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Approximation of Elliptic Control Problems in Measure Spaces with Sparse Solutions

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A joint work with Christian Clason and Karl Kunisch (University of Graz)

Sparse Controls

$$\min J(u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \alpha\|u\|_{L^1(\Omega)} + \frac{\beta}{2}\|u\|^2_{L^2(\Omega)}$$

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with $c_0 \in L^\infty(\Omega)$ and $c_0 \geq 0$. We assume that $\alpha > 0$, $y_d \in L^2(\Omega)$ and Ω is a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, which is supposed to either be convex or have a $C^{1,1}$ boundary Γ. The controls are taken in the space of regular Borel measures $\mathcal{M}(\Omega)$.

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$$\|u\|_{\mathcal{M}(\Omega)} = \sup_{\|z\|_{C_0(\Omega)} \leq 1} \langle u, z \rangle = \sup_{\|z\|_{C_0(\Omega)} \leq 1} \int_\Omega z(x)\, du = |u|(\Omega)$$

Related Papers

- C. Clason and K. Kunisch: “A duality-based approach to elliptic control problems in non-reflexive Banach spaces”, ESAIM Control Optim. Calc. Var., 17:1 (2011), pp. 243–266.

- E.C., R. Herzog and G. Wachsmuth: “Optimality conditions and error analysis of semilinear elliptic control problems with L^1 cost functional”. Submitted.

- G. Stadler: “Elliptic optimal control problems with L^1-control cost and applications for the placement of control devices”, Comp. Optim. Appls., 44:2 (2009), pp. 159–181.

- D. Wachsmuth and G. Wachsmuth: “Convergence and regularization results for optimal control problems with sparsity functional”, ESAIM Control Optim. Calc. Var. 2010, DOI: 10.1051/cocv/2010027.

The State Equation

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Given a measure $u \in \mathcal{M}(\Omega)$, we say that y is a solution to the state equation if

$$\int_\Omega (-\Delta z + c_0 z) y \, dx = \int_\Omega z \, du \qquad \text{for all } z \in H^2(\Omega) \cap H_0^1(\Omega)$$

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It is well known that there exists a unique solution in this sense. Moreover, $y \in W_0^{1,p}(\Omega)$ for every $1 \leq p < \frac{n}{n-1}$ and

$$\|y\|_{W_0^{1,p}(\Omega)} \leq C_p \|u\|_{\mathcal{M}(\Omega)}$$

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then

$$\alpha\|\bar{u}\|_{\mathcal{M}(\Omega)} + \int_\Omega \bar{\varphi}\, d\bar{u} = 0,$$

$$\left\{ \begin{array}{ll} \|\bar{\varphi}\|_{C_0(\Omega)} = \alpha & \textit{if } \bar{u} \neq 0, \\ \|\bar{\varphi}\|_{C_0(\Omega)} \leq \alpha & \textit{if } \bar{u} = 0. \end{array} \right.$$

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• Discrete States:

$$Y_h = \{y_h \in C(\bar\Omega) \mid y_{h|T} \in \mathcal{P}_1, \text{ for all } T \in \mathcal{T}_h, \text{ and } y_h = 0 \text{ on } \bar\Omega \setminus \Omega_h\},$$

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• Discrete State Equation:

$$\left\{ \begin{array}{l} \text{Find } y_h \in Y_h \text{ such that, for all } z_h \in Y_h, \\ \displaystyle\int_{\Omega_h} [\nabla y_h \nabla z_h + c_0 y_h z_h]\, dx = \int_{\Omega_h} z_h\, du. \end{array} \right.$$

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The Approximation (P_h)

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Since we have not discretized the control space, this approach is related to the variational discretization method introduced by Hinze. We will show that among all the solutions to (P_h) there is a unique one which is a finite linear combination of Dirac measures concentrated in the interior vertices of the triangulation, leading to a simple numerical implementation.

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- $D_h = \left\{ u_h \in \mathcal{M}(\Omega) : u_h = \sum_{j=1}^{N(h)} \lambda_j \delta_{x_j}, \text{ where } \{\lambda_j\}_{j=1}^{N(h)} \subset \mathbb{R} \right\}.$

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$$\bullet D_h = Y_h', \quad \langle u_h, y_h \rangle = \sum_{j=1}^{N(h)} \lambda_j y_j \quad \forall u_h \in D_h,\ \forall y_h \in Y_h.$$

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Two Linear Operators

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$$\Lambda_h u = \sum_{j=1}^{N(h)} \langle u, e_j \rangle \delta_{x_j}.$$

- For every $u \in \mathcal{M}(\Omega)$ and every $z \in C_0(\Omega)$ and $z_h \in Y_h$ we have

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- There exist a constant $C > 0$ such that for every $u \in \mathcal{M}(\Omega)$

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- Given $u \in \mathcal{M}(\Omega)$, let y_h and $\tilde{y}_h$ be the discrete solutions associated to the controls u and $\Lambda_h u$, respectively, then $y_h = \tilde{y}_h$.

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THEOREM 3 *Problem* (P_h) *admits at least one solution. Among them there exists a unique one* $\bar{u}_h$ *belonging to* D_h*. Moreover, any other solution* $\tilde{u}_h \in \mathcal{M}(\Omega)$ *of* (P_h) *satisfies that* $\Lambda_h \tilde{u}_h = \bar{u}_h$*.*

REMARK 1 *The fact that* (P_h) *has exactly one solution in* D_h *is of practical interest. Indeed, recall that, as an element of* D_h*,* $\bar{u}_h$ *has a unique representation of the form*

$$\bar{u}_h = \sum_{j=1}^{N(h)} \bar{\lambda}_j \delta_{x_j} \quad \textit{and} \quad \|\bar{u}_h\|_{\mathcal{M}(\Omega)} = \sum_{j=1}^{N(h)} |\bar{\lambda}_j|$$

Then, the numerical computation of $\bar{u}_h$ *is reduced to the computation of the coefficients* $\{\bar{\lambda}_j\}_{j=1}^{N(h)}$*.*

Convergence Analysis

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THEOREM 4 *For every* $h > 0$*, let* $\bar{u}_h$ *be the unique solution to* (P_h) *belonging to* D_h *and let* $\bar{u}$ *be the solution to* (P)*. Then, the following convergence properties hold for* $h \to 0$*:*

$$\bar{u}_h \overset{*}{\rightharpoonup} \bar{u} \ \text{ in } \mathcal{M}(\Omega)$$

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$$\begin{aligned}
&\bar{u}_h \stackrel{*}{\rightharpoonup} \bar{u} \quad \text{in } \mathcal{M}(\Omega) \\
&\|\bar{u}_h\|_{\mathcal{M}(\Omega)} \to \|\bar{u}\|_{\mathcal{M}(\Omega)} \\
&\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \to 0 \\
&J_h(\bar{u}_h) \to J(\bar{u})
\end{aligned}$$

where $\bar{y}$ and $\bar{y}_h$ are the continuous and discrete states associated to $\bar{u}$ and $\bar{u}_h$, respectively.

Some Error Estimates

Assumption:

$$y_d \in L^r(\Omega) \quad \text{with } r = \begin{cases} 4 & \text{if } n = 2 \\ \frac{8}{3} & \text{if } n = 3 \end{cases}$$

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THEOREM 5 *There exists a constant $C > 0$ independent of h such that*

$$|J(\bar{u}) - J_h(\bar{u}_h)| \leq Ch^\kappa$$

where $\kappa = 1$ if $n = 2$ and $\kappa = 1/2$ if $n = 3$.

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THEOREM 6 *There exists a constant $C > 0$ independent of h such that*

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq Ch^{\frac{\kappa}{2}}$$

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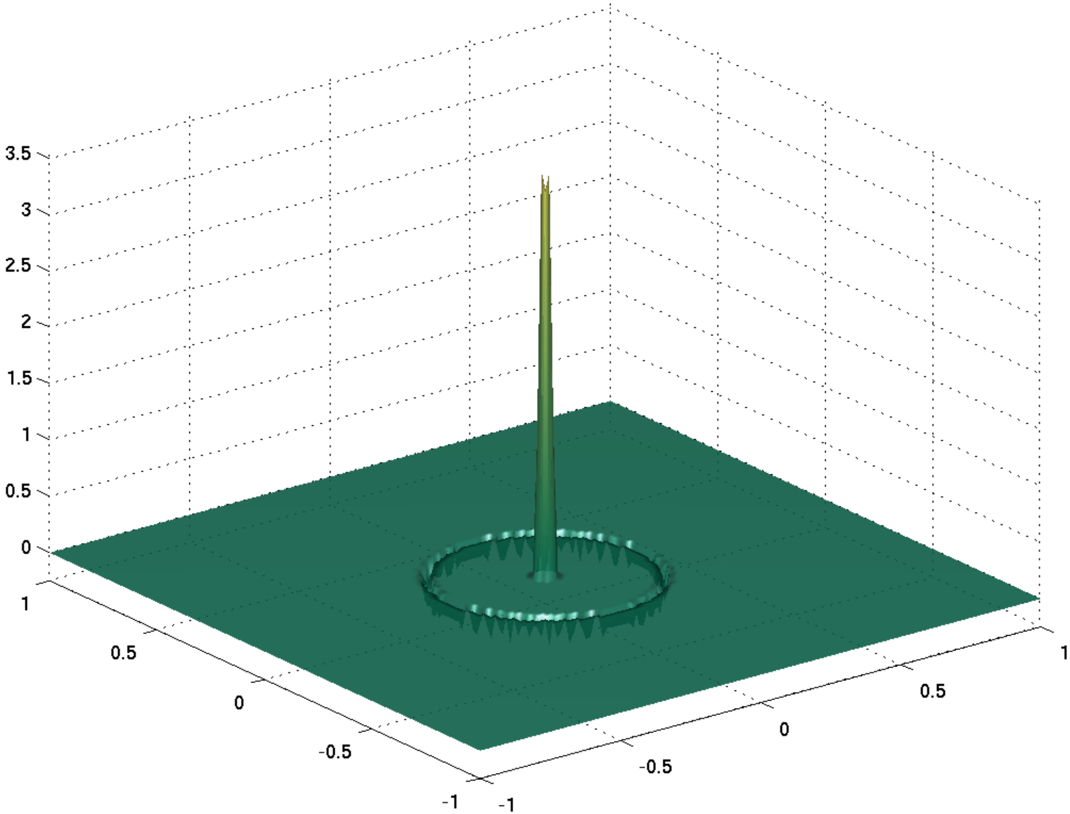
Computational Results

- $\Omega_h = \Omega = [-1, 1]^2$.
- Uniform triangulation arising from $N \times N$ equidistributed nodes.
- $N = 128$ ($h \approx 0.0157$), $c_0 = 0$, and $\alpha = 10^{-2}$.
- $y_d = 10 \exp(-50\|x\|^2)$.



Optimal Control $\bar{u}_h$

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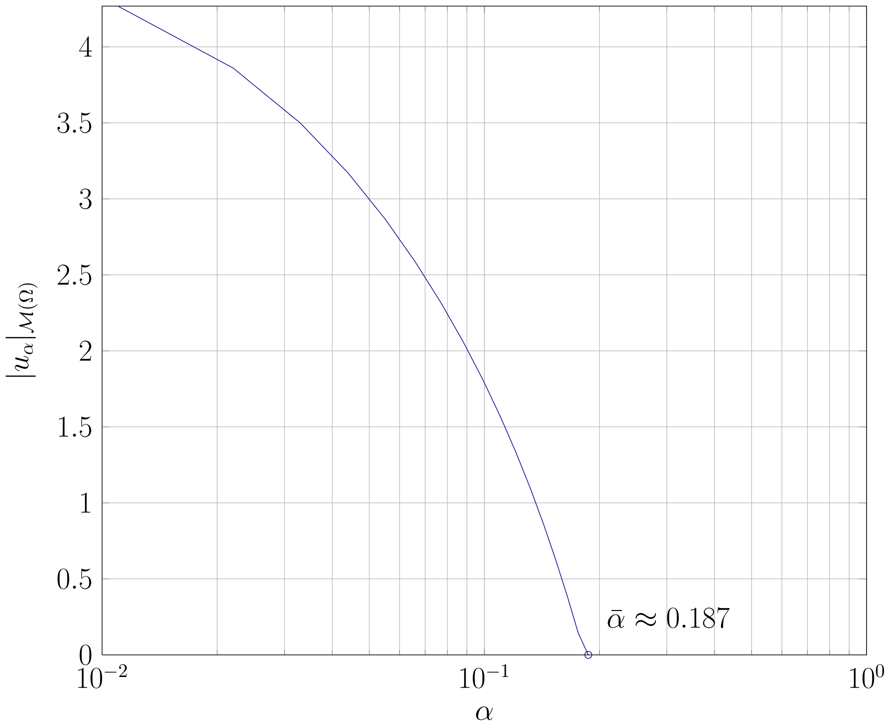
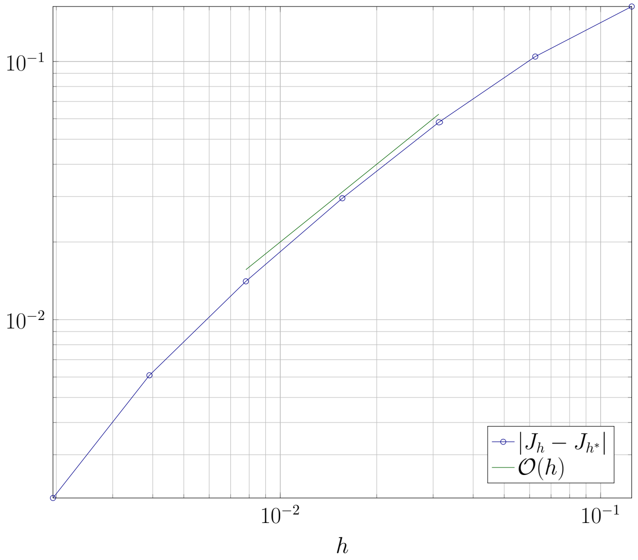


Dependence of $\|u_h\|_{M(\Omega)}$ on penalty parameter α



Convergence order for the functionals

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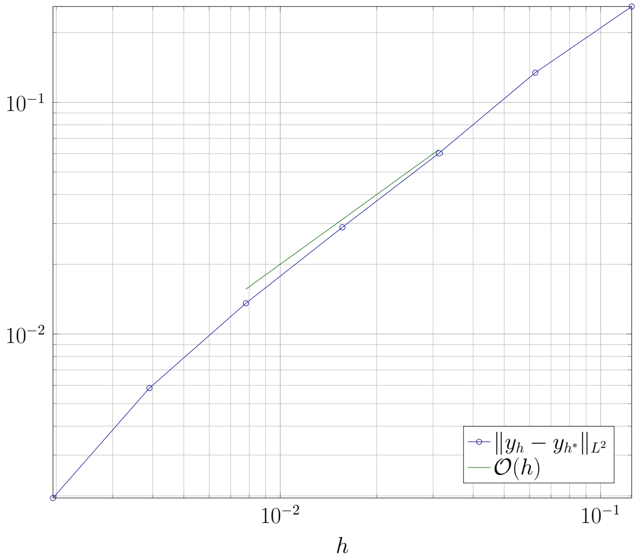

Convergence order for the states