Prescribing the motion of a set of particles in a perfect fluid

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I. Introduction

Euler's equation

- We consider a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$.
- Euler equation for perfect incompressible fluids

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0 \text{ in } [0, T] \times \Omega, \\ \text{div } u = 0 \text{ in } [0, T] \times \Omega. \end{cases}$$

- Here, $u : [0, T] \times \Omega \to \mathbb{R}^n$ is the velocity field, $p : [0, T] \times \Omega \to \mathbb{R}$ is the pressure field.
- Usual slip condition on the boundary :

$$u \cdot n = 0 \text{ on } [0, T] \times \partial\Omega.$$

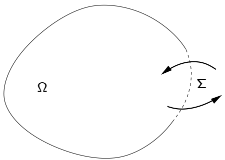
- $\rightarrow$ Global (resp. local in 3D) well-posedness, cf. Lichtenstein, Wolibner, Yudovich, Kato, ...

Boundary control

- We consider a non empty open part Σ of the boundary $\partial\Omega$.
- Non-homogeneous boundary conditions can be chosen as follows :
  - on $\partial\Omega \setminus \Sigma$, the fluid does not cross the boundary, $u \cdot n = 0$.
  - on Σ, we suppose that one can choose the boundary conditions. These can take the following form (Yudovich, Kazhikov) :

$$
\begin{aligned}
& u(t,x)\cdot n(x) \;\; \text{on} \;\; [0,T]\times\Sigma, \\
& \text{curl}\; u(t,x) \;\; \text{on} \;\; \Sigma_T^- := \{(t,x)\in[0,T]\times\Sigma \;/\; u(t,x)\cdot n(x) < 0\} \quad (2D) \\
& \text{curl}\; u(t,x)\times n \;\; \text{on} \;\; \Sigma_T^- := \{(t,x)\in[0,T]\times\Sigma \;/\; u(t,x)\cdot n(x) < 0\} \quad (3D).
\end{aligned}
$$

- This boundary condition is a control which we can choose to influence the system, in order to prescribe its behavior.



The standard problem of controlabillity

- Standard problem of exact/approximate controlabillity :

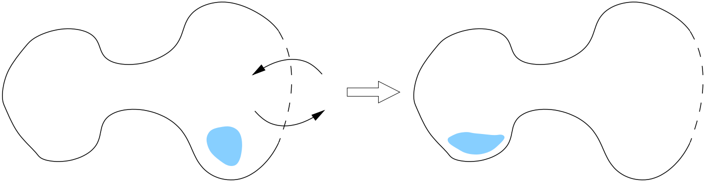
  Given two possible states of the system, say u_0 and u_1, and given a time $T > 0$, can one find a control such that the corresponding solution of the system starting from u_0 at time $t = 0$ reaches the target u_1 at time $t = T$?
  At least such that
  $$\|u(T, \cdot) - u_1\|_X \leq \varepsilon? \tag{AC}$$

- Alernative formulation : given u_0, u_1 and T, can we find a solution of the equation satisfying the constraint on the boundary :
  $$u \cdot n = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Sigma),$$
  (under-determined system) and driving u_0 to u_1 at time T ? Or to $u(T, \cdot)$ satisfying (AC) ?

- See Coron, G., for what concerns the boundary controllability of the Euler equation.

Another type of controlabillity

- Another type of controlabillity is natural for equations from fluid mechanics : is possible to drive a zone in the fluid from a given place to another by using the control ? (Based on a suggestion by J.-P. Puel)



- One can think for instance to a polluted zone in the fluid, which we would like to transfer to a zone where it can be treated.
- It is natural, in order to control the fluid zone during the whole diplacement to ask that is remains inside the domain during the whole time interval.
- Cf. Horsin in the case of the Burgers equation.

First definition

- Due to the incompressibility of the fluid, the starting zone and the target zone must have the same area.
- We have also to require that there is no topological obstruction to move a zone to the other one.

- In the sequel, we will consider fluids zones given by the interior (supposed to be inside Ω) of smooth (C^∞) Jordan curves/surface.

Definition

We will say that the system satisfies the exact Lagrangian controlabillity property, if given two smooth Jordan curves/surface γ_0, γ_1 in Ω, homotopic in Ω and surrounding the same area/volume, a time $T > 0$ and an initial datum u_0, there exists a control such that the flow given by the velocity fluid drives γ_0 to γ_1, by staying inside the domain.

An objection

The exact controlabillity Lagrangian does not hold in general, indeed :

- Let us suppose $\omega_0 := \text{curl } u_0 = 0$. In that case if the flow $\Phi(t, x)$ maintains γ_0 inside the domain, then for all t,

  $$\omega(t, \cdot) = \text{curl } u(t, \cdot) = 0,$$

  in the neighborhood of $\Phi(t, \gamma_0)$.

- Since div $u = 0$, locally around γ_0, u is the gradient of a harmonic function ; u is therefore analytic in a neighborhood $\Phi(t, \gamma_0)$.

- Hence if γ_0 is analytic, its analyticity is propagated over time.

- If γ_1 is smooth but non analytic, the exact Lagrangian controlabillity cannot hold.

Approximate Lagrangian controllability

Definition

We will say that the system satisfies the property of approximate Lagrangian controlabillity in C^k, if given two smooth Jordan curves/surface γ_0, γ_1 in Ω, homotopic in Ω and surrounding the same volume, a time $T > 0$, an initial datum u_0 and $\varepsilon > 0$, we can find a control such that the flow of the velocity field maintains γ_0 inside Ω for all time $t \in [0, T]$ and satisfies, up to reparameterization :

$$\|\Phi^u(T, \gamma_0) - \gamma_1\|_{C^k} \leq \varepsilon.$$

Here, $(t, x) \mapsto \Phi^u(t, x)$ is the flow of the vector field u.

The 2-D case

Theorem (G.-Horsin)

Consider two smooth smooth Jordan curves γ_0, γ_1 in Ω, homotopic in Ω and surrounding the same area. Let $k \in \mathbb{N}$. We consider $u_0 \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$ satisfying

$$div(u_0) = 0 \text{ in } \Omega \text{ and } u_0 \cdot n = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Sigma).$$

For any $T > 0$, $\varepsilon > 0$, there exists a solution u of the Euler equation in $C^\infty([0, T] \times \overline{\Omega}; \mathbb{R}^2)$ with

$$u \cdot n = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Sigma) \text{ and } u_{|t=0} = u_0 \text{ in } \Omega,$$

and whose flow satisfies

$$\forall t \in [0, T], \ \Phi^u(t, \gamma_0) \subset \Omega,$$

and up to reparameterization

$$\|\gamma_1 - \Phi^u(T, \gamma_0)\|_{C^k} \leq \varepsilon.$$

A connected result : vortex patches

The starting point is the following.

Theorem (Yudovich, 1961)

For any $u_0 \in C^0(\overline{\Omega}; \mathbb{R}^2)$ such that $div(u_0) = 0$ in Ω, $u_0 \cdot n = 0$ on $\partial\Omega$ and curl $u_0 \in L^\infty$, there exists a unique (weak) global solution of the Euler equation starting from u_0 and satisfying $u \cdot n = 0$ on the boundary.

A particular case of initial data with vorticity in L^∞ is the one of vortex patches.

Definition

A vortex patch is a solution of the Euler equation whose initial datum is the caracteristic function of the interior of a smooth Jordan curve (at least $C^{1,\alpha}$).

Cf. Chemin, Bertozzi-Constantin, Danchin, Depauw, Dutrifoy, Gamblin & Saint-Raymond, Hmidi, Serfati, Sueur,...

Control of the shape of a vortex patch

Theorem (G.-Horsin)

Consider two smooth Jordan curves γ_0, γ_1 in Ω, homotopic in Ω and surrounding the same area. Suppose also that the control zone Σ is in the exterior of these curves. Let $u_0 \in \mathcal{L}ip(\overline{\Omega}; \mathbb{R}^2)$ with $u_0 \cdot n \in C^\infty(\partial\Omega)$ a vortex patch initial condition corresponding to γ_0, i.e.

$$curl(u_0) = \mathbf{1}_{Int(\gamma_0)} \text{ in } \Omega, \quad div(u_0) = 0 \text{ in } \Omega, \quad u_0 \cdot n = 0 \text{ on } \partial\Omega \setminus \Sigma.$$

Then for any $T > 0$, any $k \in \mathbb{N}$, any $\varepsilon > 0$, the exists $u \in L^\infty([0,T]; \mathcal{L}ip(\overline{\Omega}))$ a solution of the Euler equation such that

$$curl\ u = 0 \text{ on } [0,T] \times \Sigma,$$
$$u \cdot n = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Sigma) \text{ and } u_{|t=0} = u_0 \text{ in } \Omega,$$

that $\Phi^u(T,0,\gamma_0)$ does not leave the domain and and that, up to reparameterization, one has

$$\|\gamma_1 - \Phi^u(T,0,\gamma_0)\|_{C^k} \leq \varepsilon.$$

Remarks

- As long as the patch stays regular, one merely has $u(t,\cdot) \in \mathcal{L}ip(\Omega)$.
- Without the regularity of the patch, the velocity field $u(t,\cdot)$ is log-Lipschitz only :

$$|u(t,x) - u(t,y)| \lesssim |x-y| \log(e + |x-y|).$$

The 3-D case

Theorem (G.-Horsin)

Let $\alpha \in (0,1)$ *and* $k \in \mathbb{N} \setminus \{0\}$*. Consider* $u_0 \in C^{k,\alpha}(\Omega;\mathbb{R}^3)$ *satisfying*

$$div\ u_0 = 0 \ \ in\ \ \Omega, \quad and \ \ u_0 \cdot n = 0 \ \ on \ \ \partial\Omega \setminus \Gamma,$$

let γ_0 *and* γ_1 *two contractible* C^∞ *embeddings of* $\mathbb{S}^2$ *in* Ω *such that*

$$\gamma_0 \text{ and } \gamma_1 \text{ are diffeotopic in } \Omega \quad and \quad |Int(\gamma_0)| = |Int(\gamma_1)|.$$

Then for any $\varepsilon > 0$*, there exist a time small enough* $T_0 > 0$*, such that for all* $T \le T_0$*, there is a solution* (u,p) *in* $L^\infty(0,T;C^{k,\alpha}(\Omega;\mathbb{R}^4))$ *of the Euler equation on* $[0,T]$ *with* $u \cdot n = 0$ *on* $\partial\Omega \setminus \Sigma$ *such that*

$$\forall t \in [0,T], \ \Phi^u(t,0,\gamma_0) \subset \Omega,$$
$$\|\Phi^u(T,0,\gamma_0) - \gamma_1\|_{C^k(\mathbb{S}^2)} < \varepsilon,$$

hold (up to reparameterization).

II. Ideas of proof (in the 2D case)

Potential flows

- For any $\theta = \theta(t, x)$ which is harmonic with respect to x for all t,

$$v(t,x) := \nabla_x \theta(t,x) \text{ is a solution of the Euler equation with}$$
$$p(t,x) = -(\theta_t + |\nabla\theta|^2/2).$$

- These are potential flows, which are classical in fluid mechanics.
- The construction of suitable potential flows is also central in the proof of the exact controlabillity of the Euler equation.
- This idea is due to J.-M. Coron, and is connected to the so-called return method.

Main proposition

Proposition

Consider two smooth Jordan curves/surface γ_0, γ_1 in Ω, diffeotopic in Ω and surrounding the same volume. For any $k \in \mathbb{N}$, $T > 0$, $\varepsilon > 0$, there exists $\theta \in C_0^\infty([0,1]; C^\infty(\overline{\Omega}; \mathbb{R}))$ such that

$$\Delta_x \theta(t, \cdot) = 0 \text{ in } \Omega, \text{ for all } t \in [0,1],$$
$$\frac{\partial \theta}{\partial n} = 0 \text{ on } [0,1] \times (\partial\Omega \setminus \Sigma),$$

whose flow satisfies

$$\forall t \in [0,1], \ \Phi^{\nabla\theta}(t, 0, \gamma_0) \subset \Omega,$$

and, up to reparameterization,

$$\|\gamma_1 - \Phi^{\nabla\theta}(T, 0, \gamma_0)\|_{C^k} \leq \varepsilon.$$

Ideas of proof for the main proposition

- One seeks a potential flow driving γ_0 to γ_1 (approximately in C^k) and fulfilling the boundary condition on $\partial\Omega \setminus \Sigma$.

- This is proven in two parts :

  - Part 1 : find a solenoidal (divergence-free) vector field driving γ_0 to γ_1.

  - Part 2 : approximate (at each time) the above vector field on the curve (or to be more precise, its normal part), by the gradient of a harmonic function defined on Ω and satisfying the constraint.

Part 1

Proposition

Consider γ_0 and γ_1 two smooth (C^∞) Jordan curves/surface diffeotopic in Ω. If γ_0 and γ_1 satisfy

$$|Int(\gamma_0)| = |Int(\gamma_1)|,$$

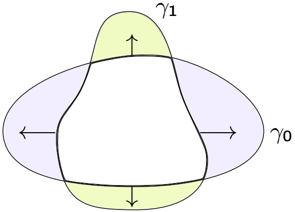
then there exists $v \in C_0^\infty((0,1) \times \Omega; \mathbb{R}^2)$ such that

$$div\ v = 0 \text{ in } (0,1) \times \Omega,$$

$$\Phi^v(1, 0, \gamma_0) = \gamma_1.$$

Idea of proof for Part 1

- In 2-D, one can make moves like the ones described below.



- But it turns out that a very general result due to A. B. Krygin (and relying on J. Moser's celebrated result on deformation of volume forms) proves the above proposition in any dimension.

Part 2

Proposition

Let γ_0 a smooth (C^∞) Jordan curve/surface ; let $X \in C^0([0,1]; C^\infty(\overline{\Omega}))$ a smooth solenoidal vector field, with $X \cdot n = 0$ on $[0,1] \times \partial\Omega$. Then for all $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\theta \in C^\infty([0,1] \times \overline{\Omega}; \mathbb{R})$ such that

$$\Delta_x \theta(t, \cdot) = 0 \text{ in } \Omega, \text{ for all } t \in [0,1],$$
$$\frac{\partial \theta}{\partial n} = 0 \text{ on } [0,1] \times (\partial\Omega \setminus \Sigma),$$

and whose flow satisfies

$$\forall t \in [0,1],\ \Phi^{\nabla\theta}(t, 0, \gamma_0) \subset \Omega,$$

and, up to reparameterization,

$$\|\Phi^X(t,0,\gamma_0) - \Phi^{\nabla\theta}(t,0,\gamma_0)\|_{C^k} \leq \varepsilon, \quad \forall t \in [0,1].$$

Ideas of proof for Part 2

The main idea is to use results from harmonic approximation. There are equivalent of Runge's theorem of approximation of holomorphic functions by rational ones, such as (see e.g. Gardner) :

Theorem

Let $\mathcal{O}$ be an open set in $\mathbb{R}^N$ and let K be a compact set in $\mathbb{R}^N$ such that that $\mathcal{O}^* \setminus K$ is connected, where $\mathcal{O}^*$ is the Alexandroff compactification of $\mathcal{O}$. Then, for each function u which is harmonic on an open set containing K and each $\varepsilon > 0$, there is a harmonic function v in Ω such that $\|v - u\|_\infty < \varepsilon$ on K.

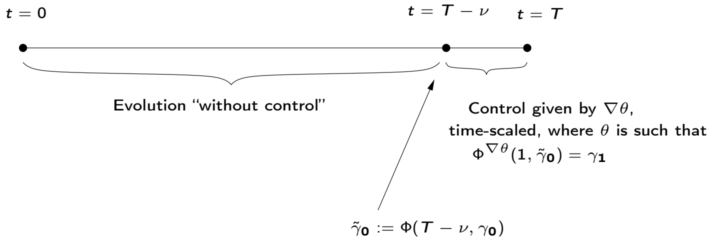
How to deduce the results from the main proposition

- Let us now prove the main theorem when $u_0 \in C^\infty$ is non zero.
- The idea (due to Coron) is to get into this situation is to use the time scale invariance of the equation : for $\lambda > 0$,

$$u(t,x) \text{ is a solution of the equation defined in } [0,T]\times\Omega \\ \iff u^\lambda(t,x) := \lambda u(\lambda t, x) \text{ is a solution of the equation} \\ \text{defined in } [0, T/\lambda]\times\Omega.$$

From the main proposition, sequel

- We cut the time interval in the following way : for ν small :



- If we change back the time scale from $[T-\nu, T]$ to $[0,1]$, the evolution is driven by the Euler equation, with :
  - As boundary condition (on the normal trace) the same as $\nabla\theta$
  - As initial condition $\nu u(T-\nu,\cdot)$.
- $\Rightarrow$ the initial datum is small !
- We show that the solution constructed on $[0, T]$:

$$\|\Phi^u(T,\gamma_0) - \gamma_1\|_{C^k} \lesssim \nu + \varepsilon.$$

Open problems

- Navier-Stokes equations. Can we obtain a similar result for incompressible Navier-Stokes equations ?

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p = 0 \text{ in } [0, T] \times \Omega, \\ \text{div } u = 0 \text{ in } [0, T] \times \Omega. \end{cases}$$

  With Dirichlet's boundary conditions ? With Navier's (cf. Coron, Chapouly) ?

- Stabilization. Can we find a feedback control :

$$\text{control(t)} = f(\gamma(t), u(t)),$$

  stabilizing a fluid zone at a fixed place ?