Four Color theorem for image segmentation

Xue-Cheng Tai, University of Bergen, Norway

Collaborations with:
Xavier Bresson, Tony F. Chan and Ruilinag Zhang

July 17, 2014

The original Munford-Shah

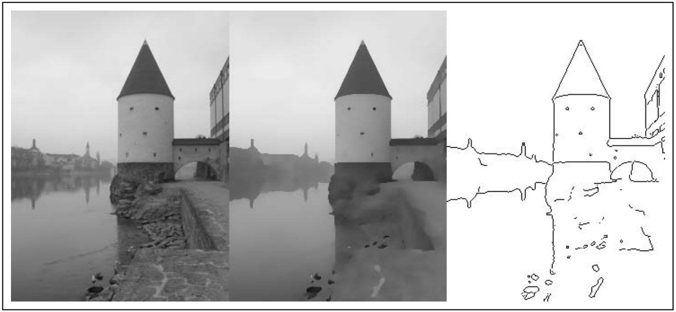
The Mumford-Shah (MS) model:

$$\min_{C,s} \ \mu|C| + \frac{\alpha}{2}\int_{\Omega}(s - s_0)^2 + \frac{\gamma}{2}\int_{\Omega\setminus C}|\nabla s|^2, \tag{1}$$

The original Munford-Shah

The Mumford-Shah (MS) model:

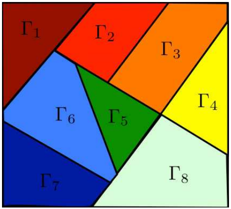
$$\min_{C,s} \; \mu|C| + \frac{\alpha}{2}\int_{\Omega}(s-s_0)^2 + \frac{\gamma}{2}\int_{\Omega\setminus C}|\nabla s|^2, \qquad (1)$$



The piecewise constant Munford-Shah

The PC Mumford-Shah (MS) model:

$$\min_{r} \min_{\{\Gamma_i\}_{i=1}^r} \min_{\{c_i\}_{i=1}^r} \sum_{i=1}^{r} \mu|\partial\Gamma_i| + \frac{\alpha}{2}\int_{\Gamma_i}(c_i - s_0)^2 \text{ s.t.}$$
$$\Omega = \cup_{i=1}^r \Gamma_i \text{ and } \Gamma_i \cap \Gamma_j = \emptyset \ \forall i \neq j \qquad (2)$$



The Potts model

Given $\{f_i\}_{i=1}^n$ for a fixed number n, the Potts models needs to:

$$\min_{\Gamma_i} \sum_{i=1}^{n} \mu|\partial\Gamma_i| + \int_{\Gamma_i} f_i, \quad \Omega = \cup_{i=1}^n \Gamma_i, \quad \Gamma_i \cap \Gamma_j = \emptyset, \forall i, j.$$

The Potts model

Given $\{f_i\}_{i=1}^n$ for a fixed number n, the Potts models needs to:

$$\min_{\Gamma_i} \sum_{i=1}^{n} \mu|\partial\Gamma_i| + \int_{\Gamma_i} f_i, \quad \Omega = \cup_{i=1}^n \Gamma_i, \quad \Gamma_i \cap \Gamma_j = \emptyset, \forall i, j.$$

So if the number n and c_i are known, the PC Mumford-Shah is reduced to the Potts model choosing $f_i = |c_i - u_0|^2$.

Combining Mumford-Shah with GAC

It is possible to combine these two popular models together and it has been shown to have superior property with no extra computational cost:

$$\min_{n,\Gamma_i,c_i} \sum_{i=1}^{n} \int_{\Gamma_i} (u_0 - c_i)^2 + \beta \int_{\partial\Gamma_i} g(|\nabla u_\sigma|) ds$$

Chan-Vese model – One of the most popular segmentation model

Given an input image u_0, the 2-phase level set representation is find ϕ and c_i from:

- $$\min_{\Gamma, c_1, c_2} \alpha|\Gamma| + \int_{\Omega_1} |c_1 - u_0|^2 + \int_{\Omega_2} |c_2 - u_0|^2.$$

Chan-Vese model – One of the most popular segmentation model

Given an input image u_0, the 2-phase level set representation is find ϕ and c_i from:

- $$\min_{\Gamma, c_1, c_2} \alpha|\Gamma| + \int_{\Omega_1} |c_1 - u_0|^2 + \int_{\Omega_2} |c_2 - u_0|^2.$$

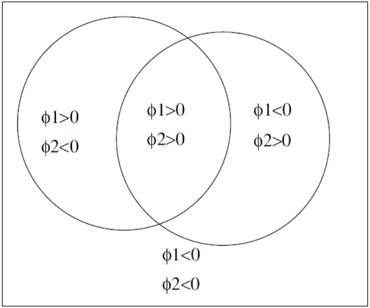
- $$\min_{\phi, c_1, c_2} \int_{\Omega} \alpha|\nabla H(\phi)| + \{H(\phi)(c_1 - u_0)^2 + (1 - H(\phi))(c_2 - u_0)^2\} dx,$$

Chan-Vese model – One of the most popular segmentation model

Given an input image u_0, the 2-phase level set representation is find ϕ and c_i from:

- $$\min_{\Gamma, c_1, c_2} \alpha|\Gamma| + \int_{\Omega_1} |c_1 - u_0|^2 + \int_{\Omega_2} |c_2 - u_0|^2.$$
- $$\min_{\phi, c_1, c_2} \int_{\Omega} \alpha|\nabla H(\phi)| + \{H(\phi)(c_1 - u_0)^2 + (1 - H(\phi))(c_2 - u_0)^2\} dx,$$
- $H(\phi) = 1$ if $\phi > 0$, $H(\phi) = 0$ if $\phi < 0$
- From ϕ, we get $\Omega_1 = \{x |\ \phi > 0\}$, $\Omega_2 = \{x |\ \phi \leq 0\}$.

More than two regions–multiple level-sets (Chan and Vese, 2000)



$$\begin{aligned}
\Omega_{++} &= \{x \in \Omega, \quad \phi_1 > 0, \quad \phi_2 > 0\} \\
\Omega_{+-} &= \{x \in \Omega, \quad \phi_1 > 0, \quad \phi_2 < 0\} \\
\Omega_{-+} &= \{x \in \Omega, \quad \phi_1 < 0, \quad \phi_2 > 0\} \\
\Omega_{--} &= \{x \in \Omega, \quad \phi_1 < 0, \quad \phi_2 < 0\}\,.
\end{aligned}$$

Multiphase level set representation of CV model

$$\min_{\phi^1,\phi^2,\{c_i\}_{i=1}^4} \alpha\int_\Omega |\nabla H(\phi^1)| + \alpha\int_\Omega |\nabla H(\phi^2)| + E^{data}(\phi^1,\phi^2),$$

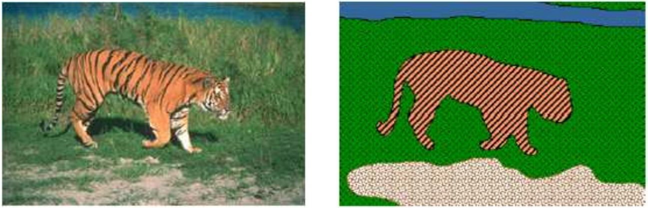
where

$$E^{data}(\phi^1,\phi^2) = \int_\Omega \{H(\phi^1)H(\phi^2)|c_2-u_0|^\beta + H(\phi^1)(1-H(\phi^2))|c_1-u_0|^\beta$$
$$+(1-H(\phi^1))H(\phi^2)|c_4-u_0|^\beta + (1-H(\phi^1))(1-H(\phi^2))|c_3-u_0|^\beta\}dx.$$

$$\begin{aligned}
\Omega_1 &= \{x\in\Omega \text{ s.t. } \phi^1(x)>0, \phi^2(x)<0\}\\
\Omega_2 &= \{x\in\Omega \text{ s.t. } \phi^1(x)>0, \phi^2(x)>0\}\\
\Omega_3 &= \{x\in\Omega \text{ s.t. } \phi^1(x)<0, \phi^2(x)<0\}\\
\Omega_4 &= \{x\in\Omega \text{ s.t. } \phi^1(x)<0, \phi^2(x)>0\}
\end{aligned}$$

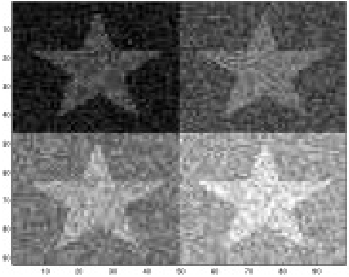
Chan-Vese model – Advantages

- Use the level set of Osher and Sethian (JCP1998): can handle very general geometries.
- Region based: robust with noise, images without edges.

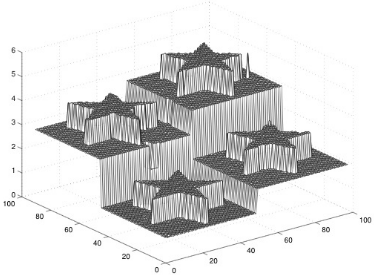


Chan-Vese model – disadvantages

- Slow in convergence.
- Each Ω_i can contain many disconnected subregions, we must have $u = c_i$ in Ω_i.



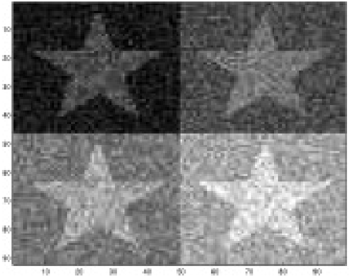
(a) Input images



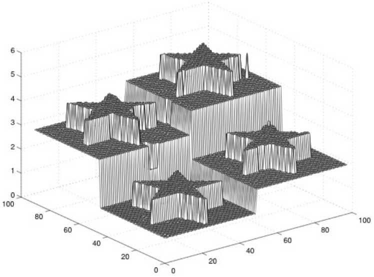
(b) Segmented image

Chan-Vese model – disadvantages

- Slow in convergence.
- Each Ω_i can contain many disconnected subregions, we must have $u = c_i$ in Ω_i.



(a) Input images

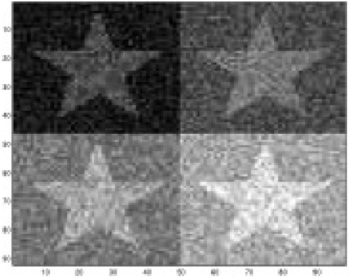


(b) Segmented image

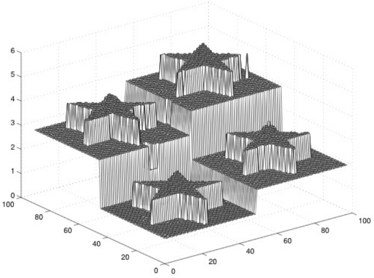
- If u needs to take n constant values, then we need n-phases for the segmentation.

Chan-Vese model – disadvantages

- Slow in convergence.
- Each Ω_i can contain many disconnected subregions, we must have $u = c_i$ in Ω_i.



(a) Input images



(b) Segmented image

- If u needs to take n constant values, then we need n-phases for the segmentation.
- Our new model allows u to take many constant values inside Ω_i.

Chan-Vese model – disadvantages

- Need to use many level set functions or labeling functions

(a) Input images

(b) Segmented image (8 phases)

Figure: Our new model never needs more than 4-phases to get the same segmentation.

Chan-Vese model – disadvantages

- Slow in convergence.
- Non-convex minimization: may get stuck with local minimums, depends on initial value.

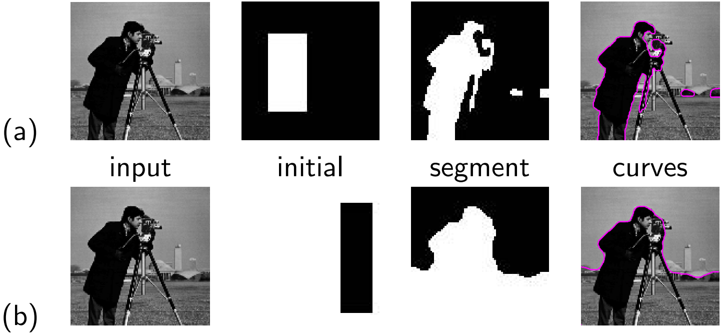


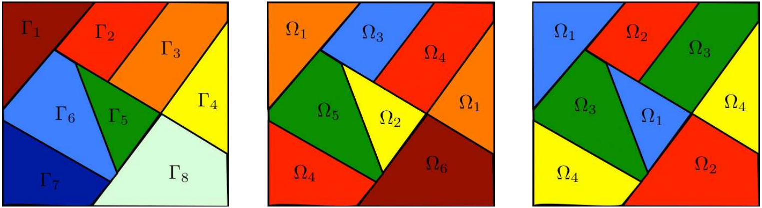
Figure: Images from Brown-Chan-Bresson (IJCV 2011).

The new 4-color model based on Munford-Shah

The New 4-Color Model Based On Munford-Shah

The new 4-color theorem

Any planar graph (map) can be painted and separated by 4 colors.



The new 4-color model based on Munford-Shah

The new model: Due to 4-color theorem:

$$\min_{\{\Omega_i\}_{i=1}^4} \min_{\{s_i\}_{i=1}^4} \sum_{i=1}^4 \mu|\partial\Omega_i| + \frac{\alpha}{2}\int_{\Omega_i}(s_i - s_0)^2 \text{ s.t.}$$
$$\Omega = \cup_{i=1}^4 \Omega_i, \Omega_i \cap \Omega_j = \emptyset \; \forall i \neq j$$
$$\text{and } |\nabla s_i(x)|^2 = 0 \text{ in } \Omega_i. \tag{6}$$

$$s_i : \Omega \mapsto R, i = 1, 2, 3, 4.$$

The new 4-color model based on Munford-Shah

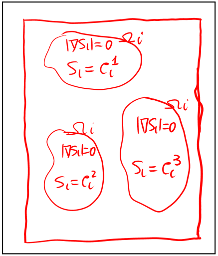
$$\nabla s_i = 0 \text{ in } \Omega_i$$
$$\not\Updownarrow$$
$$s_i = const \text{ in } \Omega_i.$$

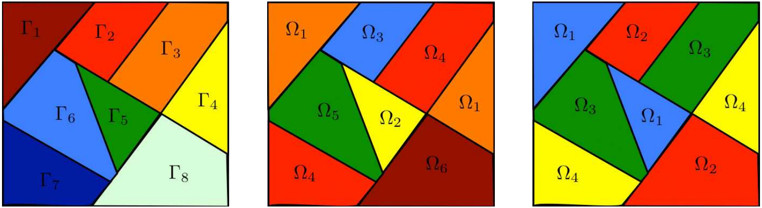
The new 4-color model based on Munford-Shah

$$s_i : \Omega \mapsto R,$$

$$\nabla s_i = 0 \text{ in } \Omega_i$$

$s_i = const$, but the constants can differ from one connected subregion to another connected subregion.

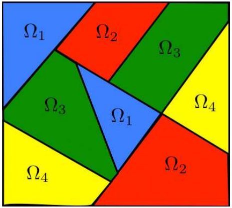




Regulatrization of s_i

$$\min_{\{\Omega_i\}_{i=1}^4} \min_{\{s_i\}_{i=1}^4} \sum_{i=1}^{4} \mu|\partial\Omega_i| + \frac{\alpha}{2}\int_{\Omega_i}(s_i - s_0)^2 + \frac{\sigma}{2}\int_{\Omega}|\nabla s_i|^2 \text{ s.t.}$$
$$\Omega = \cup_{i=1}^4 \Omega_i,\ \Omega_i \cap \Omega_j = \emptyset\ \forall i \neq j \text{ and } |\nabla s_i(x)|^2 = 0 \text{ in } \Omega_i. \quad (7)$$

Adding σ, we regularize the values of s_i outside Ω_i. In fact, the values of s_i is the harmonic extensions of the piecewise constant values of s_i inside Ω_i.



Representation of $\{\Omega_i\}_{i=1}^4$

- Using level set functions (slow, non-convex).

Representation of $\{\Omega_i\}_{i=1}^4$

- Using level set functions (slow, non-convex).
- Using binary labels: $u_1, u_2 \in \{0, 1\}$.

Representation of $\{\Omega_i\}_{i=1}^4$

- Using level set functions (slow, non-convex).
- Using binary labels: $u_1, u_2 \in \{0, 1\}$.
- Using a single label: $u \in \{1, 2, 3, 4\}$.

Representation of $\{\Omega_i\}_{i=1}^4$

- Using level set functions (slow, non-convex).
- Using binary labels: $u_1, u_2 \in \{0, 1\}$.
- Using a single label: $u \in \{1, 2, 3, 4\}$.
- Using the characteristic function $u_i = \chi_{\Omega_i},\ i = 1, 2, 3, 4$.

Representation of $\{\Omega_i\}_{i=1}^4$

- Using level set functions (slow, non-convex).
- Using binary labels: $u_1, u_2 \in \{0, 1\}$.
- Using a single label: $u \in \{1, 2, 3, 4\}$.
- Using the characteristic function $u_i = \chi_{\Omega_i}, \; i = 1, 2, 3, 4$.

All can be convexified and have fast solvers.

Chan-Vese (product) binary labelling

- Use two binary functions: $u_1, u_2 \in \{0, 1\}$.
- Corresponding characteristic functions: $\psi_1 = u_1 u_2, \psi_2 = (1 - u_1)u_2, \psi_3 = u_1(1 - u_2), \psi_4 = (1 - u_1)(1 - u_2)$.
- Convex equivalence exists: Bae-T. (EMMCVPV09), Bae-T. (JMIV 2014), Goldluecke-Cremers ECCV(2010).

PCLS labelling

- Use a single labeling functions: $u \in \{1, 2, 3, 4\}$.
- Corresponding characteristic functions:
  $\psi_i = 1$ when $u = i$, else $\psi_i = 0$.
- Convex equivalence exists: Ishikawa, Darbon-Segelle, Pock-Bremer-Chambolle-et-al, Bae-T., Brown-Bresson-Chan, Golstein-Bresson-Osher.

Characteristic function labelling

- Use labeling functions: $u_i \in \{0, 1\}, \sum_{i=1}^{4} u_i = 1$.
- Convex equivalence exists: Pock-Cremers-Chambolle-et-al (ICCV 2008, ...), Bae-Yuan-T. (IJCV2010), Lellman-et-al (2010,2011,2012), Zach-et-al (2009).

Characteristc Labelling functions (PCLS)

$$u_i(x) = \begin{cases} 1, & \forall x \in \Omega_i, \\ 0, & \text{otherwise}. \end{cases} \quad (8)$$

The problem (7) can thus be rewritten as

$$\min_{\{u_i \in \{0,1\}\}_{i=1}^4} \min_{\{s_i\}_{i=1}^4} \sum_{i=1}^4 \int_\Omega \mu(x)|\nabla u_i| + \frac{\alpha}{2} u_i (s_i - s_0)^2 + \frac{\sigma}{2}|\nabla s_i|^2$$

$$\sum_{i=1}^4 u_i(x) = 1 \ \forall x \in \Omega$$

$$\text{and } u_i(x)|\nabla s_i(x)|^2 = 0 \ \forall x \in \Omega, i = 1, \ldots, 4. \quad (9)$$

Lagrangian functional

We use Lagrangian method to deal with the constrain. The corresponding Augmented Lagrangian functional is:

$$
\begin{aligned}
& L(\{u_i\}_{i=1}^4, \{s_i\}_{i=1}^4, \lambda) = \\
& \sum_{i=1}^4 \int_\Omega \mu |\nabla u_i| + \frac{\alpha}{2} \int_\Omega u_i (s_i - s_0)^2 + \frac{\sigma}{2} \int_\Omega |\nabla s_i|^2 \\
& + \int_\Omega \lambda(x) u_i(x) |\nabla s_i(x)|^2 + \frac{r}{2} u_i(x) |\nabla s_i(x)|^2. \\
& u_i(x) \geq 0, \quad \sum_{i=1}^4 u_i(x) = 1 \ \forall x \in \Omega
\end{aligned} \tag{10}
$$

An algorithm

$$
\begin{aligned}
(u_i^{k+1}, s_i^{k+1}) = \arg \min_{\{u_i \in [0,1]\}_{i=1}^4, \{s_i\}_{i=1}^4} & \sum_{i=1}^4 \int_\Omega w_b |\nabla u_i| + \frac{\alpha}{2} \int_\Omega u_i (s_i - s_0)^2 \\
& + \frac{\sigma}{2} \int_\Omega |\nabla s_i|^2 + \int_\Omega u_i \big(\lambda_i^k |\nabla s_i|^2 + \frac{r}{2} |\nabla s_i|^2 \big) \quad \text{s.t.} \quad \sum_{i=1}^4 u_i(x) = 1. \\
\lambda_i^{k+1} = \lambda_i^k + r u_i |\nabla s_i|^2 & \qquad (11)
\end{aligned}
$$

where $w_b(x)$ can be an edge detector s.a. $w_b(x) = \frac{1}{1+\mu|\nabla s_0(x)|^2}$.

An algorithm: Subproblem I

$$\min_{u_i \in [0,1]} \int_\Omega w_b |\nabla u_i| + \int_\Omega u_i f_i \quad \text{s.t.} \quad \sum_{i=1}^{4} u_i(x) = 1. \tag{12}$$

where $f_i = \frac{\alpha}{2}(s_i - s_0)^2 + |\nabla s_i|^2(\lambda_i^k + \frac{r}{2})$.

An algorithm: Subproblem II

$$\min_{s_i} \int_\Omega \frac{h_i}{2}(s_i - s_0)^2 + \frac{g_i}{2}|\nabla s_i|^2 \tag{13}$$

where $h_i = \alpha u_i$ and $g_i = \sigma + (2\lambda_i^k + r)u_i$.

The algorithm

Algorithm for the unsupervised image segmentation model (7) using the four color theorem (with a priori unknown number of regions).

- Initialize the u_i (random initialization or k-mean). While not converged
  - s_i^{k+1} computed with Algorithm for s_i
  - u_i^{k+1} computed with Algorithm for u_i
  - $\lambda_i^{k+1} = \lambda_i^k + r u_i^{k+1} |\nabla s_i^{k+1}|^2$

Numerical Experiments

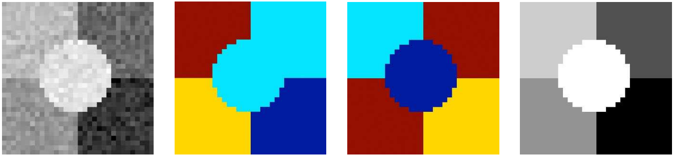


Figure: (a) Original image. (b) segmentation into four phase $\{\Omega_i\}_{i=1}^4$ (two distinct regions, central disk and upper right part are merged). (c) segmentation result after segmenting each phase into four sub-phase (this produces sixteen sub-phases $\{\Omega_{i,j}\}_{i,j=1}^4$) and recoloring into four phases (correct result). (d) piecewise constant approximation of (a).

Local minimizers and re-coloring

Two-level recursive algorithm for the unsupervised image segmentation model (2) using the four color theorem (with a priori unknown number of regions).

- Initialize the u_i (random initialization or k-mean), select the scale parameter α which controls the number of regions. While not converged.
  - Compute four phases $\{\Omega_i\}_{i=1}^4$ with Algorithm 1
  - Partition each phase $\{\Omega_i\}_{i=1}^4$ into 4 sub-phases $\{\Omega_{i,j}\}_{i,j=1}^4$ with Algorithm 2
  - Recolor the 16 sub-phases into 4 phases

Numerical Experiments

Figure: Comparison between the standard recursive bi-partitioning method and our method: (a) Original image. (b) segmentation after 1st bi-partitioning. (c) segmentation after 2nd/final bi-partitioning (over-segmentation). (d) Our algorithm (correct segmentation).

Numerical Experiments

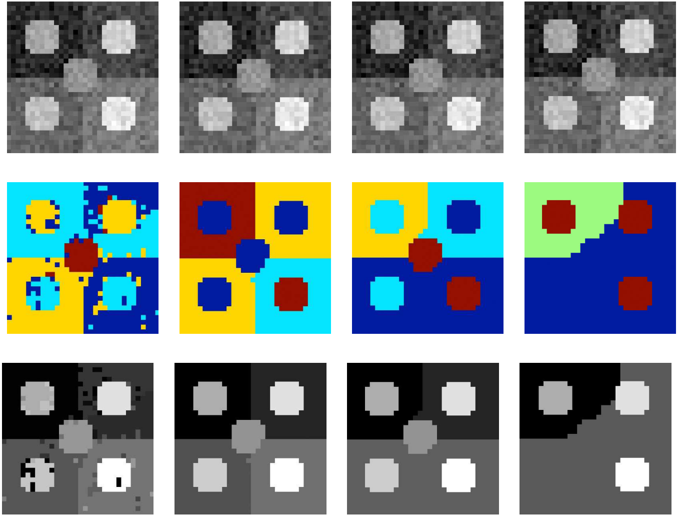


Figure: Influence of the regularization parameter α. First row is the original image. Second row is the four-color segmentation result. Third row is the piecewise constant approximation of the image. First column $\alpha = 1.5e5/255^2$, second column $\alpha = 3e4/255^2$, third column $\alpha = 1e4/255^2$, fourth column $\alpha = 1e3/255^2$.

Numerical Experiments

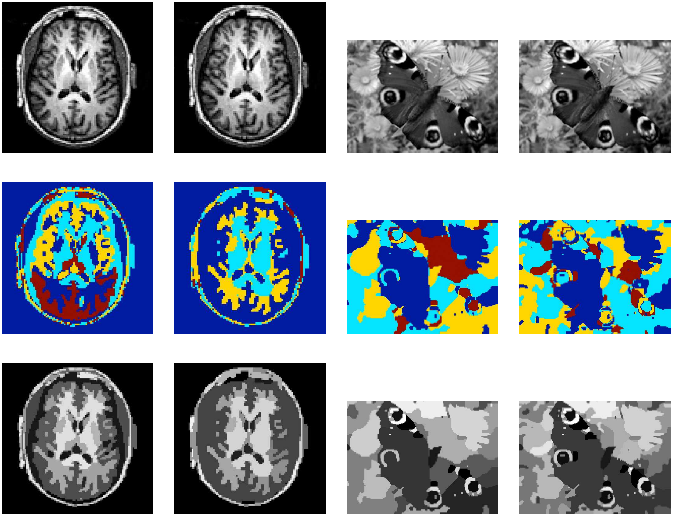


Figure: First and fourth rows present the original image. Second and fifth rows show the four-color segmentation result. Third and last rows display the piecewise constant approximation of the image s_0. Each column present a different value of α, which controls the number of final segmented regions.

Numerical Experiments

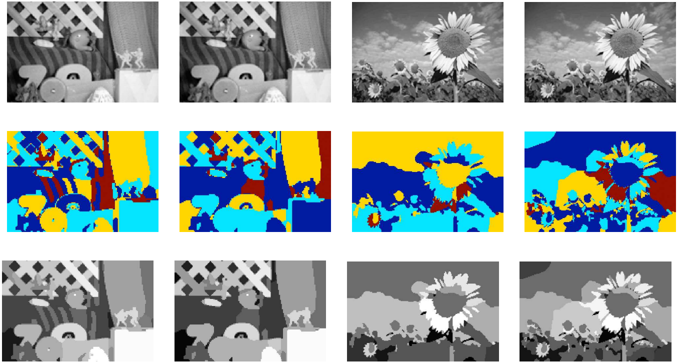


Figure: First and fourth rows present the original image. Second and fifth rows show the four-color segmentation result. Third and last rows display the piecewise constant approximation of the image s_0. Each column present a different value of α, which controls the number of final segmented regions.

Advantages

- Can automatically determine how many regions.

Advantages

- Can automatically determine how many regions.
- Only uses 4 labels.

Advantages

- Can automatically determine how many regions.
- Only uses 4 labels.
- Can take as many constants as is needed for a given regularization parameters.

Advantages

- Can automatically determine how many regions.
- Only uses 4 labels.
- Can take as many constants as is needed for a given regularization parameters.
- Regularization parameter controls how many regions and constants.

Advantages

- Can automatically determine how many regions.
- Only uses 4 labels.
- Can take as many constants as is needed for a given regularization parameters.
- Regularization parameter controls how many regions and constants.
- Fast to compute.

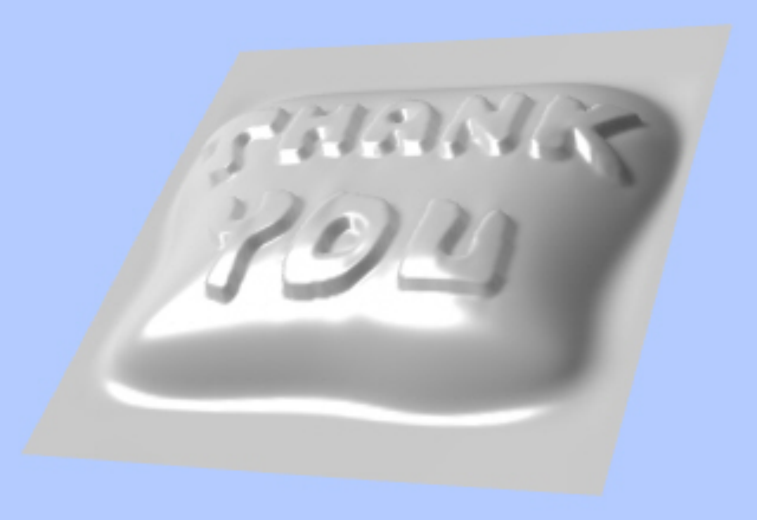
Advantages

- Can automatically determine how many regions.
- Only uses 4 labels.
- Can take as many constants as is needed for a given regularization parameters.
- Regularization parameter controls how many regions and constants.
- Fast to compute.

Historical overview

- Vese-Chan (2002, IJCV) has proposed to use 4-color theorem for the original Mumford-Shal model.
- Hodneland-Tai-Gerdes (2009, IJCV), watersehd+levelset+4color.
- Liu-Tao (2011, PR), Tao-Tai (UCLA-CAM-09-13).

Ref: Four color theorem and convex relaxation for image segmentation with any number of regions. Inverse Problems & Imaging 7 (3), 1099-1113.



Ref: Four color theorem and convex relaxation for image segmentation with any number of regions. Inverse Problems & Imaging 7 (3), 1099-1113.