Dennis-Moré Theorem Revisited

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Dennis-Moré Theorem

Quasi-Newton method for solving $f(x) = 0$:

$$f(x_k) + B_k(x_{k+1} - x_k) = 0,$$

where $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ and B_k is a sequence of matrices.
Let $s_k = x_{k+1} - x_k$, $E_k = B_k - Df(\bar{x})$.
Recall that $\{x_k\}$ converges superlinearly when $\|e_{k+1}\|/\|e_k\| \to 0$.

Theorem [Dennis-Moré, 1974]. Suppose that f is differentiable in an open convex set D in $\boldsymbol{R}^n$ containing $\bar{x}$, a zero of f, the derivative Df is continuous at $\bar{x}$ and $Df(\bar{x})$ is nonsingular. Let $\{B_k\}$ be a sequence of nonsingular matrices and let for some starting point x_0 in D the sequence $\{x_k\}$ be generated by the method, remain in D for all k and $x_k \neq \bar{x}$ for all k. Then $x_k \to \bar{x}$ superlinearly if and only if

$$x_k \to \bar{x} \quad \text{and} \quad \lim_{k\to\infty} \frac{\|E_k s_k\|}{\|s_k\|} = 0.$$

Outline:

1. Strong metric subregularity
2. Dennis-Moré theorem for Newton differentiable functions
3. Dennis-Moré theorem for generalized equations

Strong metric subregularity

Definition. Consider a mapping $H : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y}) \in X \times Y$. Then H is said to be **strongly metrically subregular** at $\bar{x}$ for $\bar{y}$ when $\bar{y} \in H(\bar{x})$ and there is a constant $\kappa > 0$ together with a neighborhood U of $\bar{x}$ such that

$$\|x - \bar{x}\| \leq \kappa d(\bar{y}, H(x)) \quad \text{for all } x \in U.$$

Obeys the paradigm of the implicit function theorem: f is s.m.s. if and only if the linearization $x \mapsto f(\bar{x}) + Df(\bar{x})(x - \bar{x})$ is s.m.s., which is the same as $\kappa\|h\| \leq \|Df(\bar{x})h\|$ for all $h \in X$.

Every mapping $T : \boldsymbol{R}^n \rightrightarrows \boldsymbol{R}^m$, whose graph is the union of finitely many polyhedral convex sets, is s.m.s at $\bar{x}$ for $\bar{y}$ if and only if $\bar{x}$ is an isolated point in $T^{-1}(\bar{y})$.

Strong metric subregularity in optimization

Convex optimization

$$\text{minimize} \quad g(x) - \langle p, x\rangle \qquad \text{over} \quad x \in C,$$

where $g : \boldsymbol{R}^n \to \boldsymbol{R}$ is convex and C^2, $p \in \boldsymbol{R}^n$, and C is a convex polyhedral set in $\boldsymbol{R}^n$.
First-order optimality condition

$$\nabla g(x) + N_C(x) \ni p$$

The mapping $\nabla g + N_C$ is strongly metrically subregular at $\bar{x}$ for $\bar{p}$ if and only if the standard second-order sufficient condition holds at $\bar{x}$ for $\bar{p}$: $\langle \nabla^2 g(\bar{x})u, u\rangle > 0$ for all nonzero u in the critical cone $K_C(\bar{x}, \bar{p} - \nabla g(\bar{x}))$.

Asen L. Dontchev · R. Tyrrell Rockafellar
Implicit Functions and Solution Mappings

The implicit function theorem is one of the most important theorems in analysis and its many variants are basic tools in partial differential equations and numerical analysis. This book treats the implicit function paradigm in the classical framework and beyond, focusing largely on properties of solution mappings of variational problems.

The purpose of this self-contained work is to provide a reference on the topic and to provide a unified collection of a number of results which are currently scattered throughout the literature. The first chapter of the book treats the classical implicit function theorem in a way that will be useful for students and teachers of undergraduate calculus. The remaining part becomes gradually more advanced, and considers implicit mappings defined by relations other than equations, e.g., variational problems. Applications to numerical analysis and optimization are also provided.

This valuable book is a major achievement and is sure to become a standard reference on the topic.

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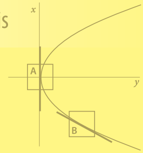
Implicit Functions and Solution Mappings

A View from Variational Analysis



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Convergence under strong subregularity

Main Lemma. Let $f : X \to Y$, for X and Y Banach spaces, be Lipschitz continuous in a neighborhood U of $\bar{x}$ and strongly metrically subregular at $\bar{x}$ with a neighborhood U. Consider any sequence $\{x_k\}$ the elements of which are in U for all $k = 0, 1, \ldots$ and $x_k \neq \bar{x}$ for all k. Then $x_k \to \bar{x}$ superlinearly if and only if

$$x_k \to \bar{x} \quad \text{and} \quad \lim_{k\to\infty} \frac{\|f(x_{k+1}) - f(\bar{x})\|}{\|s_k\|} = 0.$$

Proof. Consider any $\{x_k\}$ with elements in U such that $x_{k+1} \neq \bar{x}$ for all k and $x_k \to \bar{x}$ superlinearly. Then $\|e_k\|/\|s_k\| \to 1$. Let $\varepsilon > 0$ and choose k_0 large enough so that

$$\|e_{k+1}\| \leq \varepsilon\|s_k\|.$$

Let L be a Lipschitz constant of f in U. We have

$$\frac{\|f(x_{k+1}) - f(\bar{x})\|}{\|s_k\|} \leq \frac{L\|x_{k+1} - \bar{x}\|}{\|s_k\|} = \frac{L\|e_{k+1}\|}{\|s_k\|} \leq L\varepsilon.$$

Proof continued.

Let $\varepsilon > 0$ satisfy $\varepsilon < 1/\kappa$ and let k_0 be so large that for $k \geq k_0$, one has $x_k \in U$ and

$$\|f(x_{k+1}) - f(\bar{x})\| \leq \varepsilon \|s_k\| \qquad \text{for} \quad k \geq k_0.$$

The assumed strong subregularity yields

$$\|x_{k+1} - \bar{x}\| \leq \kappa \|f(x_{k+1}) - f(\bar{x})\|$$

and hence, for all $k \geq k_0$, $\|e_{k+1}\| \leq \kappa\varepsilon \|s_k\|$. But then, for such k,

$$\|e_{k+1}\| \leq \kappa\varepsilon \|s_k\| \leq \kappa\varepsilon(\|e_k\| + \|e_{k+1}\|).$$

Hence

$$\frac{\|e_{k+1}\|}{\|e_k\|} \leq \frac{\kappa\varepsilon}{1 - \kappa\varepsilon}$$

for all sufficiently large k. Hence $x_k \to \bar{x}$ superlinearly. □

Proof of Dennis-Moré Theorem

Let

$$V_k = \int_0^1 Df(\bar{x} + \tau e_k) d\tau - Df(\bar{x}) e_k.$$

By elementary calculus

$$\begin{aligned}
f(x_{k+1}) &= f(\bar{x}) + \int_0^1 Df(\bar{x} + \tau e_{k+1}) e_{k+1} d\tau = Df(\bar{x}) e_{k+1} + V_{k+1} \\
&= -f(x_k) - B_k s_k + Df(\bar{x}) s_k + Df(\bar{x}) e_k + V_{k+1} \\
&= -f(x_k) - E_k s_k + Df(\bar{x}) e_k + V_{k+1} \\
&= -E_k s_k - f(x_k) + f(\bar{x}) + Df(\bar{x}) e_k + V_{k+1} \\
&= -E_k s_k - V_k + V_{k+1}.
\end{aligned}$$

Apply Main Lemma.

Newton differentiable functions

Definition. A function f is Newton differentiable at $\bar{x} \in \operatorname{int} \operatorname{dom} f$ when for each $\varepsilon > 0$ there exists a neighborhood U of $\bar{x}$ and such that for each $x \in U$ there exists a mapping $G(x) \in \mathcal{L}(X, Y)$, called the N-derivative of f at x, such that

$$\|f(x) - f(\bar{x}) - G(x)(x - \bar{x})\| \le \varepsilon \|x - \bar{x}\|.$$

Reference: K. Ito and K. Kunisch, On the Lagrange multiplier approach to variational problems and applications, SIAM, Philadelphia, PA, 2008.

f is strongly metrically subregular at $\bar{x}$ if and only if there is a constant $\kappa > 0$ such that for any h near 0 and any N-derivative $G(\bar{x} + h)$ one has

$$\|G(\bar{x} + h)h\| \ge \kappa \|h\|.$$

Dennis-Moré for Newton differentiable functions

Theorem. Suppose that f has a zero $\bar{x}$, is Lipschitz continuous in a neighborhood $\bar{x}$, Newton differentiable at $\bar{x}$ and strongly subregular at $\bar{x}$. Let $\{B_k\}$ be a sequence of linear and bounded mappings and suppose that there is a neighborhood of $\bar{x}$ such that for any starting point x_0 in U a sequence $\{x_k\}$ be generated by

$$f(x_k) + B_k(x_{k+1} - x_k) = 0,$$

remain in U and satisfy $x_k \neq \bar{x}$ for all k. Let $G_k(x_k)$ be any Newton derivative of f for x_k and denote $E_k = B_k - G_k(x_k)$. Then $x_k \to \bar{x}$ superlinearly if and only if

$$x_k \to \bar{x} \quad \text{and} \quad \lim_{k\to\infty} \frac{\|E_k s_k\|}{\|s_k\|} = 0.$$

Quasi-Newton method for generalized equations

$f : X \to Y$, $F : X \rightrightarrows Y$
Generalized equation

$$f(x) + F(x) \ni 0,$$

Quasi-Newton method

$$\text{(QN)} \qquad f(x_k) + B_k(x_{k+1} - x_k) + F(x_{k+1}) \ni 0,$$

where $f : X \to Y$, $F : X \rightrightarrows Y$, $B_k \in \mathcal{L}(X, Y)$.

Dennis-Moré Theorem for generalized equations

Let $s_k = x_{k+1} - x_k$, $E_k = B_k - Df(\bar{x})$.

Theorem [extended Dennis-Moré]. Suppose that f is Fréchet differentiable in an open convex set D in X containing a solution $\bar{x}$ and Df is continuous at $\bar{x}$. Let for some starting point x_0 in D the sequence $\{x_k\}$ be generated by (QN), remain in D and satisfy $x_k \neq \bar{x}$ for all k. If $x_k \to \bar{x}$ superlinearly, then

$$\lim_{k\to\infty} \frac{d(0, f(\bar{x}) + E_k s_k + F(x_{k+1}))}{\|s_k\|} = 0.$$

Conversely, suppose that $x \mapsto G(x) = f(\bar{x}) + Df(\bar{x})(x - \bar{x}) + F(x)$ is strongly metrically subregular at $\bar{x}$ for 0 and consider a sequence $\{x_k\}$ generated by (QN) for some x_0 in D, which remain in D, satisfy $x_k \neq \bar{x}$ for all k,

$$x_k \to \bar{x} \quad \text{and} \quad \lim_{k\to\infty} \frac{\|E_k s_k\|}{\|s_k\|} = 0.$$

Then $x_k \to \bar{x}$ superlinearly.