Prescribing the motion of a set of particles in a perfect fluid

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I. Introduction
Euler’s equation

- We consider a smooth bounded domain \( \Omega \subset \mathbb{R}^n \), \( n = 2, 3 \).
- Euler equation for perfect incompressible fluids

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla p &= 0 \text{ in } [0, T] \times \Omega, \\
\text{div } u &= 0 \text{ in } [0, T] \times \Omega.
\end{align*}
\]

- Here, \( u : [0, T] \times \Omega \to \mathbb{R}^n \) is the velocity field, \( p : [0, T] \times \Omega \to \mathbb{R} \) is the pressure field.

- Usual slip condition on the boundary:

\[
u \cdot n = 0 \text{ on } [0, T] \times \partial \Omega.
\]

Boundary control

- We consider a non empty open part $\Sigma$ of the boundary $\partial\Omega$.

- Non-homogeneous boundary conditions can be chosen as follows:
  - on $\partial\Omega \setminus \Sigma$, the fluid does not cross the boundary, $u \cdot n = 0$.
  - on $\Sigma$, we suppose that one can choose the boundary conditions. These can take the following form (Yudovich, Kazhikov):

$$u(t, x) \cdot n(x) \text{ on } [0, T] \times \Sigma,$$
$$\text{curl } u(t, x) \text{ on } \Sigma^- := \{(t, x) \in [0, T] \times \Sigma / u(t, x) \cdot n(x) < 0\} \quad (2D)$$
$$\text{curl } u(t, x) \times n \text{ on } \Sigma^- := \{(t, x) \in [0, T] \times \Sigma / u(t, x) \cdot n(x) < 0\} \quad (3D).$$

- This boundary condition is a control which we can choose to influence the system, in order to prescribe its behavior.
The standard problem of controlability

- Standard problem of exact/approximate controlability:

  Given two possible states of the system, say $u_0$ and $u_1$, and given a time $T > 0$, can one find a control such that the corresponding solution of the system starting from $u_0$ at time $t = 0$ reaches the target $u_1$ at time $t = T$? At least such that

  $$\|u(T, \cdot) - u_1\|_X \leq \varepsilon? \quad (AC)$$

- Alternative formulation: given $u_0$, $u_1$ and $T$, can we find a solution of the equation satisfying the constraint on the boundary:

  $$u \cdot n = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Sigma),$$

  (under-determined system) and driving $u_0$ to $u_1$ at time $T$? Or to $u(T, \cdot)$ satisfying (AC)?

- See Coron, G., for what concerns the boundary controllability of the Euler equation.
Another type of controlability

▶ Another type of controlability is natural for equations from fluid mechanics: is possible to drive a zone in the fluid from a given place to another by using the control? (Based on a suggestion by J.-P. Puel)

▶ One can think for instance to a polluted zone in the fluid, which we would like to transfer to a zone where it can be treated.

▶ It is natural, in order to control the fluid zone during the whole displacement to ask that it remains inside the domain during the whole time interval.

▶ Cf. Horsin in the case of the Burgers equation.
First definition

- Due to the incompressibility of the fluid, the starting zone and the target zone must have the same area.
- We have also to require that there is no topological obstruction to move a zone to the other one.

- In the sequel, we will consider fluids zones given by the interior (supposed to be inside $\Omega$) of smooth ($C^\infty$) Jordan curves/surface.

Definition

We will say that the system satisfies the exact Lagrangian controlability property, if given two smooth Jordan curves/surface $\gamma_0, \gamma_1$ in $\Omega$, homotopic in $\Omega$ and surrounding the same area/volume, a time $T > 0$ and an initial datum $u_0$, there exists a control such that the flow given by the velocity fluid drives $\gamma_0$ to $\gamma_1$, by staying inside the domain.
An objection

The exact controlability Lagrangian does not hold in general, indeed:

- Let us suppose $\omega_0 := \text{curl } u_0 = 0$. In that case if the flow $\Phi(t, x)$ maintains $\gamma_0$ inside the domain, then for all $t$,

$$
\omega(t, \cdot) = \text{curl } u(t, \cdot) = 0,
$$

in the neighborhood of $\Phi(t, \gamma_0)$.

- Since $\text{div } u = 0$, locally around $\gamma_0$, $u$ is the gradient of a harmonic function; $u$ is therefore analytic in a neighborhood $\Phi(t, \gamma_0)$.

- Hence if $\gamma_0$ is analytic, its analyticity is propagated over time.

- If $\gamma_1$ is smooth but non analytic, the exact Lagrangian controlability cannot hold.
Approximate Lagrangian controllability

Definition
We will say that the system satisfies the property of approximate Lagrangian controllability in $C^k$, if given two smooth Jordan curves/surface $\gamma_0, \gamma_1$ in $\Omega$, homotopic in $\Omega$ and surrounding the same volume, a time $T > 0$, an initial datum $u_0$ and $\varepsilon > 0$, we can find a control such that the flow of the velocity field maintains $\gamma_0$ inside $\Omega$ for all time $t \in [0, T]$ and satisfies, up to reparameterization:

$$\|\Phi^u(T, \gamma_0) - \gamma_1\|_{C^k} \leq \varepsilon.$$ 

Here, $(t, x) \mapsto \Phi^u(t, x)$ is the flow of the vector field $u$. 
The 2-D case

Theorem (G.-Horsin)

Consider two smooth smooth Jordan curves $\gamma_0, \gamma_1$ in $\Omega$, homotopic in $\Omega$ and surrounding the same area. Let $k \in \mathbb{N}$. We consider $u_0 \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$ satisfying

$$\text{div}(u_0) = 0 \text{ in } \Omega \text{ and } u_0 \cdot n = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Sigma).$$

For any $T > 0$, $\varepsilon > 0$, there exists a solution $u$ of the Euler equation in $C^\infty([0, T] \times \overline{\Omega}; \mathbb{R}^2)$ with

$$u \cdot n = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Sigma) \text{ and } u|_{t=0} = u_0 \text{ in } \Omega,$$

and whose flow satisfies

$$\forall t \in [0, T], \Phi^u(t, \gamma_0) \subset \Omega,$$

and up to reparameterization

$$\|\gamma_1 - \Phi^u(T, \gamma_0)\|_{C^k} \leq \varepsilon.$$
A connected result: vortex patches

The starting point is the following.

**Theorem (Yudovich, 1961)**

For any $u_0 \in C^0(\overline{\Omega}; \mathbb{R}^2)$ such that $\text{div}(u_0) = 0$ in $\Omega$, $u_0 \cdot n = 0$ on $\partial \Omega$ and $\text{curl } u_0 \in L^\infty$, there exists a unique (weak) global solution of the Euler equation starting from $u_0$ and satisfying $u \cdot n = 0$ on the boundary.

A particular case of initial data with vorticity in $L^\infty$ is the one of vortex patches.

**Definition**

A vortex patch is a solution of the Euler equation whose initial datum is the characteristic function of the interior of a smooth Jordan curve (at least $C^{1, \alpha}$).

Cf. Chemin, Bertozzi-Constantin, Danchin, Depauw, Dutrifoy, Gamblin & Saint-Raymond, Hmidi, Serfati, Sueur,…
Control of the shape of a vortex patch

Theorem (G.-Horsin)
Consider two smooth Jordan curves $\gamma_0, \gamma_1$ in $\Omega$, homotopic in $\Omega$ and surrounding the same area. Suppose also that the control zone $\Sigma$ is in the exterior of these curves. Let $u_0 \in \mathcal{Lip}(\Omega; \mathbb{R}^2)$ with $u_0 \cdot n \in C^\infty(\partial \Omega)$ a vortex patch initial condition corresponding to $\gamma_0$, i.e.

$$\text{curl}(u_0) = 1_{\text{Int}(\gamma_0)} \text{ in } \Omega, \quad \text{div}(u_0) = 0 \text{ in } \Omega, \quad u_0 \cdot n = 0 \text{ on } \partial \Omega \setminus \Sigma.$$ 

Then for any $T > 0$, any $k \in \mathbb{N}$, any $\varepsilon > 0$, the exists $u \in L^\infty([0, T]; \mathcal{Lip}(\Omega))$ a solution of the Euler equation such that

$$\text{curl } u = 0 \text{ on } [0, T] \times \Sigma,$$
$$u \cdot n = 0 \text{ on } [0, T] \times (\partial \Omega \setminus \Sigma) \text{ and } u|_{t=0} = u_0 \text{ in } \Omega,$$

that $\Phi^u(T, 0, \gamma_0)$ does not leave the domain and and that, up to reparameterization, one has

$$\|\gamma_1 - \Phi^u(T, 0, \gamma_0)\|_{C^k} \leq \varepsilon.$$
Remarks

- As long as the patch stays regular, one merely has $u(t, \cdot) \in \mathcal{L}ip(\Omega)$.
- Without the regularity of the patch, the velocity field $u(t, \cdot)$ is log-Lipschitz only:

$$|u(t, x) - u(t, y)| \lesssim |x - y| \log(e + |x - y|).$$
The 3-D case

Theorem (G.-Horsin)

Let $\alpha \in (0, 1)$ and $k \in \mathbb{N} \setminus \{0\}$. Consider $u_0 \in C^{k,\alpha}(\Omega; \mathbb{R}^3)$ satisfying

$$\text{div} \, u_0 = 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad u_0 \cdot n = 0 \quad \text{on} \quad \partial \Omega \setminus \Gamma,$$

let $\gamma_0$ and $\gamma_1$ two contractible $C^\infty$ embeddings of $\mathbb{S}^2$ in $\Omega$ such that

$\gamma_0$ and $\gamma_1$ are diffeotopic in $\Omega$ and $|\text{Int}(\gamma_0)| = |\text{Int}(\gamma_1)|$.

Then for any $\varepsilon > 0$, there exist a time small enough $T_0 > 0$, such that for all $T \leq T_0$, there is a solution $(u, p)$ in $L^\infty(0, T; C^{k,\alpha}(\Omega; \mathbb{R}^4))$ of the Euler equation on $[0, T]$ with $u \cdot n = 0$ on $\partial\Omega \setminus \Sigma$ such that

$$\forall t \in [0, T], \quad \Phi^u(t, 0, \gamma_0) \subset \Omega,$$

$$\|\Phi^u(T, 0, \gamma_0) - \gamma_1\|_{C^k(\mathbb{S}^2)} < \varepsilon,$$

hold (up to reparameterization).
II. Ideas of proof (in the 2D case)
Potential flows

- For any \( \theta = \theta(t, x) \) which is harmonic with respect to \( x \) for all \( t \),

\[
\nu(t, x) := \nabla_x \theta(t, x)
\]

is a solution of the Euler equation with

\[
p(t, x) = - (\theta_t + |\nabla \theta|^2 / 2).
\]

- These are potential flows, which are classical in fluid mechanics.

- The construction of suitable potential flows is also central in the proof of the exact controlability of the Euler equation.

- This idea is due to J.-M. Coron, and is connected to the so-called return method.
Main proposition

Proposition

Consider two smooth Jordan curves/surface \( \gamma_0, \gamma_1 \) in \( \Omega \), diffeotopic in \( \Omega \) and surrounding the same volume. For any \( k \in \mathbb{N}, \; T > 0, \; \varepsilon > 0 \), there exists \( \theta \in C_0^\infty([0, 1]; C^\infty(\overline{\Omega}; \mathbb{R})) \) such that

\[
\Delta_x \theta(t, \cdot) = 0 \text{ in } \Omega, \text{ for all } t \in [0, 1],
\]

\[
\frac{\partial \theta}{\partial n} = 0 \text{ on } [0, 1] \times (\partial \Omega \setminus \Sigma),
\]

whose flow satisfies

\[
\forall t \in [0, 1], \; \Phi^{\nabla \theta}(t, 0, \gamma_0) \subset \Omega,
\]

and, up to reparameterization,

\[
\| \gamma_1 - \Phi^{\nabla \theta}(T, 0, \gamma_0) \|_{C^k} \leq \varepsilon.
\]
Ideas of proof for the main proposition

- One seeks a potential flow driving $\gamma_0$ to $\gamma_1$ (approximately in $C^k$) and fulfilling the boundary condition on $\partial \Omega \setminus \Sigma$.

- This is proven in two parts:

  - **Part 1**: find a solenoidal (divergence-free) vector field driving $\gamma_0$ to $\gamma_1$.

  - **Part 2**: approximate (at each time) the above vector field on the curve (or to be more precise, its normal part), by the gradient of a harmonic function defined on $\Omega$ and satisfying the constraint.
Part 1

Proposition

Consider \( \gamma_0 \) and \( \gamma_1 \) two smooth \( (C^\infty) \) Jordan curves/surface diffeotopic in \( \Omega \). If \( \gamma_0 \) and \( \gamma_1 \) satisfy

\[
|\text{Int}(\gamma_0)| = |\text{Int}(\gamma_1)|,
\]

then there exists \( v \in C_0^\infty((0,1) \times \Omega; \mathbb{R}^2) \) such that

\[
\text{div } v = 0 \text{ in } (0,1) \times \Omega,
\]

\[
\Phi^v(1,0,\gamma_0) = \gamma_1.
\]
Idea of proof for Part 1

In 2-D, one can make moves like the ones described below.

But it turns out that a very general result due to A. B. Krygin (and relying on J. Moser’s celebrated result on deformation of volume forms) proves the above proposition in any dimension.
Proposition

Let $\gamma_0$ a smooth ($C^\infty$) Jordan curve/surface; let $X \in C^0([0, 1]; C^\infty(\overline{\Omega}))$ a smooth solenoidal vector field, with $X \cdot n = 0$ on $[0, 1] \times \partial\Omega$. Then for all $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\theta \in C^\infty([0, 1] \times \overline{\Omega}; \mathbb{R})$ such that

$$\Delta_X \theta(t, \cdot) = 0 \text{ in } \Omega, \text{ for all } t \in [0, 1],$$

$$\frac{\partial \theta}{\partial n} = 0 \text{ on } [0, 1] \times (\partial\Omega \setminus \Sigma),$$

and whose flow satisfies

$$\forall t \in [0, 1], \quad \Phi^{\nabla \theta}(t, 0, \gamma_0) \subset \Omega,$$

and, up to reparameterization,

$$\|\Phi^X(t, 0, \gamma_0) - \Phi^{\nabla \theta}(t, 0, \gamma_0)\|_{C^k} \leq \varepsilon, \quad \forall t \in [0, 1].$$
The main idea is to use results from harmonic approximation. There are equivalent of Runge’s theorem of approximation of holomorphic functions by rational ones, such as (see e.g. Gardner):

**Theorem**

Let \( \mathcal{O} \) be an open set in \( \mathbb{R}^N \) and let \( K \) be a compact set in \( \mathbb{R}^N \) such that \( \mathcal{O}^* \setminus K \) is connected, where \( \mathcal{O}^* \) is the Alexandroff compactification of \( \mathcal{O} \). Then, for each function \( u \) which is harmonic on an open set containing \( K \) and each \( \varepsilon > 0 \), there is a harmonic function \( v \) in \( \Omega \) such that \( \| v - u \|_\infty < \varepsilon \) on \( K \).
How to deduce the results from the main proposition

- Let us now prove the main theorem when $u_0 \in C^\infty$ is non zero.

- The idea (due to Coron) is to get into this situation is to use the time scale invariance of the equation: for $\lambda > 0$,

\[
\begin{align*}
  u(t, x) &\text{ is a solution of the equation defined in } [0, T] \times \Omega \\
  \iff u^\lambda(t, x) := \lambda u(\lambda t, x) &\text{ is a solution of the equation defined in } [0, T/\lambda] \times \Omega.
\end{align*}
\]
From the main proposition, sequel

- We cut the time interval in the following way: for $\nu$ small:

\[
\begin{align*}
t &= 0 & t &= T - \nu & t &= T \\
\bullet & & \bullet & & \bullet \\
\text{Evolution “without control”} & & & & \text{Control given by } \nabla \theta, \\
& & & & \text{time-scaled, where } \theta \text{ is such that} \\
\tilde{\gamma}_0 := \Phi(T - \nu, \gamma_0) & & & & \Phi \nabla \theta (1, \tilde{\gamma}_0) = \gamma_1
\end{align*}
\]

- If we change back the time scale from $[T - \nu, T]$ to $[0, 1]$, the evolution is driven by the Euler equation, with:
  - As boundary condition (on the normal trace) the same as $\nabla \theta$
  - As initial condition $\nu u(T - \nu, \cdot)$.

- $\Rightarrow$ the initial datum is small!

- We show that the solution constructed on $[0, T]$:

\[
\| \Phi^u(T, \gamma_0) - \gamma_1 \|_{C^k} \lesssim \nu + \varepsilon.
\]
Open problems

- **Navier-Stokes equations.** Can we obtain a similar result for incompressible Navier-Stokes equations?

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p &= 0 \text{ in } [0, T] \times \Omega, \\
\text{div } u &= 0 \text{ in } [0, T] \times \Omega.
\end{aligned}
\]

With Dirichlet’s boundary conditions? With Navier’s (cf. Coron, Chapouly)?

- **Stabilization.** Can we find a feedback control :

\[
\text{control}(t) = f(\gamma(t), u(t)),
\]

stabilizing a fluid zone at a fixed place?