Four Color theorem for image segmentation

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The original Munford-Shah

The Mumford-Shah (MS) model:

$$\min_{C,s} \mu |C| + \frac{\alpha}{2} \int_{\Omega} (s - s_0)^2 + \frac{\gamma}{2} \int_{\Omega \setminus C} |\nabla s|^2,$$  \hspace{1cm} (1)
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The PC Mumford-Shah (MS) model:

\[
\min_{r} \min_{\{\Gamma_i\}_{i=1}^{r}} \min_{\{c_i\}_{i=1}^{r}} \sum_{i=1}^{r} \mu |\partial \Gamma_i| + \frac{\alpha}{2} \int_{\Gamma_i} (c_i - s_0)^2 \quad \text{s.t.} \quad \Omega = \bigcup_{i=1}^{r} \Gamma_i \text{ and } \Gamma_i \cap \Gamma_j = \emptyset \ \forall i \neq j
\]
Given \( \{ f_i \}_{i=1}^n \) for a fixed number \( n \), the Potts models needs to:

\[
\min_{\Gamma_i} \sum_{i=1}^{n} \mu |\partial \Gamma_i| + \int_{\Gamma_i} f_i, \quad \Omega = \bigcup_{i=1}^{n} \Gamma_i, \quad \Gamma_i \cap \Gamma_j = \emptyset, \forall i, j.
\]
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\]

So if the number \( n \) and \( c_i \) are known, the PC Mumford-Shah is reduced to the Potts model choosing \( f_i = |c_i - u_0|^2 \).
Combining Mumford-Shah with GAC

It is possible to combine these two popular models together and it has been shown to have superior property with no extra computational cost:

$$\min_{n, \Gamma_i, c_i} \sum_{i=1}^{n} \int_{\Gamma_i} (u_0 - c_i)^2 + \beta \int_{\partial\Gamma_i} g(|\nabla u_\sigma|)ds$$
Given an input image $u_0$, the 2-phase level set representation is find $\phi$ and $c_i$ from:

$$\min_{\Gamma, c_1, c_2} \alpha |\Gamma| + \int_{\Omega_1} |c_1 - u_0|^2 + \int_{\Omega_2} |c_2 - u_0|^2.$$
Chan-Vese model – One of the most popular segmentation model

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$$\min_{\phi, c_1, c_2} \int_{\Omega} \alpha |\nabla H(\phi)| + \{H(\phi)(c_1 - u_0)^2 + (1 - H(\phi))(c_2 - u_0)^2\} dx,$$
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$\begin{align*}
\text{min}_{\Gamma, c_1, c_2} & \alpha |\Gamma| + \int_{\Omega_1} |c_1 - u_0|^2 + \int_{\Omega_2} |c_2 - u_0|^2. \\
\text{min}_{\phi, c_1, c_2} & \int_{\Omega} \alpha |\nabla H(\phi)| + \{H(\phi)(c_1 - u_0)^2 + (1 - H(\phi))(c_2 - u_0)^2\} \, dx,
\end{align*}$

$H(\phi) = 1 \text{ if } \phi > 0, \quad H(\phi) = 0 \text{ if } \phi < 0$

$\text{From } \phi, \text{ we get } \Omega_1 = \{x \mid \phi > 0\}, \quad \Omega_2 = \{x \mid \phi \leq 0\}.$
More than two regions–multiple level-sets (Chan and Vese, 2000)

\[ \Omega_{++} = \{ x \in \Omega, \ \phi_1 > 0, \ \phi_2 > 0 \} \]
\[ \Omega_{+-} = \{ x \in \Omega, \ \phi_1 > 0, \ \phi_2 < 0 \} \]
\[ \Omega_{-+} = \{ x \in \Omega, \ \phi_1 < 0, \ \phi_2 > 0 \} \]
\[ \Omega_{--} = \{ x \in \Omega, \ \phi_1 < 0, \ \phi_2 < 0 \} . \]
Multiphase level set representation of CV model

\[
\min_{\phi^1, \phi^2, \{c_i\}_{i=1}^4} \alpha \int_{\Omega} |\nabla H(\phi^1)| + \alpha \int_{\Omega} |\nabla H(\phi^2)| + E^{data}(\phi^1, \phi^2),
\]

where

\[
E^{data}(\phi^1, \phi^2) = \int_{\Omega} \left\{ H(\phi^1)H(\phi^2)|c_2-u_0|^\beta + H(\phi^1)(1-H(\phi^2))|c_1-u_0|^\beta \\
+ (1-H(\phi^1))H(\phi^2)|c_4-u_0|^\beta + (1-H(\phi^1))(1-H(\phi^2))|c_3-u_0|^\beta \right\} dx.
\]

\[
\begin{align*}
\Omega_1 &= \{ x \in \Omega \text{ s.t. } \phi^1(x) > 0, \phi^2(x) < 0 \} \\
\Omega_2 &= \{ x \in \Omega \text{ s.t. } \phi^1(x) > 0, \phi^2(x) > 0 \} \\
\Omega_3 &= \{ x \in \Omega \text{ s.t. } \phi^1(x) < 0, \phi^2(x) < 0 \} \\
\Omega_4 &= \{ x \in \Omega \text{ s.t. } \phi^1(x) < 0, \phi^2(x) > 0 \}
\end{align*}
\]
-chan-Vese model – Advantages

- Use the level set of Osher and Sethian (JCP1998): can handle very general geometries.
- Region based: robust with noise, images without edges.
Chan-Vese model – disadvantages

- Slow in convergence.
- Each $\Omega_i$ can contain many disconnected subregions, we must have $u = c_i$ in $\Omega_i$. 

(a) Input images  
(b) Segmented image
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(a) Input images  
(b) Segmented image

- If $u$ needs to take $n$ constant values, then we need $n$-phases for the segmentation.
- Our new model allows $u$ to take many constant values inside $\Omega_i$. 
Chan-Vese model – disadvantages

- Need to use many level set functions or labeling functions

![Input images](a) Input images ![Segmented image](b) Segmented image (8 phases)

Figure: Our new model never needs more than 4-phases to get the same segmentation.
Chan-Vese model – disadvantages

- Slow in convergence.
- Non-convex minimization: may get stuck with local minimums, depends on initial value.

Figure: Images from Brown-Chan-Bresson (IJCV 2011).
The new 4-color model based on Munford-Shah

The New 4-Color Model Based On Munford-Shah
The new 4-color theorem

Any planar graph (map) can be painted and separated by 4 colors.
The new model: Due to 4-color theorem:

\[
\min_{\{\Omega_i\}_{i=1}^4} \min_{\{s_i\}_{i=1}^4} \sum_{i=1}^4 \mu |\partial \Omega_i| + \frac{\alpha}{2} \int_{\Omega_i} (s_i - s_0)^2 \quad \text{s.t.}
\]

\[
\Omega = \bigcup_{i=1}^4 \Omega_i, \Omega_i \cap \Omega_j = \emptyset \ \forall i \neq j
\]

and \(|\nabla s_i(x)|^2 = 0\) in \(\Omega_i\). \hspace{1cm} (6)

\[
s_i : \Omega \mapsto R, i = 1, 2, 3, 4.
\]
The new 4-color model based on Munford-Shah

\[ \nabla s_i = 0 \text{ in } \Omega_i \]

\[ \Rightarrow \]

\[ s_i = \text{const} \text{ in } \Omega_i. \]
The new 4-color model based on Munford-Shah

\[ s_i : \Omega \mapsto R, \]

\[ \nabla s_i = 0 \text{ in } \Omega_i \]

\( s_i = \text{const} \), but the constants can differ from one connected subregion to another connected subregion.
Regulatrization of $s_i$

$$\min_{\{\Omega_i\}_{i=1}^4} \min_{\{s_i\}_{i=1}^4} \sum_{i=1}^4 \mu |\partial \Omega_i| + \frac{\alpha}{2} \int_{\Omega_i} (s_i - s_0)^2 + \frac{\sigma}{2} \int_{\Omega} |\nabla s_i|^2 \quad \text{s.t.}$$

$$\Omega = \bigcup_{i=1}^4 \Omega_i, \quad \Omega_i \cap \Omega_j = \emptyset \quad \forall \ i \neq j \quad \text{and} \quad |\nabla s_i(x)|^2 = 0 \quad \text{in} \ \Omega_i. \ (7)$$

Adding $\sigma$, we regularize the values of $s_i$ outside $\Omega_i$. In fact, the values of $s_i$ is the harmonic extensions of the piecewise constant values of $s_i$ inside $\Omega_i$. 

![Diagram of overlapping regions $\Omega_1, \Omega_2, \Omega_3, \Omega_4$]
Representation of \( \{\Omega_i\}_{i=1}^{4} \)

- Using level set functions (slow, non-convex).

- Using binary labels: \( u_1, u_2 \in \{0, 1\} \).

- Using a single label: \( u \in \{1, 2, 3, 4\} \).

- Using the characteristic function \( u_i = \chi_{\Omega_i}, i = 1, 2, 3, 4 \).

All can be convexified and have fast solvers.
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All can be convexified and have fast solvers.
Chan-Vese (product) binary labelling

- Use two binary functions: $u_1, u_2 \in \{0, 1\}$.
- Corresponding characteristic functions:
  \[ \psi_1 = u_1 u_2, \psi_2 = (1 - u_1) u_2, \psi_3 = u_1 (1 - u_2), \psi_4 = (1 - u_1) (1 - u_2). \]
Use a single labeling functions: \( u \in \{1, 2, 3, 4\} \).

Corresponding characteristic functions:
\[
\psi_i = 1 \text{ when } u = i, \text{ else } \psi_i = 0.
\]

Use labeling functions: $u_i \in \{0, 1\}$, $\sum_{i=1}^{4} u_i = 1$.

Characteristic Labelling functions (PCLS)

\[ u_i(x) = \begin{cases} 1, & \forall x \in \Omega_i, \\ 0, & \text{otherwise.} \end{cases} \tag{8} \]

The problem (7) can thus be rewritten as

\[
\min_{\{u_i \in \{0,1\}\}_{i=1}^4} \min_{\{s_i\}_{i=1}^4} \sum_{i=1}^4 \int_{\Omega} \mu(x)|\nabla u_i| + \frac{\alpha}{2} u_i(s_i - s_0)^2 + \frac{\sigma}{2} |\nabla s_i|^2 \\
\sum_{i=1}^4 u_i(x) = 1 \ \forall x \in \Omega \\
\text{and } u_i(x)|\nabla s_i(x)|^2 = 0 \ \forall x \in \Omega, \ i = 1, \ldots, 4. \tag{9} \]
We use Lagrangian method to deal with the constrain. The corresponding Augmented Lagrangian functional is:

\[ L\left(\{u_i\}_{i=1}^4, \{s_i\}_{i=1}^4, \lambda\right) = \]
\[ \sum_{i=1}^4 \int_{\Omega} \mu |\nabla u_i| + \frac{\alpha}{2} \int_{\Omega} u_i(s_i - s_0)^2 + \frac{\sigma}{2} \int_{\Omega} |\nabla s_i|^2 \]
\[ + \int_{\Omega} \lambda(x) u_i(x) |\nabla s_i(x)|^2 + \frac{r}{2} u_i(x) |\nabla s_i(x)|^2. \]

\[ u_i(x) \geq 0, \quad \sum_{i=1}^4 u_i(x) = 1 \quad \forall x \in \Omega \quad (10) \]
An algorithm

\[(u_{i}^{k+1}, s_{i}^{k+1}) = \arg \min_{\{u_{i} \in [0,1]\}^{4}_{i=1}, \{s_{i}\}^{4}_{i=1}} \sum_{i=1}^{4} \int_{\Omega} w_{b} |\nabla u_{i}| + \frac{\alpha}{2} \int_{\Omega} u_{i}(s_{i} - s_{0})^{2} \]

\[+ \frac{\sigma}{2} \int_{\Omega} |\nabla s_{i}|^{2} + \int_{\Omega} u_{i}(\lambda_{i}^{k} |\nabla s_{i}|^{2} + \frac{r}{2} |\nabla s_{i}|^{2}) \quad \text{s.t.} \quad \sum_{i=1}^{4} u_{i}(x) = 1.\]

\[\lambda_{i}^{k+1} = \lambda_{i}^{k} + ru_{i} |\nabla s_{i}|^{2} \quad \text{(11)}\]

where \(w_{b}(x)\) can be an edge detector s.a. \(w_{b}(x) = \frac{1}{1+\mu |\nabla s_{0}(x)|^{2}}.\)
An algorithm: Subproblem I

\[
\min_{u_i \in [0,1]} \int_{\Omega} w_b |\nabla u_i| + \int_{\Omega} u_i f_i \quad \text{s.t.} \quad \sum_{i=1}^{4} u_i(x) = 1. \tag{12}
\]

where \( f_i = \frac{\alpha}{2}(s_i - s_0)^2 + |\nabla s_i|^2(\lambda_i^k + \frac{r}{2}). \)
An algorithm: Subproblem II

\[
\min_{s_i} \int_{\Omega} \frac{h_i}{2} (s_i - s_0)^2 + \frac{g_i}{2} |\nabla s_i|^2
\]  \hspace{1cm} (13)

where \( h_i = \alpha u_i \) and \( g_i = \sigma + (2\lambda_i^k + r) u_i \).
The algorithm

Algorithm for the unsupervised image segmentation model (7) using the four color theorem (with a priori unknown number of regions).

- Initialize the $u_i$ (random initialization or k-mean). While not converged
  - $s_i^{k+1}$ computed with Algorithm for $s_i$
  - $u_i^{k+1}$ computed with Algorithm for $u_i$
  - $\lambda_i^{k+1} = \lambda_i^k + ru_i^{k+1} \left| \nabla s_i^{k+1} \right|^2$
Numerical Experiments

**Figure:** (a) Original image. (b) segmentation into four phase $\{\Omega_i\}_{i=1}^4$ (two distinct regions, central disk and upper right part are merged). (c) segmentation result after segmenting each phase into four sub-phase (this produces sixteen sub-phases $\{\Omega_{i,j}\}_{i,j=1}^4$) and recoloring into four phases (correct result). (d) piecewise constant approximation of (a).
Two-level recursive algorithm for the unsupervised image segmentation model (2) using the four color theorem (with a priori unknown number of regions).

- Initialize the $u_i$ (random initialization or k-mean), select the scale parameter $\alpha$ which controls the number of regions.
  While not converged.
  - Compute four phases $\{\Omega_i\}_{i=1}^4$ with Algorithm 1
  - Partition each phase $\{\Omega_i\}_{i=1}^4$ into 4 sub-phases $\{\Omega_{i,j}\}_{i,j=1}^4$ with Algorithm 2
  - Recolor the 16 sub-phases into 4 phases
Figure: Comparison between the standard recursive bi-partitioning method and our method: (a) Original image. (b) segmentation after 1st bi-partitioning. (c) segmentation after 2nd/final bi-partitioning (over-segmentation). (d) Our algorithm (correct segmentation).
Numerical Experiments

Figure: Influence of the regularization parameter $\alpha$. First row is the original image. Second row is the four-color segmentation result. Third row is the piecewise constant approximation of the image. First column $\alpha = 1.5e5/255^2$, second column $\alpha = 3e4/255^2$, third column $\alpha = 1e4/255^2$, fourth column $\alpha = 1e3/255^2$. 
Numerical Experiments

Figure: First and fourth rows present the original image. Second and fifth rows show the four-color segmentation result. Third and last rows display the piecewise constant approximation of the image $s_0$. Each column presents a different value of $\alpha$, which controls the number of final segmented regions.
Numerical Experiments

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Historical overview

- Vese-Chan (2002, IJCV) has proposed to use 4-color theorem for the original Mumford-Shah model.
- Liu-Tao (2011, PR), Tao-Tai (UCLA-CAM-09-13).

Ref: Four color theorem and convex relaxation for image segmentation with any number of regions. Inverse Problems & Imaging 7 (3), 1099-1113.
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