \times V$ is characterized by $\boldsymbol{\Sigma} \in \mathcal{K}$,

$$\begin{aligned} a(\boldsymbol{\Sigma}, \mathbf{T} - \boldsymbol{\Sigma}) + b(\mathbf{T} - \boldsymbol{\Sigma}, \mathbf{u}) &\geq 0 \quad \text{for all } \mathbf{T} = (\boldsymbol{\tau}, \boldsymbol{\mu}) \in \mathcal{K} \\ b(\boldsymbol{\Sigma}, \mathbf{v}) &= \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V \end{aligned}$$

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b(\boldsymbol{\Sigma}, \mathbf{v}) &= \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V
\end{aligned}
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Note: The displacement field $\mathbf{u}$ acts as a Lagrange multiplier.

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Equivalently, there exists $\lambda \in L^2(\Omega)$ (plastic multiplier) such that

$$\begin{aligned} a(\boldsymbol{\Sigma}, \mathbf{T}) + b(\mathbf{T}, \mathbf{u}) + c(\lambda, \boldsymbol{\Sigma}, \mathbf{T}) = 0 \quad &\text{for all } \mathbf{T} = (\boldsymbol{\tau}, \boldsymbol{\mu}) \in S \times S \\ b(\boldsymbol{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad &\text{for all } \mathbf{v} \in V \\ 0 \leq \lambda \quad \perp \quad \tfrac{1}{2}\left(|\mathcal{D}\boldsymbol{\Sigma}|^2 - \widetilde{\sigma}_0^2\right) \leq 0 \quad &\text{a.e. in } \Omega \end{aligned}$$

$$c(\lambda, \boldsymbol{\Sigma}, \mathbf{T}) = \int_\Omega \lambda \quad \mathcal{D}\boldsymbol{\Sigma} : \mathcal{D}\mathbf{T} \, dx$$

[Herzog, Meyer, Wachsmuth (GAMM, 2011)]

Two Ways to Write the Forward Problem

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$$c(\lambda, \boldsymbol{\Sigma}, \mathbf{T}) = \int_\Omega \underbrace{\lambda}_{L^2} \, \underbrace{\mathcal{D}\boldsymbol{\Sigma}}_{L^\infty} : \underbrace{\mathcal{D}\mathbf{T}}_{L^2} \, dx$$

[Herzog, Meyer, Wachsmuth (GAMM, 2011)]

Comparison to the Obstacle Problem

Static Plasticity Problem

$$a(\boldsymbol{\Sigma}, \mathbf{T} - \boldsymbol{\Sigma}) + b(\boldsymbol{\tau} - \boldsymbol{\sigma}, \mathbf{u}) \ge 0 \quad \forall \mathbf{T} \in \mathcal{K}$$

$$b(\boldsymbol{\sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V$$

with $\quad \langle \ell, \mathbf{v} \rangle = -\big(\mathbf{f}, \mathbf{v}\big)_\Omega - \big(\mathbf{g}, \mathbf{v}\big)_{\Gamma_N}$

$$\mathcal{K} = \big\{ \boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \chi) : |\mathcal{D}\boldsymbol{\Sigma}| \le \widetilde{\sigma}_0 \big\}$$

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Obstacle Problem

$$a(y, z - y) \geq (f, z - y)_\Omega \quad \forall z \in \mathcal{K}$$

with $a(y, z) = (\nabla y, \nabla z)_\Omega$

$$\mathcal{K} = \{y \in H_0^1(\Omega) : y \geq 0\}$$

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- VI in mixed form

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- VI in mixed form
- a algebraic, b 1st order

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- elliptic VI
- a 2nd order

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- VI in mixed form
- a algebraic, b 1st order
- no regularity gain: $L^2 \ni \mathbf{f} \mapsto \mathbf{\Sigma} \in L^2$

Obstacle Problem

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- elliptic VI
- a 2nd order
- substantial regularity gain: $L^2 \ni f \mapsto y \in H_0^1$ or even H^2

Comparison to the Obstacle Problem

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- VI in mixed form
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- moderate regularity gain: $L^2 \ni \mathbf{f} \mapsto \boldsymbol{\Sigma} \in L^p$

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[Herzog, Meyer, Wachsmuth (JMAA, 2011)]

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- VI in mixed form
- a algebraic, b 1st order
- moderate regularity gain: $L^2 \ni \mathbf{f} \mapsto \boldsymbol{\Sigma} \in L^p$
- admissible set $\mathcal{K}$ more involved

Obstacle Problem

$$a(y, z - y) \geq (f, z - y)_\Omega \quad \forall z \in \mathcal{K}$$
with $a(y, z) = (\nabla y, \nabla z)_\Omega$

$$\mathcal{K} = \{y \in H^1_0(\Omega) : y \geq 0\}$$

- elliptic VI
- a 2nd order
- substantial regularity gain: $L^2 \ni f \mapsto y \in H^1_0$ or even H^2
- admissible set $\mathcal{K}$ simple

[Herzog, Meyer, Wachsmuth (JMAA, 2011)]

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- VI in mixed form
- a algebraic, b 1st order
- moderate regularity gain: $L^2 \ni \mathbf{f} \mapsto \mathbf{\Sigma} \in L^p$
- admissible set $\mathcal{K}$ more involved
- Lagr. multiplier λ non-trivial

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with $a(y, z) = (\nabla y, \nabla z)_\Omega$

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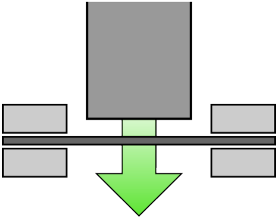
- elliptic VI
- a 2nd order
- substantial regularity gain: $L^2 \ni f \mapsto y \in H_0^1$ or even H^2
- admissible set $\mathcal{K}$ simple
- Lagr. multiplier $\lambda := f + \triangle y$

[Herzog, Meyer, Wachsmuth (JMAA, 2011)]

Outline

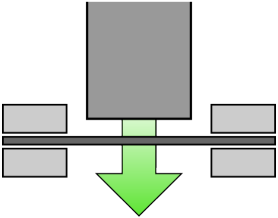
Application: Control of Springback



Deep drawing

- car body parts
- plane body parts
- packings

Application: Control of Springback



Deep drawing

- car body parts
- plane body parts
- packings

Springback

- release of stored elastic energy once the loads are withdrawn
- partial restoration away from the desired shape

A Control Problem in Plasticity

Upper-level problem

$$\text{Minimize} \quad \frac{1}{2}\|\mathbf{u}-\mathbf{u}_d\|^2_{L^2(\Omega;\mathbb{R}^d)} + \frac{\nu_1}{2}\|\mathbf{f}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \frac{\nu_2}{2}\|\mathbf{g}\|^2_{L^2(\Gamma_N;\mathbb{R}^d)}$$

$$\text{s.t.} \quad \langle \ell, \mathbf{v}\rangle = -\int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx - \int_{\Gamma_N}\mathbf{g}\cdot\mathbf{v}\,ds \quad \text{and} \quad \cdots$$

Lower-level problem

$$\text{Minimize} \quad \frac{1}{2}a(\mathbf{\Sigma},\mathbf{\Sigma}), \quad \mathbf{\Sigma}\in\mathcal{K}$$

$$\text{s.t.} \quad b(\mathbf{\Sigma},\mathbf{v}) = \langle \ell, \mathbf{v}\rangle \quad \text{for all } \mathbf{v}\in V$$

A Control Problem in Plasticity

Upper-level problem

$$\text{Minimize} \quad \frac{1}{2}\|\mathbf{u}-\mathbf{u}_d\|^2_{L^2(\Omega;\mathbb{R}^d)} + \frac{\nu_1}{2}\|\mathbf{f}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \frac{\nu_2}{2}\|\mathbf{g}\|^2_{L^2(\Gamma_N;\mathbb{R}^d)}$$

$$\text{s.t.} \quad \langle \ell, \mathbf{v}\rangle = -\int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx - \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\,ds \quad \text{and} \quad \cdots$$

$$\mathcal{K} = \{\mathbf{\Sigma} : \phi(\mathbf{\Sigma}) \le 0\}$$

Variational inequality

$$a(\mathbf{\Sigma}, \mathbf{T}-\mathbf{\Sigma}) + b(\mathbf{T}-\mathbf{\Sigma}, \mathbf{u}) \ge 0$$
$$\text{for all } \mathbf{T} = (\boldsymbol{\tau}, \boldsymbol{\mu}) \in \mathcal{K}$$
$$b(\mathbf{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in V$$

MPEC

Complementarity system

$$a(\mathbf{\Sigma}, \mathbf{T}) + b(\mathbf{T}, \mathbf{u}) + (\lambda\,\phi'(\mathbf{\Sigma}), \mathbf{T}) = 0$$
$$\text{for all } \mathbf{T} = (\boldsymbol{\tau}, \boldsymbol{\mu}) \in S \times S$$
$$b(\mathbf{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in V$$
$$0 \le \lambda \quad \perp \quad \phi(\mathbf{\Sigma}) \le 0$$

MPCC

MPEC: Non-smooth control-to-state map. **MPCC:** Classical Lagrange multiplier approach for upper-level problem unsuitable, several stationarity concepts exist.

MPECs & MPCCs in Function Space

Contributors:

- Mignot, Puel
- Barbu
- Bermúdez, Saguez

- Bonnans, Casas
- Bergounioux

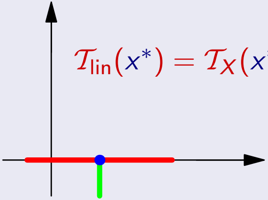
- Mordukhovich
- Ito, Kunisch
- Hintermüller, Kopacka, Rautenberg, Surowiec, Tber, Wegner
- D. Wachsmuth
- Farshbaf-Shaker

Example (MPCC)

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \ge 0, \quad x_2 \ge 0, \quad x_1\, x_2 = 0$$

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 \, x_2 = 0$$

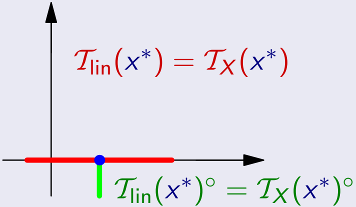
singly active point



Example (MPCC)

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$$

singly active point



Example (MPCC)

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$$

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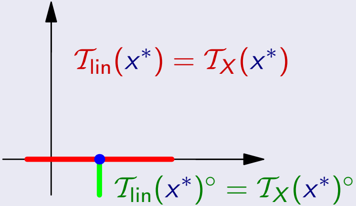
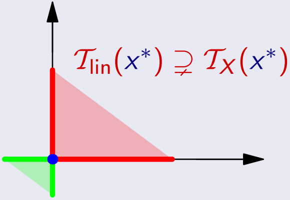


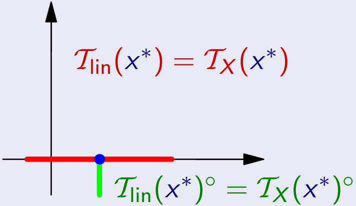
bi-active point



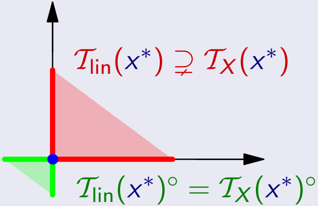
Example (MPCC)

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$$

singly active point



bi-active point

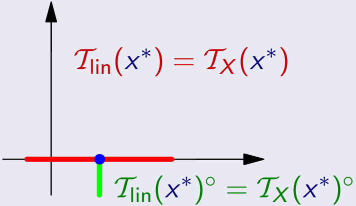


Example (MPCC)

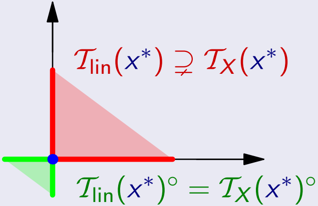
$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$$

$$\mathcal{L}(x, \mu, \lambda) = \quad f(x) - \mu^\top x \quad + \lambda\, x_1 x_2$$

singly active point



bi-active point

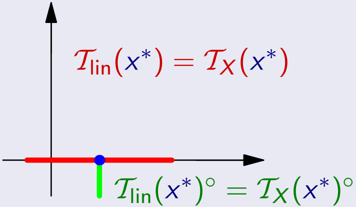


Example (MPCC)

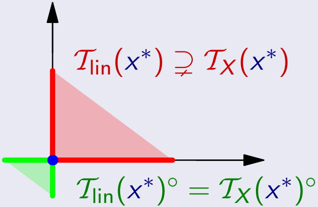
$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 x_2 = 0$$

$$\nabla_x \mathcal{L}(x, \mu, \lambda) = \nabla f(x) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

singly active point



bi-active point

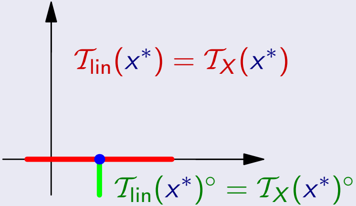


Example (MPCC)

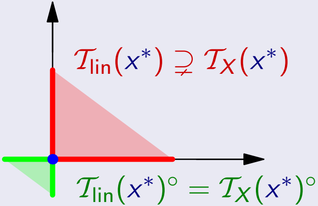
Minimize $f(x)$ s.t. $x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$

$$\nabla_x \mathcal{L}(x, \mu, \lambda) = \nabla f(x) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

singly active point



bi-active point



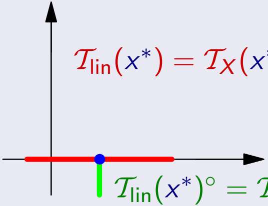
$$\nabla f(x^*) - \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ x_1^* \end{pmatrix} = 0$$

Example (MPCC)

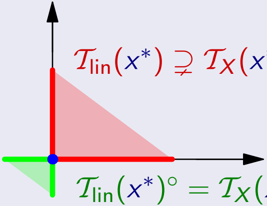
$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$$

$$\nabla_x \mathcal{L}(x, \mu, \lambda) = \nabla f(x) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

singly active point



bi-active point



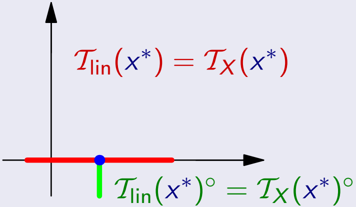
$$\nabla f(x^*) - \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ x_1^* \end{pmatrix} = 0 \qquad \nabla f(x^*) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

Example (MPCC)

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$$

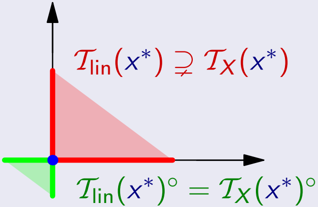
$$\nabla_x \mathcal{L}(x, \mu, \lambda) = \nabla f(x) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

singly active point



$$\nabla f(x^*) - \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ x_1^* \end{pmatrix} = 0$$

bi-active point



$$\nabla f(x^*) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

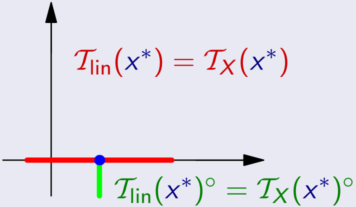
redundant multiplier λ, MFCQ does not hold

Example (MPCC)

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$$

$$\nabla_x \mathcal{L}(x,\mu,\lambda) = \nabla f(x) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

singly active point



$$\nabla f(x^*) - \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ x_1^* \end{pmatrix} = 0$$

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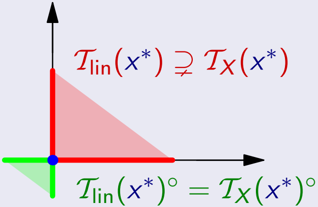

$$\nabla f(x^*) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

$\rightsquigarrow$ algorithmic difficulties

[Fletcher, Leyffer (2004)]

Some Stationarity Concepts for MPCCs

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$$

KKT conditions

$$\mathcal{L} = f(x) - \mu^\top x + \lambda\, x_1 x_2$$

$$\nabla_x \mathcal{L} = \nabla f(x) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$\mu_1 \geq 0, \quad x_1 \geq 0, \quad \mu_1\, x_1 = 0$$

$$\mu_2 \geq 0, \quad x_2 \geq 0, \quad \mu_2\, x_2 = 0$$

Some Stationarity Concepts for MPCCs

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$$

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$$\mu_1 \geq 0, \quad x_1 \geq 0, \quad \mu_1\, x_1 = 0$$

$$\mu_2 \geq 0, \quad x_2 \geq 0, \quad \mu_2\, x_2 = 0$$

Some Stationarity Concepts for MPCCs

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$$

KKT conditions

$$\mathcal{L} = f(x) - \mu^\top x + \lambda\, x_1 x_2$$

$$\nabla_x \mathcal{L} = \nabla f(x) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$\mu_1 \geq 0, \quad x_1 \geq 0, \quad \mu_1\, x_1 = 0$$

$$\mu_2 \geq 0, \quad x_2 \geq 0, \quad \mu_2\, x_2 = 0$$

Some Stationarity Concepts for MPCCs

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \ge 0, \quad x_2 \ge 0, \quad x_1\,x_2 = 0$$

KKT conditions

$$\mathcal{L} = f(x) - \mu^\top x + \lambda\, x_1 x_2$$

$$\nabla_x \mathcal{L} = \nabla f(x) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$\mu_1 \ge 0, \quad x_1 \ge 0, \quad \mu_1\, x_1 = 0$$

$$\mu_2 \ge 0, \quad x_2 \ge 0, \quad \mu_2\, x_2 = 0$$

MPCC: stationarity

$$\widehat{\mathcal{L}} = f(x) - \widehat{\mu}^\top x$$

$$\nabla_x \widehat{\mathcal{L}} = \nabla f(x) - \begin{pmatrix} \widehat{\mu}_1 \\ \widehat{\mu}_2 \end{pmatrix}$$

$$\widehat{\mu}_1 \in \mathbb{R}, \quad x_1 \ge 0, \quad \widehat{\mu}_1\, x_1 = 0$$

$$\widehat{\mu}_2 \in \mathbb{R}, \quad x_2 \ge 0, \quad \widehat{\mu}_2\, x_2 = 0$$

Some Stationarity Concepts for MPCCs

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \ge 0, \quad x_2 \ge 0, \quad x_1\, x_2 = 0$$

KKT conditions

$$\mathcal{L} = f(x) - \mu^\top x + \lambda\, x_1 x_2$$

$$\nabla_x \mathcal{L} = \nabla f(x) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$\mu_1 \ge 0, \quad x_1 \ge 0, \quad \mu_1\, x_1 = 0$$

$$\mu_2 \ge 0, \quad x_2 \ge 0, \quad \mu_2\, x_2 = 0$$

MPCC: weak stationarity

$$\widehat{\mathcal{L}} = f(x) - \widehat{\mu}^\top x$$

$$\nabla_x \widehat{\mathcal{L}} = \nabla f(x) - \begin{pmatrix} \widehat{\mu}_1 \\ \widehat{\mu}_2 \end{pmatrix}$$

$$\widehat{\mu}_1 \in \mathbb{R}, \quad x_1 \ge 0, \quad \widehat{\mu}_1\, x_1 = 0$$

$$\widehat{\mu}_2 \in \mathbb{R}, \quad x_2 \ge 0, \quad \widehat{\mu}_2\, x_2 = 0$$

MPCC stationarity concepts

weak stat.

Some Stationarity Concepts for MPCCs

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$$

KKT conditions

$$\mathcal{L} = f(x) - \mu^\top x + \lambda\, x_1 x_2$$

$$\nabla_x \mathcal{L} = \nabla f(x) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$\mu_1 \geq 0, \quad x_1 \geq 0, \quad \mu_1\, x_1 = 0$$

$$\mu_2 \geq 0, \quad x_2 \geq 0, \quad \mu_2\, x_2 = 0$$

MPCC: weak stationarity

$$\widehat{\mathcal{L}} = f(x) - \widehat{\mu}^\top x$$

$$\nabla_x \widehat{\mathcal{L}} = \nabla f(x) - \begin{pmatrix} \widehat{\mu}_1 \\ \widehat{\mu}_2 \end{pmatrix}$$

$$\widehat{\mu}_1 \in \mathbb{R}, \quad x_1 \geq 0, \quad \widehat{\mu}_1\, x_1 = 0$$

$$\widehat{\mu}_2 \in \mathbb{R}, \quad x_2 \geq 0, \quad \widehat{\mu}_2\, x_2 = 0$$

MPCC stationarity concepts

strong stat. $\Rightarrow$ M-stationarity $\Rightarrow$ C-stationarity $\Rightarrow$ weak stat.

- differ only in conditions for $\widehat{\mu}_1$, $\widehat{\mu}_2$, if $x_1 = x_2 = 0$

[Scheel, Scholtes (2000)]

Some Stationarity Concepts for MPCCs

$$\text{Minimize} \quad f(x) \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1\, x_2 = 0$$

KKT conditions

$$\mathcal{L} = f(x) - \mu^\top x + \lambda\, x_1 x_2$$

$$\nabla_x \mathcal{L} = \nabla f(x) - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \lambda \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$\mu_1 \geq 0, \quad x_1 \geq 0, \quad \mu_1\, x_1 = 0$$

$$\mu_2 \geq 0, \quad x_2 \geq 0, \quad \mu_2\, x_2 = 0$$

MPCC: weak stationarity

$$\widehat{\mathcal{L}} = f(x) - \widehat{\mu}^\top x$$

$$\nabla_x \widehat{\mathcal{L}} = \nabla f(x) - \begin{pmatrix} \widehat{\mu}_1 \\ \widehat{\mu}_2 \end{pmatrix}$$

$$\widehat{\mu}_1 \in \mathbb{R}, \quad x_1 \geq 0, \quad \widehat{\mu}_1\, x_1 = 0$$

$$\widehat{\mu}_2 \in \mathbb{R}, \quad x_2 \geq 0, \quad \widehat{\mu}_2\, x_2 = 0$$

MPCC stationarity concepts

strong stat. $\Rightarrow$ M-stationarity $\Rightarrow$ C-stationarity $\Rightarrow$ weak stat.

- differ only in conditions for $\widehat{\mu}_1$, $\widehat{\mu}_2$, if $x_1 = x_2 = 0$
- "Limits of regularized MPCCs satisfy C-(or M-)stationarity"

[Scheel, Scholtes (2000); Kanzow, Schwartz (2010)]

Regularization by Penalization

Lower level (forward) problem

$$\begin{aligned} \text{Minimize} \quad & \tfrac{1}{2} a(\boldsymbol{\Sigma}, \boldsymbol{\Sigma}), \quad \boldsymbol{\Sigma} \in \mathcal{K} \\ \text{s.t.} \quad & b(\boldsymbol{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V \end{aligned}$$

Regularization by Penalization

Lower level (forward) problem

$$\text{Minimize} \quad \frac{1}{2}a(\boldsymbol{\Sigma}, \boldsymbol{\Sigma}) + I_\gamma(\boldsymbol{\Sigma}) \quad \text{(penalize constraint violation)}$$

$$\text{s.t.} \quad b(\boldsymbol{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V$$

Regularization by Penalization

Lower level (forward) problem

$$\begin{aligned} \text{Minimize} \quad & \tfrac{1}{2} a(\boldsymbol{\Sigma}, \boldsymbol{\Sigma}) + I_\gamma(\boldsymbol{\Sigma}) \quad \text{(penalize constraint violation)} \\ \text{s.t.} \quad & b(\boldsymbol{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V \end{aligned}$$

Regularized optimality conditions

$$\begin{aligned} a(\boldsymbol{\Sigma}, \mathbf{T}) + b(\mathbf{T}, \mathbf{u}) + I'_\gamma(\boldsymbol{\Sigma})\, \mathbf{T} = 0 \quad & \text{for all } \mathbf{T} = (\boldsymbol{\tau}, \boldsymbol{\mu}) \in S^2 \\ b(\boldsymbol{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad & \text{for all } \mathbf{v} \in V \end{aligned}$$

Regularization by Penalization

Lower level (forward) problem

$$\text{Minimize} \quad \tfrac{1}{2} a(\boldsymbol{\Sigma}, \boldsymbol{\Sigma}) + I_\gamma(\boldsymbol{\Sigma}) \quad \text{(penalize constraint violation)}$$

$$\text{s.t.} \quad b(\boldsymbol{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V$$

Regularized optimality conditions

$$a(\boldsymbol{\Sigma}, \mathbf{T}) + b(\mathbf{T}, \mathbf{u}) + I_\gamma'(\boldsymbol{\Sigma})\, \mathbf{T} = 0 \quad \text{for all } \mathbf{T} = (\boldsymbol{\tau}, \boldsymbol{\mu}) \in S^2$$

$$b(\boldsymbol{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V$$

$$I_\gamma(\boldsymbol{\Sigma}) = \tfrac{\gamma}{2} \| \boldsymbol{\Sigma} - P_{\mathcal{K}}(\boldsymbol{\Sigma}) \|^2$$ — Moreau-Yosida regularization

$$I_\gamma'(\boldsymbol{\Sigma}) = \gamma \, (\boldsymbol{\Sigma} - P_{\mathcal{K}}(\boldsymbol{\Sigma}))$$

Regularization by Penalization

Lower level (forward) problem

$$\begin{aligned} \text{Minimize} \quad & \tfrac{1}{2} a(\boldsymbol{\Sigma}, \boldsymbol{\Sigma}) + I_\gamma(\boldsymbol{\Sigma}) \quad \text{(penalize constraint violation)} \\ \text{s.t.} \quad & b(\boldsymbol{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V \end{aligned}$$

Regularized optimality conditions

$$\begin{aligned} a(\boldsymbol{\Sigma}, \mathbf{T}) + b(\mathbf{T}, \mathbf{u}) + I_\gamma'(\boldsymbol{\Sigma})\ \mathbf{T} &= 0 \quad \text{for all } \mathbf{T} = (\boldsymbol{\tau}, \boldsymbol{\mu}) \in S^2 \\ b(\boldsymbol{\Sigma}, \mathbf{v}) &= \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V \end{aligned}$$

$I_\gamma(\boldsymbol{\Sigma}) = \frac{\gamma}{2} \|\boldsymbol{\Sigma} - P_{\mathcal{K}}(\boldsymbol{\Sigma})\|^2$ — Moreau-Yosida regularization

$$I_\gamma'(\boldsymbol{\Sigma}) = \gamma\,(\boldsymbol{\Sigma} - P_{\mathcal{K}}(\boldsymbol{\Sigma})) = \gamma \max\left\{0, 1 - \tilde{\sigma}_0 \left|\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\right|^{-1}\right\} \begin{pmatrix} \boldsymbol{\sigma}^D + \boldsymbol{\chi}^D \\ \boldsymbol{\sigma}^D + \boldsymbol{\chi}^D \end{pmatrix}$$

Regularization by Penalization

This regularization corresponds to a visco-plastic model!

Regularized optimality conditions

$$a(\boldsymbol{\Sigma}, \mathbf{T}) + b(\mathbf{T}, \mathbf{u}) + J_\gamma(\boldsymbol{\Sigma})\,\mathbf{T} = 0 \quad \text{for all } \mathbf{T} = (\boldsymbol{\tau}, \boldsymbol{\mu}) \in S^2$$

$$b(\boldsymbol{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V$$

$$I_\gamma(\boldsymbol{\Sigma}) = \frac{\gamma}{2} \|\boldsymbol{\Sigma} - P_{\mathcal{K}}(\boldsymbol{\Sigma})\|^2 \quad \text{— Moreau-Yosida regularization}$$

$$I'_\gamma(\boldsymbol{\Sigma}) = \gamma\,(\boldsymbol{\Sigma} - P_{\mathcal{K}}(\boldsymbol{\Sigma})) = \gamma \max\left\{0, 1 - \tilde{\sigma}_0\, |\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D|^{-1}\right\} \begin{pmatrix} \boldsymbol{\sigma}^D + \boldsymbol{\chi}^D \\ \boldsymbol{\sigma}^D + \boldsymbol{\chi}^D \end{pmatrix}$$

$$J_\gamma(\boldsymbol{\Sigma}) = \max{}_\varepsilon \left\{0, \gamma\,(1 - \tilde{\sigma}_0\, |\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D|^{-1})\right\} \begin{pmatrix} \boldsymbol{\sigma}^D + \boldsymbol{\chi}^D \\ \boldsymbol{\sigma}^D + \boldsymbol{\chi}^D \end{pmatrix}$$

Regularization by Penalization

This regularization corresponds to a visco-plastic model!

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Regularized optimality conditions

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Differentiability of the Control-to-State Map

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Facts

- Quasi-linear elasticity system

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[Herzog, Meyer, Wachsmuth (JMAA, 2011)]

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Derivative

The derivative $(\delta\boldsymbol{\Sigma},\delta\mathbf{u})$ of $(\boldsymbol{\Sigma},\mathbf{u})$ in the direction $\delta\ell$ solves the system

$$a(\delta\boldsymbol{\Sigma},\mathbf{T}) + b(\mathbf{T},\delta\mathbf{u}) + J_\gamma'(\boldsymbol{\Sigma})(\delta\boldsymbol{\Sigma},\mathbf{T}) = 0 \quad \text{for all } \mathbf{T} = (\boldsymbol{\tau},\boldsymbol{\mu}) \in S^2$$
$$b(\delta\boldsymbol{\Sigma},\mathbf{v}) = \langle \delta\ell, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in V$$

[Herzog, Meyer, Wachsmuth (JMAA, 2011); Herzog, Meyer, Wachsmuth (in revision)]

Optimality Cond. for Regularized Problem

Regularized optimal control problem

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2}\|\mathbf{u}-\mathbf{u}_d\|^2_{L^2(\Omega;\mathbb{R}^d)} + \frac{\nu}{2}\|\mathbf{g}\|^2_{L^2(\Gamma_N;\mathbb{R}^d)} \\ \text{s.t.} \quad & \text{the regularized static plasticity problem} \\ \text{with} \quad & \langle \ell, \mathbf{v}\rangle = -\int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\,\mathrm{d}s =: R\,\mathbf{g} \end{aligned} \qquad (\mathbf{P}_\gamma)$$

Optimality Conditions

$$\begin{aligned} A\boldsymbol{\Sigma}_\gamma + J_\gamma(\boldsymbol{\Sigma}_\gamma) + B^\star\mathbf{u}_\gamma &= 0 \\ B\boldsymbol{\Sigma}_\gamma &= R\,\mathbf{g}_\gamma \\ (A + J'_\gamma(\boldsymbol{\Sigma}_\gamma))\boldsymbol{\Upsilon}_\gamma + B^\star\mathbf{w}_\gamma &= 0 \\ B\boldsymbol{\Upsilon}_\gamma &= \mathbf{u}_\gamma - \mathbf{u}_d \\ R^\star\mathbf{w}_\gamma + \nu\,\mathbf{g}_\gamma &= 0 \end{aligned}$$

Approximability of Solutions

Global Minimizers

If $\{\mathbf{g}_k\}$ are global solutions to regularized problems with

$$\gamma_k \to \infty \quad \text{and} \quad \varepsilon_k \to 0 \quad \text{as } k \to \infty$$

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If $\mathbf{g}$ is a strict local minimizer of the unregularized problem, then there exists a sequence $\{\mathbf{g}_k\}$ of local optimal solutions to regularized problems which converges to $\mathbf{g}$.

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Local Minimizers

If $\mathbf{g}$ is a local minimizer of the unregularized problem then there exists a sequence $\{\mathbf{g}_k\}$ of local optimal solutions to a *perturbed* and regularized problem which converges to $\mathbf{g}$.

Passage to the Limit

$$\begin{aligned} A\boldsymbol{\Sigma} + \lambda\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Sigma} + B^\star\mathbf{u} &= 0 \\ B\boldsymbol{\Sigma} &= R\,\mathbf{g} \\ 0 \le \lambda \quad \perp \quad \phi(\boldsymbol{\Sigma}) &\le 0 \end{aligned}$$

$$\mathcal{D}\boldsymbol{\Sigma} := \boldsymbol{\sigma}^D + \chi^D, \quad \langle R\,\mathbf{g}, \mathbf{v}\rangle := -\int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\,ds$$

[Herzog, Meyer, Wachsmuth (submitted)]

Passage to the Limit

$$
\begin{aligned}
A\boldsymbol{\Upsilon} + \lambda\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Upsilon} + \theta\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Sigma} + B^\star\mathbf{w} &= 0 \\
B\boldsymbol{\Upsilon} &= \mathbf{u} - \mathbf{u}_d \\
\mathcal{D}\boldsymbol{\Upsilon} : \mathcal{D}\boldsymbol{\Sigma} - \mu &= 0
\end{aligned}
\qquad
\begin{aligned}
\mathcal{L}_{\boldsymbol{\Sigma}} &= 0 \\
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$$\perp \qquad \perp$$

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$$\perp \qquad \perp$$

$$\mu \quad \cdot \quad \theta \quad \ge 0$$

C-stationarity

$$R^\star\mathbf{w} + \nu\,\mathbf{g} = 0 \qquad \mathcal{L}_{\mathbf{g}} = 0$$

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[Herzog, Meyer, Wachsmuth (submitted)]

Some Remarks on the Proof

- derive bounds on adjoint states and 'regularized multipliers':

$$\|\boldsymbol{\Upsilon}_\gamma\|_{S^2} + \|\mathbf{w}_\gamma\|_V \le C\left(\|(\mathbf{f}_\gamma, \mathbf{g}_\gamma)\|_U + 1\right)$$
$$\|\theta_\gamma\|_{L^2(\Omega)} \le \frac{1}{\sqrt{2}} \frac{\gamma+\varepsilon}{\tilde{\sigma}_0\,\gamma} \|\mathbf{Q}_\gamma\|_{S^2} \quad \text{where } \mathbf{Q}_\gamma = -A\boldsymbol{\Upsilon}_\gamma - B^\star \mathbf{w}_\gamma$$
$$\|\theta_\gamma\, \mathcal{D}^\star\mathcal{D}\boldsymbol{\Sigma}_\gamma\|_{S^2}^2 + \|\lambda_\gamma\, \mathcal{D}^\star\mathcal{D}\boldsymbol{\Upsilon}_\gamma\|_{S^2}^2 \le \|\mathbf{Q}_\gamma\|_{S^2}^2$$
$$\|\lambda_\gamma\, \mu_\gamma\|_{L^1(\Omega)} \le C\,(\varepsilon + \gamma^{-1})\, \|(\mathbf{f}_\gamma, \mathbf{g}_\gamma)\|_U \left(\|\mathcal{D}\boldsymbol{\Upsilon}_\gamma\|_S + \|\mathbf{Q}_\gamma\|_{S^2}\right)$$
$$\|\theta_\gamma\, \phi(\boldsymbol{\Sigma}_\gamma)\|_{L^1(\Omega)} \le C\, \frac{\varepsilon^2}{\gamma^2} \|\theta_\gamma\|_{L^2(A^0_\gamma)} + C\,\gamma^{-1} \|(\mathbf{f}_\gamma, \mathbf{g}_\gamma)\|_U\, \|\mathbf{Q}_\gamma\|_{S^2}$$

- it is particularly hard to prove the C-stationarity relation

$$\mu\,\theta \ge 0 \quad \text{a.e. in } \Omega$$

since only $\mu_k \rightharpoonup \mu$ and $\theta_k \rightharpoonup \theta$ in $L^2(\Omega)$

[Herzog, Meyer, Wachsmuth (submitted)]

CHEMNITZ UNIVERSITY OF TECHNOLOGY

Outline

Differentiability of the Control-to-State Map?

Optimal control problem

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Unregularized forward problem

$$\begin{aligned} \langle A\boldsymbol{\Sigma},\, \mathbf{T}-\boldsymbol{\Sigma}\rangle + \langle B^\star\mathbf{u},\, \mathbf{T}-\boldsymbol{\Sigma}\rangle \ge 0 \quad & \text{for all } \mathbf{T} = (\boldsymbol{\tau},\boldsymbol{\mu}) \in \mathcal{K} \\ \langle B\boldsymbol{\Sigma},\, \mathbf{v}\rangle = \langle \ell, \mathbf{v}\rangle \quad & \text{for all } \mathbf{v} \in V \end{aligned}$$

MPEC point of view (implicit approach):

- exploit properties of the control-to-state map $\ell \mapsto (\boldsymbol{\Sigma}, \mathbf{u})$
- to show that the reduced objective j is directionally differentiable
- then B-stationarity holds:

$$\delta j(\overline{\mathbf{g}}; \mathbf{g}-\overline{\mathbf{g}}) \ge 0 \quad \text{for all } \mathbf{g} \text{ admissible}$$

Weak Directional Differentiability

Theorem

For some $p > 2$, the map

$$W^{-1,p}_{\Gamma_D}(\Omega;\mathbb{R}^3) \ni \ell \mapsto (\boldsymbol{\Sigma}, \mathbf{u}) \in S^2 \times V$$

is weakly directionally differentiable (even for all directions $\delta\ell \in V'$).

[Herzog, Meyer, Wachsmuth (in revision)]

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is weakly directionally differentiable (even for all directions $\delta\ell \in V'$). This derivative is the unique solution $(\boldsymbol{\Sigma}', \mathbf{u}') \in \mathcal{S}_\ell \times V$ of

$$\langle A\boldsymbol{\Sigma}', \mathbf{T} - \boldsymbol{\Sigma}'\rangle + \langle B^\star \mathbf{u}', \mathbf{T} - \boldsymbol{\Sigma}'\rangle + \big(\lambda, \mathcal{D}\boldsymbol{\Sigma}' : \mathcal{D}(\mathbf{T} - \boldsymbol{\Sigma}')\big)_\Omega \geq 0$$
$$B\boldsymbol{\Sigma}' = \delta\ell$$

for all $\mathbf{T}$ in the convex cone

$$\mathcal{S}_\ell := \{\mathbf{T} \in S^2 : \sqrt{\lambda}\,\mathcal{D}\mathbf{T} \in S, \quad \mathcal{D}\boldsymbol{\Sigma} : \mathcal{D}\mathbf{T} \leq 0 \text{ where } \phi(\boldsymbol{\Sigma}) = \lambda = 0, \quad \mathcal{D}\boldsymbol{\Sigma} : \mathcal{D}\mathbf{T} = 0 \text{ where } \lambda > 0\}.$$

[Herzog, Meyer, Wachsmuth (in revision)]

B-Stationarity

Theorem (B-stationariy)

Let $\bar{\mathbf{g}}$ *be a local optimal solution of* $(\mathbf{P})$*. Then*

$$\int_\Omega (\overline{\mathbf{u}} - \mathbf{u}_d) \cdot \mathbf{u}' \, \mathrm{dx} + \nu \int_{\Gamma_N} \overline{\mathbf{g}} \cdot (\mathbf{g} - \overline{\mathbf{g}}) \, \mathrm{ds} \geq 0 \quad \textit{for all } \mathbf{g} \textit{ admissible},$$

where $(\boldsymbol{\Sigma}', \mathbf{u}')$ *solves the derivative problem with* $\delta\ell$ *generated by* $\mathbf{g} - \overline{\mathbf{g}}$*.*

Remark

- purely primal concept
- equivalent to notion of B-stationarity, e.g., in [Scheel, Scholtes]
- weak directional derivative of $\ell \mapsto \lambda$ exists as well
- algorithmic exploitation unknown

Concluding Remarks

Conclusions

- optimal control problem for a static plasticity problem
- regularization (γ) and smoothing (ε) of the lower-level problem
- passage to the limit $\rightsquigarrow$ optimality conditions of C-stationary type
- analysis more involved than for obstacle control problems
- B-stationarity based on weak directional differentiability of the control-to-state map

Both results required extra regularity for nonlinear elasticity systems shown in [Herzog, Meyer, Wachsmuth (JMAA, 2011)].

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This talk: static (incremental) setting
See talk by Gerd Wachsmuth (Tue, 9:30) for quasi-static setting and numerics

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