UNIVERSITÄT ZU LÜBECK
INSTITUTE OF MATHEMATICS
AND IMAGE COMPUTING

Fraunhofer
MEVIS

Hyperelastic Image Registration with an Application to PET Reconstruction

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Outline

- Introduction to image registration
- Mathematical model: $\mathcal{D}[\mathcal{T}[y],\mathcal{R}] + \mathcal{S}[y] \xrightarrow{y} \min$
- A case study: Hyperelasticity and mass preservation
- Numerical analysis: Stabilizing the Hessian

Image Registration

$$\mathcal{D}[\mathcal{T}[y], \mathcal{R}] + \mathcal{S}[y] \xrightarrow{y} \min$$

Introduction

Mathematical Modelling

Image Registration

Given a reference image $\mathcal{R}$ and a template image $\mathcal{T}$, find a reasonable transformation y, such that the transformed image $\mathcal{T}[y]$ is similar to $\mathcal{R}$

reference $\mathcal{R}$ | $\mathcal{T}[y]$ | template $\mathcal{T}$

Mathematical Modelling

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Questions:

- Transformed image $\mathcal{T}[y]$? $\rightsquigarrow$ image model $\mathcal{T}[y]$
- Similarity of $\mathcal{T}[y]$ and $\mathcal{R}$? $\rightsquigarrow$ $\mathcal{D}[\mathcal{T}[y], \mathcal{R}]$
- Reasonability of y ? $\rightsquigarrow$ $\mathcal{S}[y]$
- Constraints on y ? $\rightsquigarrow$ $y \in \mathcal{A}$

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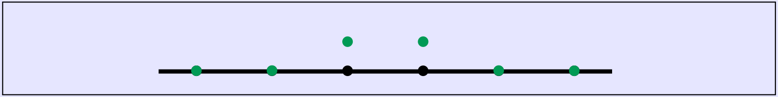
Image Registration: Variational Formulation

$$\mathcal{D}[\mathcal{T}[y], \mathcal{R}] + \mathcal{S}[y] \xrightarrow{y} \min, \qquad y \in \mathcal{A}$$

Simplified Image Registration Model

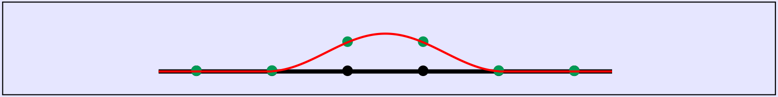
- Continuous model for images

Data and Transformation Model



- Given: discrete data $\mathtt{T}_i \in \mathbb{R}$ at locations $\mathtt{x}_i \in \Omega \subset \mathbb{R}^d$

Data and Transformation Model



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- Interpolation yields continuous model $\mathcal{T} : \Omega \subset \mathbb{R}^d \to \mathbb{R}$

$$\mathcal{T}(x) = \text{interpolation}(\mathtt{X}, \mathtt{T}, x)$$

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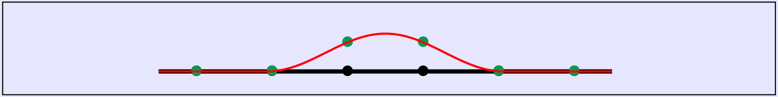

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- Transformed image (Eulerian framework)

$$\mathcal{T}[y](x) = \mathcal{T}(y(x)) = \text{interpolation}(\mathtt{X}, \mathtt{T}, y(x))$$

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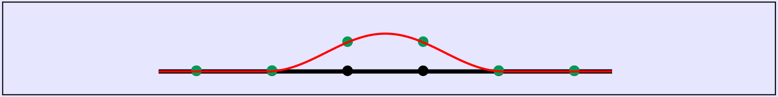
Hyperelastic Image Registration

Data and Transformation Model



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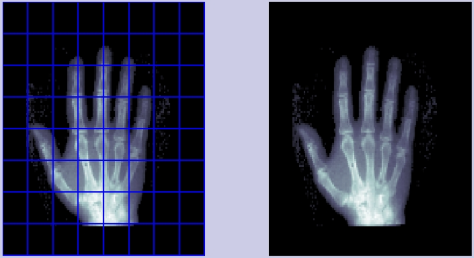
- Differentiability: analytic derivatives a.e.
- Multi-scale framework
- Multi-resolution framework

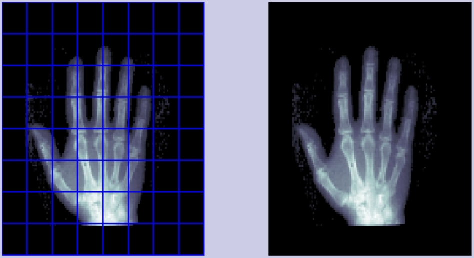
Transforming Images: Scaling

$$\mathcal{T}[y](x) = \mathcal{T}(y(x)) = \text{interpolation}(\mathtt{X}, \mathtt{T}, y(x))$$



Transforming Images: Non-linear

$$\mathcal{T}[y](x) = \mathcal{T}(y(x)) = \text{interpolation}(\mathtt{X}, \mathtt{T}, y(x))$$



Simplified Image Registration Model

- Continuous model for images, transformed image $\mathcal{T}[y]$

$$
\begin{aligned}
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\end{aligned}
$$

Simplified Image Registration Model

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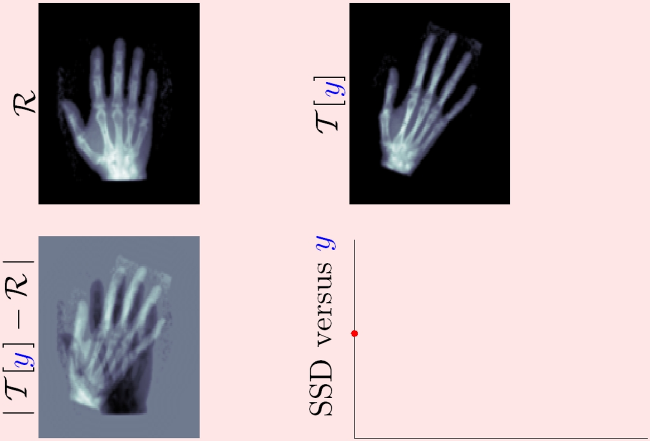
- Similarity of $\mathcal{T}[y]$ and $\mathcal{R}$, for example

$$\mathcal{D}^{\text{SSD}}[\,\mathcal{T}[y],\mathcal{R}\,] = \tfrac{1}{2}\int_\Omega [\,\mathcal{T}(y(x)) - \mathcal{R}(x)\,]^2\,dx,$$

Sum of Squared Differences



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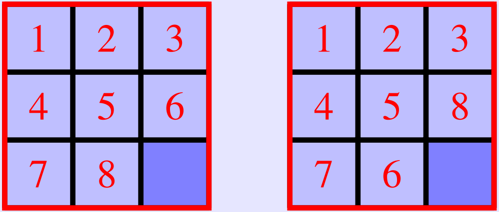
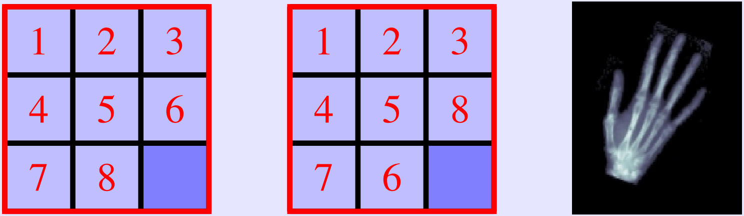
- Reasonability

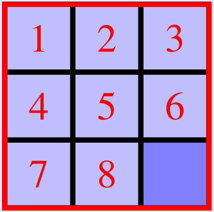
Reasonability of y

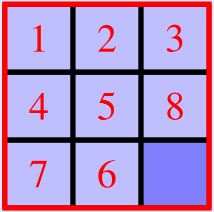


Reasonability of y



Reasonability of y





- Registration is severely ill-posed
- Restrictions onto the transformation y required
- Goal: explicit physical restrictions

Simplified Image Registration Model

- Continuous model for images, transformed image $\mathcal{T}[y]$

$$\begin{aligned} \mathcal{T}(x) &= \text{interpolation}(\mathtt{X}, \mathtt{T}, x) \\ \mathcal{T}[y](x) &= \mathcal{T}(y(x)) = \text{interpolation}(\mathtt{X}, \mathtt{T}, y(x)) \end{aligned}$$

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$$\mathcal{D}^{\text{SSD}}[\,\mathcal{T}[y], \mathcal{R}\,] = \frac{1}{2} \int_\Omega [\, \mathcal{T}(y(x)) - \mathcal{R}(x)\,]^2 \, dx,$$

- Reasonability $\rightsquigarrow$ Regularization, for example

$$\mathcal{S}^{\text{diff}}[\,y\,] = \int_\Omega \|\nabla y\|^2_{\text{Fro}} \; dx$$

Simplified Image Registration Model

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- Reasonability $\rightsquigarrow$ Regularization, for example

$$\mathcal{S}^{\text{diff}}[y] = \int_\Omega \|\nabla y\|_{\text{Fro}}^2 \, dx$$

- Objective: $\mathcal{D}[\mathcal{T}[y], \mathcal{R}] + \mathcal{S}[y] \xrightarrow{y} \min$

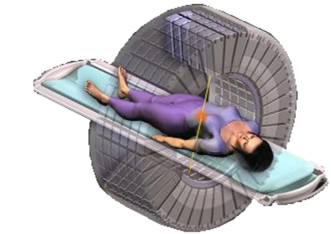
Hyperelasticity

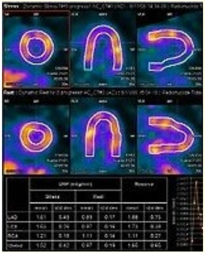
in Correspondence Problems

DFG Grant MO 1053/2-1

- Prof. Dr. **Martin Burger**
  Institute for Computational and Applied Mathematics,
  University of Münster
- Dr. **Lars Ruthotto**, PostDoc at UBC, Vancouver
- Dipl.-math. **Sebastian Suhr**, Lübeck and Münster
- Burger, Modersitzki, Ruthotto: *A hyperelastic regularization energy for image registration.* SIAM SISC, 35(1), 2013.

Motivation: PET Cardiac Imaging

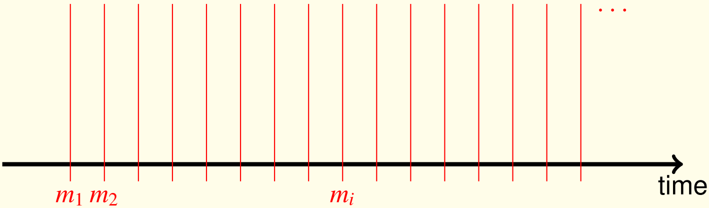




- http://www.siemens.com
- http://cardiacpetsolutions.com
- http://www.medical.siemens.com

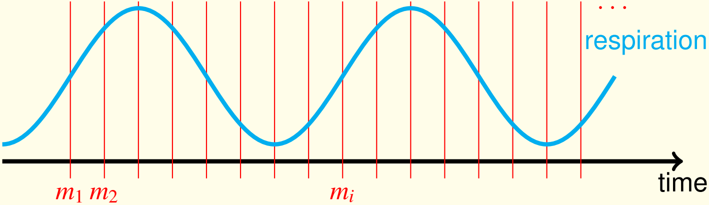
Goal: Produce the "best" 3D image

Data Acquisition



- measurement takes several minutes
- reconstruction: $\hat{I} = R(m_i,\ i \in M)$

Data Acquisition



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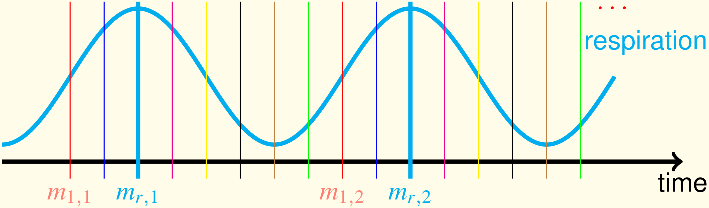
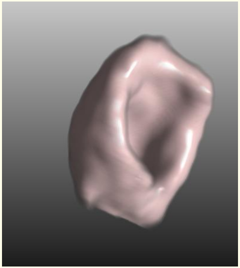
- respiratory challenge

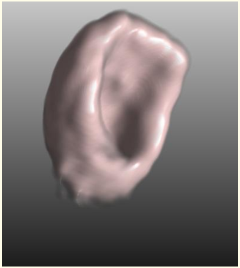
Data Acquisition



- measurement takes several minutes
- reconstruction: $\hat{I} = R(m_i,\ i \in M)$
- respiratory challenge resolved via gating
- sort m_i into B gates: $(m_{r,i},\ i \in M_r),\ r = 1, \ldots, B$
- B reconstructions: $I_r = R(m_{r,i},\ i \in M_r)$

Respiratory Challenge

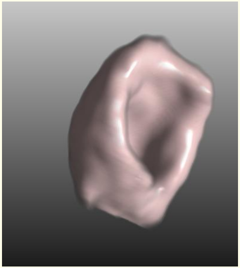


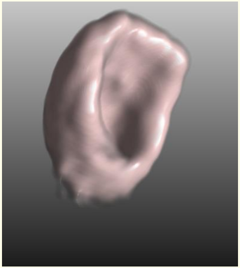


- gated images: $I_r = R(m_{r,i},\ i \in M_r)$

  PET cardiac images (human)
  European Institute for Molecular Imaging, Münster

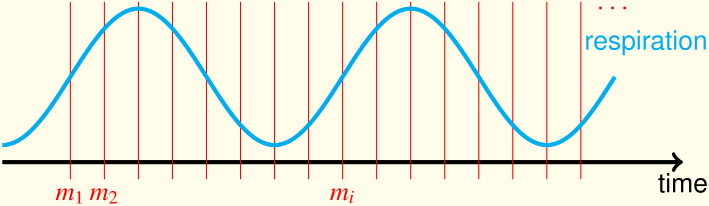
Respiratory Challenge





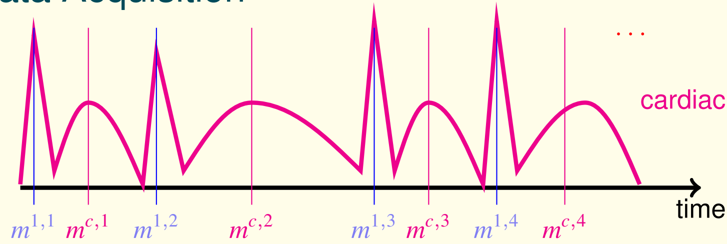
- gated images: $I_r = R(m_{r,i},\ i \in M_r)$
- compensates motion, compromises quality: fewer events per gate
- estimate transformations y_r: such that $I_0 \approx I_r \circ y_r$
- reconstruction: $\hat{I} = R(m_{r,i} \circ y_r,\ i \in M_r,\ \text{all } r\ )$

Data Acquisition



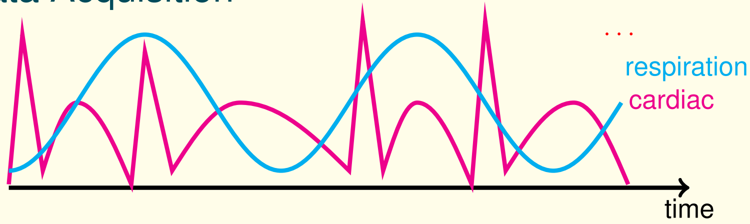
- measurement takes several minutes
- reconstruction: $\hat{I} = R(m_i,\ i \in M)$
- respiratory challenge $I_r = R(m_{r,i},\ i \in M_r)$

Data Acquisition



- measurement takes several minutes
- reconstruction: $\hat{I} = R(m_i,\ i \in M)$
- respiratory challenge $I_r = R(m_{r,i},\ i \in M_r)$
- cardiac challenge: $I^c = R(m^{c,i},\ i \in M^c)$

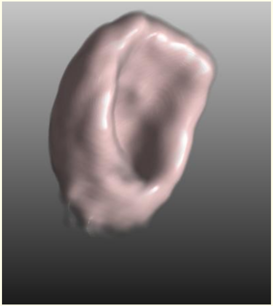
Data Acquisition

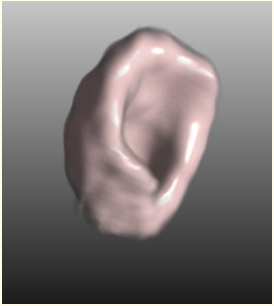


- measurement takes several minutes
- reconstruction: $\hat{I} = R(m_i,\ i \in M)$
- respiratory challenge $I_r = R(m_{r,i},\ i \in M_r)$
- cardiac challenge: $I^c = R(m^{c,i},\ i \in M^c)$
- overall goal: $I_r^c = R(m_{r,i}^{c,i},\ i \in M_r \cap\ M^c)$

Cardiac Challenge

respiratory



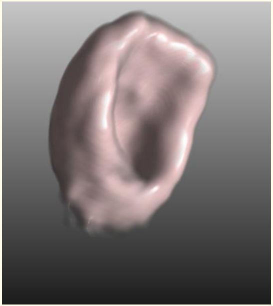


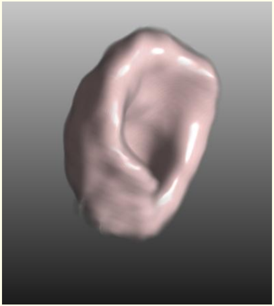
cardiac

- gated: $\color{cyan}{I_r} = R(\color{cyan}{m_{r,i}}, \ \color{cyan}{i} \in M_{\color{cyan}{r}}), \quad \color{magenta}{I^c} = R(\color{magenta}{m^{c,i}}, \ \color{magenta}{i} \in M^{\color{magenta}{c}})$

Cardiac Challenge

respiratory





cardiac

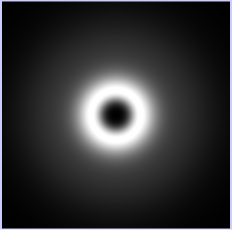
- gated: $I_r = R(m_{r,i},\ i \in M_r), \quad I^c = R(m^{c,i},\ i \in M^c)$
- y_r almost rigid
- y^c highly non-linear

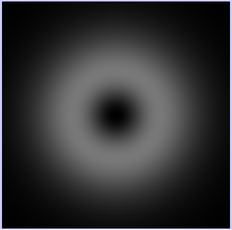
PET Image Registration

Registration of PET Data (simplified)

- Given images I^0 and I^c
- Find y, such that ideally $I^0(x) \approx I^c(y(x))$

$$J[y] = \int \left[I^0(x) - I^c(y(x))\right]^2 dx + \int \|\nabla y\|_{\text{Fro}}^2 dx$$





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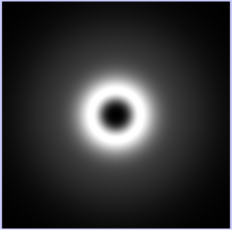

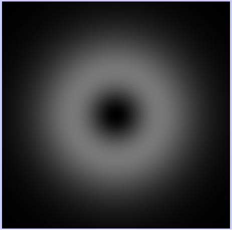

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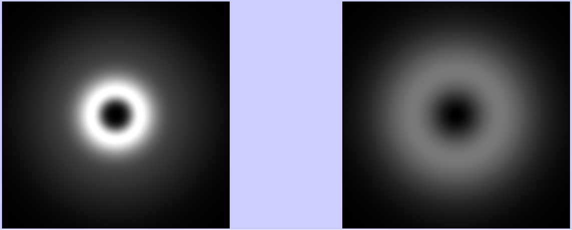
Hyperelastic Image Registration

Registration of PET Data (simplified)

- Given densities I^0 and I^c
- Find y, such that ideally $I^0(x) \approx I^c\big(y(x)\big) \cdot \det \nabla y$

$$J[y] = \int \left[I^0(x) - I^c(y(x)) \cdot \det \nabla y\right]^2 dx + \int \|\nabla y\|_{\text{Fro}}^2 \, dx$$



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- data-fit non-convex in ∇y, regularization insufficient,
- standard approach requires 6th order regularization

Registration of PET Data (simplified)

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- Find y, such that ideally $I^0(x) \approx I^c\big(y(x)\big) \cdot \det \nabla y$

$$
\begin{aligned}
J[y] = &\\
&\int \left[I^0(x) - I^c(y(x)) \cdot \det \nabla y\right]^2 dx\\
&+ \int \|\nabla y\|^2_{\mathrm{Fro}} + \varphi(\|\operatorname{cof} \nabla y\|^2_{\mathrm{Fro}}) + \psi(\det \nabla y)\ dx
\end{aligned}
$$

- data-fit non-convex in ∇y, regularization insufficient,
- standard approach requires 6th order regularization
- new approach involves gradient, cofactor, and determinant

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- Given densities I^0 and I^c
- Find y, such that ideally $I^0(x) \approx I^c\big(y(x)\big) \cdot \det \nabla y$

$$J[y, \nabla y, \operatorname{cof} \nabla y, \det \nabla y] = \int \left[I^0(x) - I^c(y(x)) \cdot \det \nabla y\right]^2 dx + \int \|\nabla y\|^2_{\mathrm{Fro}} + \varphi(\|\operatorname{cof} \nabla y\|^2_{\mathrm{Fro}}) + \psi(\det \nabla y)\, dx$$

- data-fit non-convex in ∇y, regularization insufficient,
- standard approach requires 6th order regularization
- new approach involves gradient, cofactor, and determinant
- poly-convex, convex in ∇y, $\operatorname{cof} \nabla y$, and $\det \nabla y$

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- standard approach requires 6th order regularization
- new approach involves gradient, cofactor, and determinant
- poly-convex, convex in ∇y, $\operatorname{cof} \nabla y$, and $\det \nabla y$
  $\rightsquigarrow$ hyperelasticity, non-linear elasticity model

Regularization

$$\mathcal{S}[\nabla y, \operatorname{cof} \nabla y, \det \nabla y] = \int \|\nabla y\|_{\mathrm{Fro}}^2 + \varphi(\|\operatorname{cof} \nabla y\|_{\mathrm{Fro}}^2) + \psi(\det \nabla y)\, dx$$

- $\|\nabla y\|_{\mathrm{Fro}}^2$ controls lengths

$$\nabla y = \begin{pmatrix} \partial_1 y^1 & \partial_2 y^1 & \partial_3 y^1 \\ \partial_1 y^2 & \partial_2 y^2 & \partial_3 y^2 \\ \partial_1 y^3 & \partial_2 y^3 & \partial_3 y^3 \end{pmatrix}$$

- $\operatorname{cof} \nabla y$ controls areas

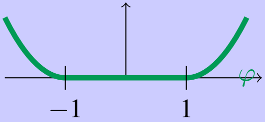
$$\begin{pmatrix} \partial_2 y^2 \partial_3 y^3 - \partial_3 y^2 \partial_2 y^3 & \partial_1 y^2 \partial_3 y^3 - \partial_3 y^2 \partial_1 y^3 & \partial_1 y^2 \partial_2 y^3 - \partial_2 y^2 \partial_1 y^3 \\ \partial_3 y^1 \partial_2 y^3 - \partial_2 y^1 \partial_3 y^3 & \partial_3 y^1 \partial_1 y^3 - \partial_1 y^1 \partial_3 y^3 & \partial_2 y^1 \partial_1 y^3 - \partial_1 y^1 \partial_2 y^3 \\ \partial_2 y^1 \partial_3 y^2 - \partial_2 y^2 \partial_3 y^2 & \partial_3 y^1 \partial_1 y^2 - \partial_1 y^1 \partial_3 y^1 & \partial_1 y^1 \partial_2 y^2 - \partial_2 y^1 \partial_1 y^2 \end{pmatrix}$$

- $\det \nabla y$ controls volumes

$$\begin{aligned} \det \nabla y \quad = \quad & \partial_1 y^1 \partial_2 y^2 \partial_3 y^3 + \partial_2 y^1 \partial_3 y^2 \partial_1 y^3 + \partial_3 y^1 \partial_1 y^2 \partial_2 y^3 \\ & - \partial_1 y^1 \partial_3 y^2 \partial_2 y^3 - \partial_2 y^1 \partial_1 y^2 \partial_3 y^3 - \partial_3 y^1 \partial_2 y^2 \partial_1 y^3 \end{aligned}$$

Penalties $\varphi, \psi : \mathbb{R} \to [0, \infty]$

- $C := \operatorname{cof} \nabla y$
- $\varphi(\pm\infty) = \infty$
- $\varphi(C) = \sum_{j=1}^{3} \max\{\|C_{:,j}\|^2 - 1, 0\}$



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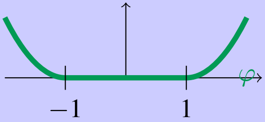
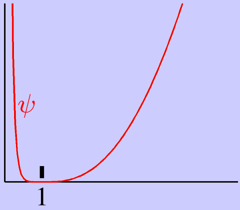


- $v = \det \nabla y$
- $\psi(-|v|) = \infty$
- $\psi(\infty) = \infty$
- $\psi(v) = \psi(1/v)$
- $\psi(v) = (v-1)^4/v^2$
- enforces diffeomorphism



Hyperelasticity: energy depends on ∇u

- displacement u, $\quad y(x) = x + u(x) \quad \Rightarrow \quad \nabla y = I_d + \nabla u$
- Cauchy strain tensor: $V = V(y) = \nabla u + \nabla u^\top$, for $\|\nabla u\| \ll 1$

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Material constants, Lamé constants ν and μ

- linear elasticity: $\mathcal{S}^{\text{elas}}[y] = \int \nu\ (\text{trace} V)^2 + \mu\ \text{trace}(V^2)\ dx$

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Hyperelasticity: Ogden Materials

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Ogden materials

$$\begin{aligned} \mathcal{S}^{\text{Ogden}}[y] &= \int \|\nabla y\|^2 + \varphi_O(\|\text{cof}\ \nabla y\|^2_{\text{Fro}}) + \psi_O(\det \nabla y) dx \\ &= \mathcal{S}^{\text{quad}}[y] + \mathcal{O}(\|\nabla y\|^3) \end{aligned}$$

- $\varphi_O(s) = s, \qquad \psi_O(v) = v^2 - \log v$

Hyperelasticity: Ogden Materials

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Ogden materials

$$\mathcal{S}^{\text{hyper}}[y] = \int \|\nabla y\|^2 + \varphi\,(\|\text{cof}\,\nabla y\|^2_{\text{Fro}}) + \psi\,(\det \nabla y)dx$$

- $\varphi_O(s) = s, \qquad \psi_O(v) = v^2 - \log v$
- $\varphi(s) = (s-3)^2, \qquad \psi(v) = (v-1)^4/v^2 = \psi(1/v)$

Hyperelasticity: Extremal Stress

Ogden materials

$$\mathcal{S}^{\text{hyper}}[y] \quad = \quad \int \|\nabla y\|^2 + \varphi(\|\text{cof}\,\nabla y\|^2_{\text{Fro}}) + \psi(\det \nabla y)dx$$

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Ogden materials

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extremal stress and coercivity

$$\mathcal{S}^{\text{Ogden/hyper}}[y] \longrightarrow \infty \quad \text{for} \quad \det \nabla y \to 0,$$

$$\mathcal{S}^{\text{Ogden/hyper}}[y] \geq c_1\{\|\nabla y\|^p + \|\text{cof } \nabla y\|^q + (\det \nabla y)^r\} + c_2,$$

- price: $\mathcal{S}^{\text{Ogden/hyper}}$ non-convex in ∇y but poly-convex

Existence of Minimizer

$$\begin{aligned} \mathcal{A}_0 &:= \{y \in W^{1,2}(\Omega, \mathbb{R}^3) : \\ &\qquad \operatorname{cof} \nabla y \in L_4(\Omega, \mathbb{R}^{3\times 3}),\ \det \nabla y \in L_2(\Omega, \mathbb{R}),\ \det \nabla y > 0\ a.e.\ \} \\ \mathcal{A} &:= \left\{y \in \mathcal{A} :\ \left|\int y(x)\, dx\right| \le |\Omega|\ (M + \operatorname{diam}(\Omega))\right\} \end{aligned}$$

Theorem (Burger, Modersitzki, Ruthotto 2013)

Given are images $\mathcal{R}, \mathcal{T} \in C(\Omega, \mathbb{R})$, a polyconvex distance measure $\mathcal{D} = \mathcal{D}[y] = \mathcal{D}[\mathcal{T}, \mathcal{R}; y, \nabla y, \det \nabla y]$ with $\mathcal{D} \ge 0$, $\mathcal{S}^{\text{hyper}}$ the hyperelastic regularizer with convex penalties φ and ψ, the feasible set $\mathcal{A}$.
We assume that the registration functional $\mathcal{J} = \mathcal{D} + \mathcal{S}$ satisfies $\mathcal{J}[\mathrm{Id}] < \infty$ for $\mathrm{Id}(x) := x$ on Ω.

Then there exists at least one minimizer $y^* \in \mathcal{A}$ of $\mathcal{J}$.

Remarks on Proof:

- Problem: $\nabla y \mapsto \mathcal{J}[y, \nabla y]$ is non-convex
- Splitting:

$$\{y^k\} \rightsquigarrow \{(y^k, \operatorname{cof} \nabla y^k, \det \nabla y^k)\} \subset X = W^{1,2} \times L^4 \times L^2$$

- Coercivity: $\exists\, C > 0, K \in \mathbb{R}$ such that

$$\forall y \in \mathcal{A}: \;\; \mathcal{J}[y] \geq C\|y\|_X + K$$

- Lower semi-continuity:

$$(y^k, \operatorname{cof} \nabla y^k, \det \nabla y^k) \rightharpoonup (y, H, v)$$
$$\Rightarrow \liminf_k \mathcal{J}[y^k, \operatorname{cof} \nabla y^k, \det \nabla y^k] \geq \mathcal{J}[y, H, v]$$

- Existence of minimizing sequence in X

$$(y^k, \operatorname{cof} \nabla y^k, \det \nabla y^k) \rightarrow (y, H, v)$$

- Undo splitting: Weak continuity of cof and $\det$ implies $H = \operatorname{cof} \nabla y$ and $v = \det \nabla y$
- Verify that $\det \nabla y > 0$ a.e.

Numerical Scheme

- Discretize then optimize $\rightsquigarrow$ nodal discretization
- Multi-level approach
- Gauss-Newton
- Armijo line search with backtracking $\rightsquigarrow$ ensures $\det \nabla y > 0$
- Conjugate gradient for linear systems

Discretization

Measuring Volumes

$$\mathrm{vol}(V) = \int_V dx$$

$$\mathrm{vol}(y(V)) = \int_{y(V)} dx = \int_V \det(\nabla y)\, dx$$

Measuring Volumes

$$\mathrm{vol}(V) = \int_V dx$$

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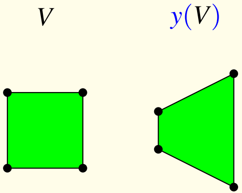

Measuring Volumes

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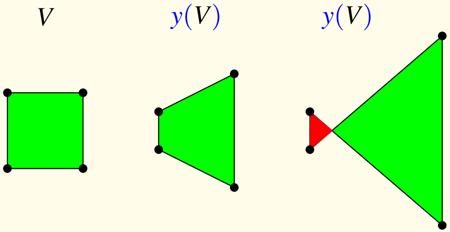

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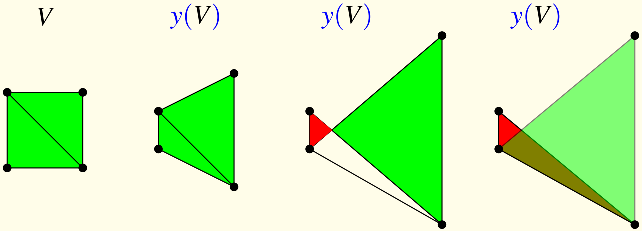

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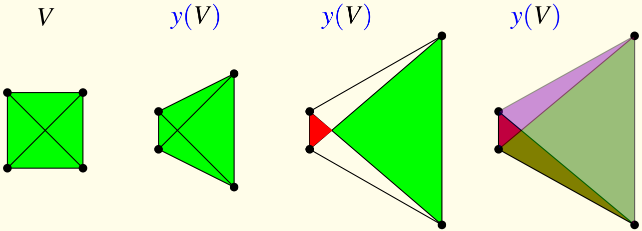

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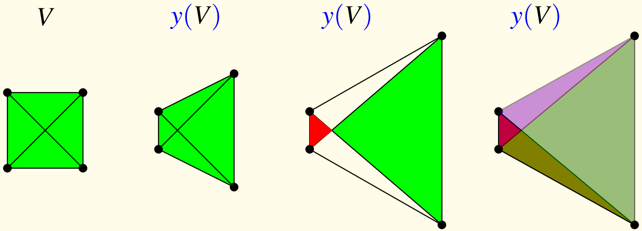
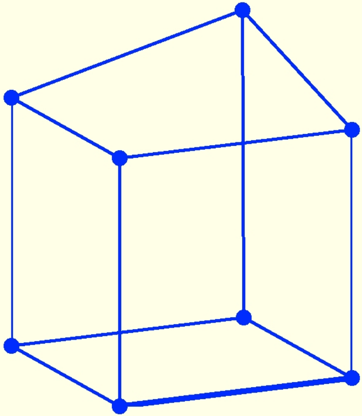
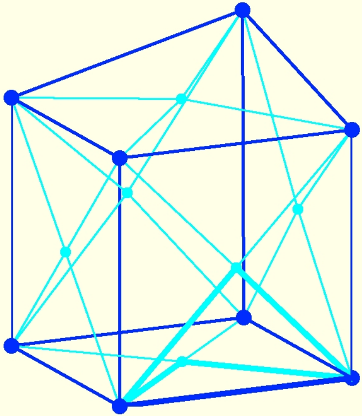


y continuous, piecewise linear on triangles/tetrahedron

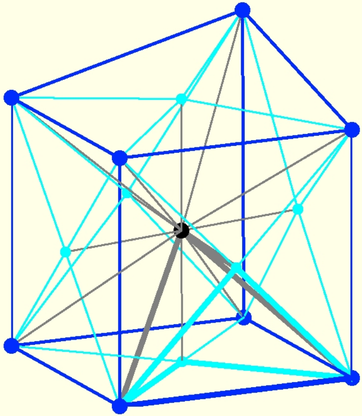
Voxel-based Discretization, Model



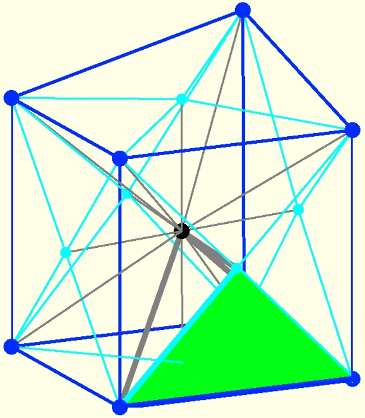
Voxel-based Discretization, Model



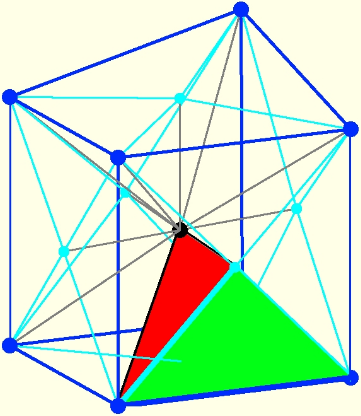
Voxel-based Discretization, Model



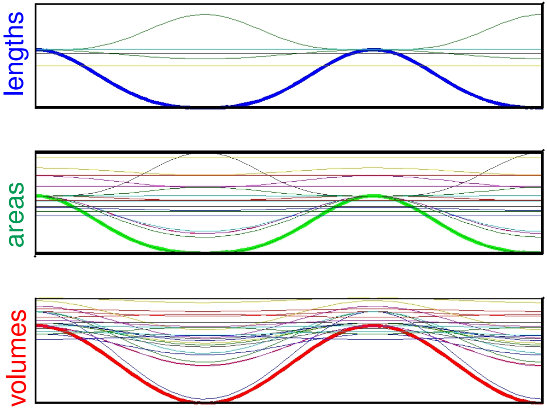
Voxel-based Discretization, Model

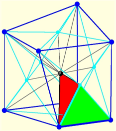


Voxel-based Discretization, Deformation



Voxel-based Discretization, Controls





Properties of the Discretization

Theorem (Burger, Modersitzki, Ruthotto 2013)

Let V *be a voxel and* $\{T_j,\ j \in J\}$ *be a tetrahedral partition of* V *with* $\mathrm{vol}(T_j) > 0$ *for all* $j \in J$. *Let* $y : \bar{\Omega} \to \mathbb{R}^3$ *be a vector field such that* $y\big|_{T_j}$ *is linear. It holds*

$$\det \nabla y\big|_V > 0 \quad \text{a.e.} \qquad \Longleftrightarrow \qquad \forall\, j \in J : \quad \mathrm{vol}(y(T_j)) > 0.$$

Results

Results

- Hyperelasticity makes a difference
- Mass-preservation makes a difference
- Cardiac motion compensation

  3D PET images
  European Institute for Molecular Imaging
  Münster, Germany

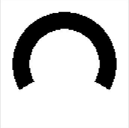
Jan Modersitzki

Elasticity versus Hyperelasticity

data

template

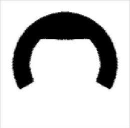
reference



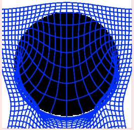
$\det(\nabla y)$	min	max
elas	-1.2	19.4

elastic

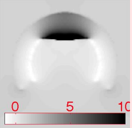
$T(y)$



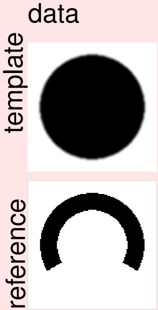
T+grid

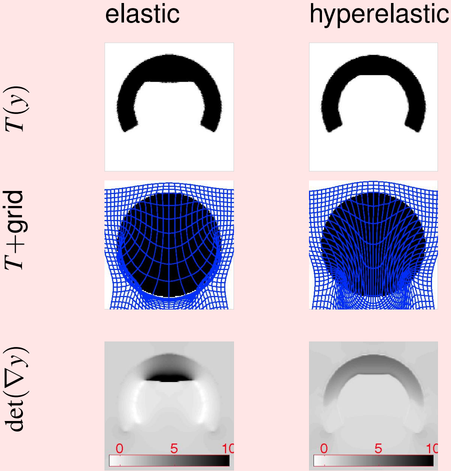


$\det(\nabla y)$



Elasticity versus Hyperelasticity





$\det(\nabla y)$	min	max
elas	−1.2	19.4
hyper	0.4	5.5

Plain versus Mass-Preservation

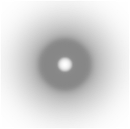
data

template

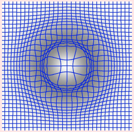
reference

plain

$T(y)$

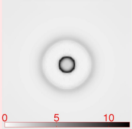


T+grid

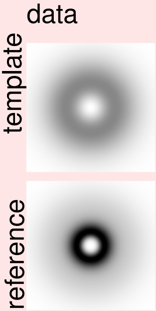


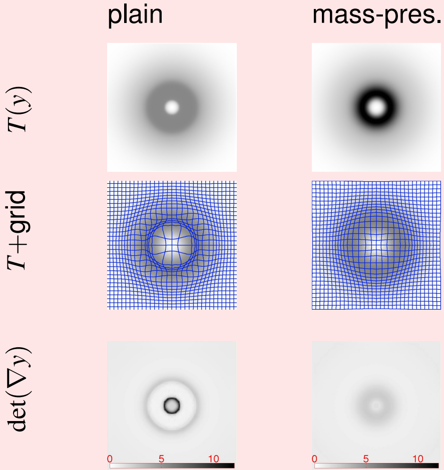
$\det(\nabla y)$	min	max
plain	0.44	11.00

$\det(\nabla y)$



Plain versus Mass-Preservation

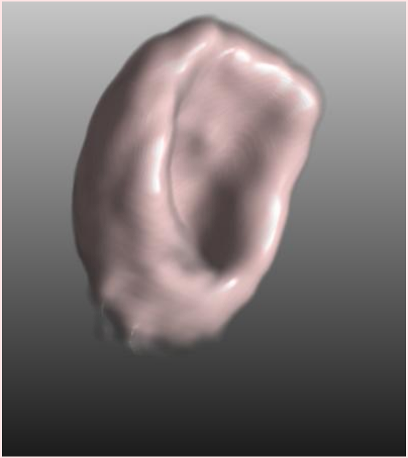




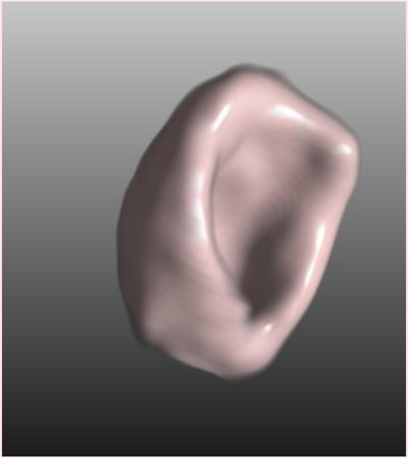
$\det(\nabla y)$	min	max
plain	0.44	11.00
MP	0.75	2.45

Respiratory Motion Compensation

gated images

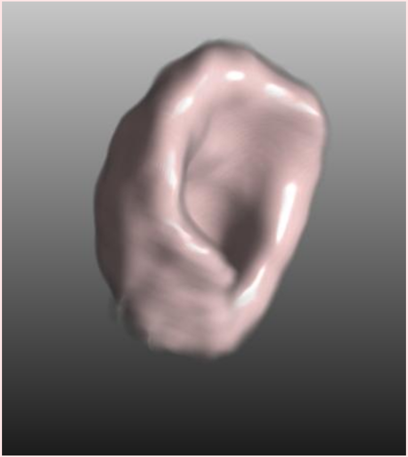


motion compensated

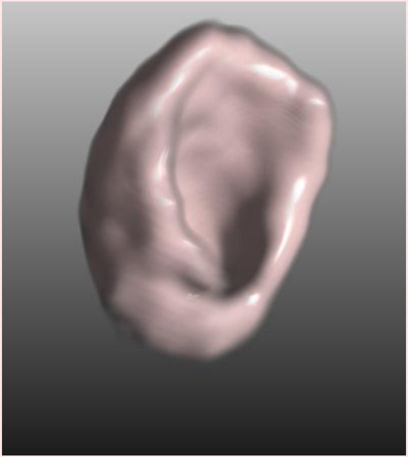


Cardiac Motion Compensation

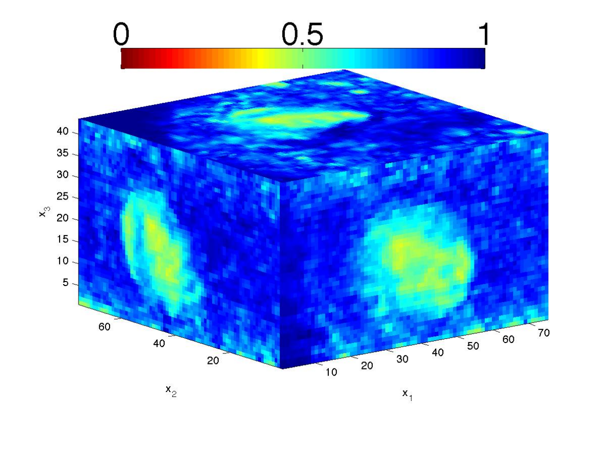
gated images



motion compensated

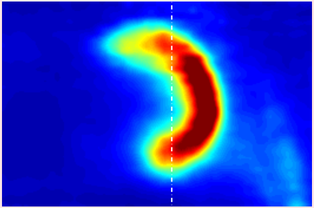


Minimum Intensity Projection of $\det(\nabla y)$



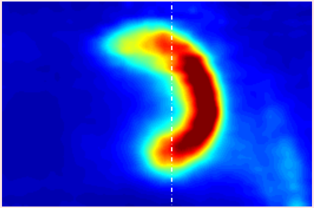
The “Best” Image

no gating

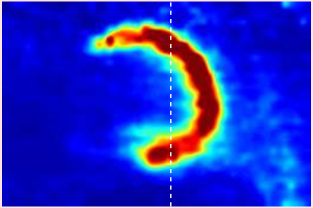


The “Best” Image

no gating

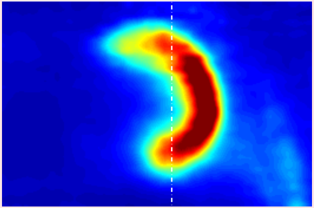


single gate

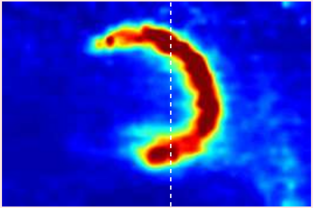


The “Best” Image

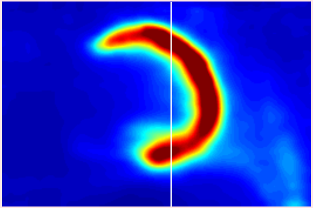
no gating



single gate

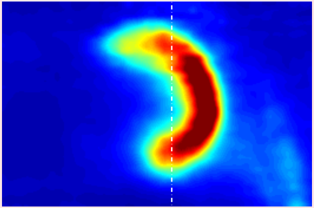


VAMPIRE

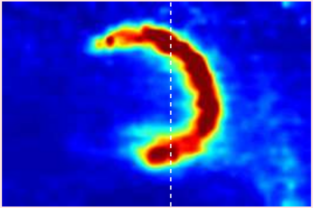


The “Best” Image

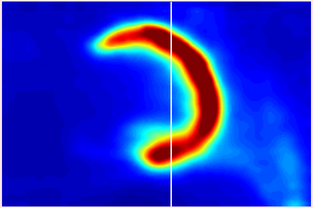
no gating



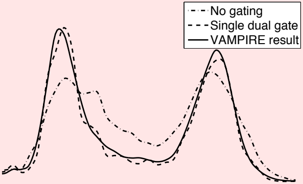
single gate



VAMPIRE



profiles

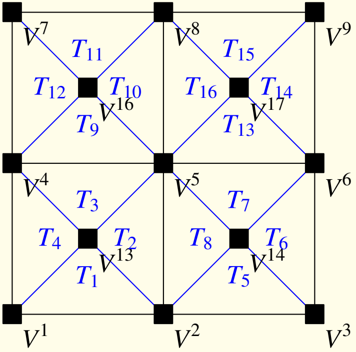


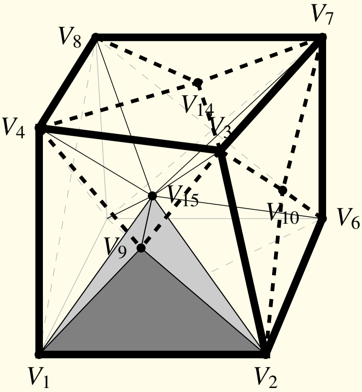
Multigrid

Numerical Analysis

- Prof. Dr. **Chen Greif**
  Department of Computer Science, UBC, Vancouver
- Prof Dr. **Lars Ruthotto**, Emory University, Atlanta (starting 9-2014)
- Ruthotto, Greif, Modersitzki: *A Multigrid Solver for Hyperelastic Image Registration.* SIAM SISC, under revision.

Discretization: Meshes in 2D and 3D





Discretization of Data Fit I

FE spaces: vertices $V^1, \dots, V^{n_V}$, tetrahedra $T_1, \dots, T_{n_T}$

$$\mathcal{A}^h = \left\{ y \in \mathcal{C}(\Omega, \mathbb{R}^3) : \; y\big|_{T_i} \in \Pi^1(T_i, \mathbb{R}^3) \text{ for } i = 1, \dots, n_T \right\} \subset \mathcal{A},$$

Nodal Lagrange hat-functions: b^j

$$y^{\mathcal{A},h}(x) = B(x)y = \sum \eta_j b^j(x), \quad \eta_j = y^{\mathcal{A},h}(V^j) \in \mathbb{R}^3$$

Gradient: $Gy = \nabla B(x) y \in \mathbb{R}^{9n_T}$, constant on T_i

$$G = I_3 \otimes \begin{pmatrix} \partial_1^h \\ \partial_2^h \\ \partial_3^h \end{pmatrix}, \quad \partial_k^h \in \mathbb{R}^{n_T, n_V} \text{ with } (\partial_k^h)_{i,j} = \partial_k b^i(V^j)$$

Discretization of Data Fit II

Averaging: $A = A_T^V = I_3 \otimes A \in \mathbb{R}^{n_T, n_V}$

$$(A)_{i,j} = \begin{cases} 1/4, & \text{if } V^j \text{ is node of } T_i \\ 0 & \text{otherwise} \end{cases}$$

Volume: $v_i = \mathrm{vol}(T_i)$, $v = (v_i) \in \mathbb{R}^{n_T}$, $V = \mathrm{diag}(v)$

Data fit:

$$\mathcal{D}[y^{\mathcal{A},h}] = 0.5\, \|\mathcal{T} \circ y^{\mathcal{A},h} - R\|^2$$

$$D(y) = 0.5\, \mathrm{res}(y)^\top V\, \mathrm{res}(y), \quad \mathrm{res}(y) = \mathcal{T}(Ay) - \mathcal{R}(x)$$

$$dD(y) = \mathrm{res}(y)^\top\, V\, (\nabla\mathcal{T}(Ay))\, A$$

$$d^2D(y) \overset{\mathrm{GN}}{\approx} A^\top (\nabla\mathcal{T}(Ay))^\top\, V\, (\nabla\mathcal{T}(Ay))\, A$$

Discretization of Data Fit III

Hyperelasticity:

$$\mathcal{S}[y^{\mathcal{A},h}] = \mathcal{S}^{\text{length}}[y^{\mathcal{A},h}] + \mathcal{S}^{\text{area}}[y^{\mathcal{A},h}] + \mathcal{S}^{\text{volume}}[y^{\mathcal{A},h}]$$

Discretization of Hyperelasticity I

Length:

$$\mathcal{S}^{\text{length}}[y^{\mathcal{A},h}] = \frac{\alpha}{2}\,\|\nabla(y^{\mathcal{A},h} - y_{\text{ref}}^{\mathcal{A},h})\|^2$$

$$S^{\text{length}}(y) = \frac{\alpha}{2}\,(y - y_{\text{ref}})^\top\, G^\top (I_9 \otimes V) G\,(y - y_{\text{ref}})$$

$$dS^{\text{length}}(y) = \alpha\,(y - y_{\text{ref}})^\top\, G^\top (I_9 \otimes V) G$$

$$d^2 S^{\text{length}}(y) = \alpha\, G^\top (I_9 \otimes V) G$$

Discretization of Hyperelasticity II

Area: $D_i^j = \operatorname{diag}(\partial_i^h y^j) \in \mathbb{R}^{n_T, n_T}$

$$\mathcal{S}^{\text{area}}[y^{\mathcal{A},h}] = \int \varphi(\operatorname{cof} \nabla y^{\mathcal{A},h}) dx$$

$$S^{\text{area}}(y) = v^\top \varphi(\operatorname{cof} Gy)$$

$$dS^{\text{area}}(y) = (I_9 \otimes V)\, \varphi'(\operatorname{cof} Gy)^\top\ d \operatorname{cof} Gy$$

$$d^2 S^{\text{area}}(y) \overset{\text{GN}}{\approx} (d \operatorname{cof} Gy)^\top ((I_9 \otimes V)\varphi''(\operatorname{cof} Gy))\ d \operatorname{cof} Gy$$

$$d \operatorname{cof} Gy = \begin{pmatrix} & & & & D_3^3 & -D_2^3 & & -D_3^2 & D_2^2 \\ & & & D_3^3 & & -D_2^3 & & -D_3^1 & D_1^2 \\ & & & D_2^3 & -D_1^3 & & -D_2^2 & D_1^2 & \\ & D_3^3 & -D_2^3 & & & & & -D_3^1 & D_2^1 \\ D_3^3 & & -D_1^3 & & & & -D_3^1 & & D_1^1 \\ D_2^3 & -D_1^3 & & & & & -D_1^2 & D_1^1 & \\ & D_3^2 & -D_2^2 & & -D_3^1 & D_2^1 & & & \\ D_3^2 & & -D_1^2 & -D_3^1 & & D_1^1 & & & \\ D_2^2 & -D_1^2 & & D_2^1 & D_1^1 & & & & \end{pmatrix} \quad G \in \mathbb{R}^{9n_T \times 3n_V}.$$

Discretization of Hyperelasticity III

Volume:

$$\mathcal{S}^{\text{volume}}[y^{\mathcal{A},h}] = \int \psi(\det \nabla y^{\mathcal{A},h})dx$$

$$S^{\text{volume}}(y) = v^\top \psi(\det Gy)$$

$$dS^{\text{volume}}(y) = v^\top \ \psi'(\det Gy)^\top \ d \det Gy$$

$$d^2S^{\text{volume}}(y) \overset{\text{GN}}{\approx} (d \det Gy)^\top \operatorname{diag}(V \ \psi''(\det Gy)) \ d \det Gy$$

$$d \det Gy = (C_1^1, C_1^2, C_1^3, C_2^1, C_2^2, C_2^3, C_3^1, C_3^2, C_3^3,) \ G \in \mathbb{R}^{n_T, 3n_V},$$

$$C_i^j = \operatorname{diag}((\operatorname{cof} Gy)_{i,j}) \in \mathbb{R}^{n_T, n_T}.$$

Remarks on Discretization

- Exact gradients
- Gauss-Newton approximation of Hessians
- Main problem:

$$\psi''(\det Gy) \longrightarrow \infty \quad \text{for} \quad \det Gy \longrightarrow \infty \quad \text{or} \quad \det Gy \longrightarrow 0$$

conditioning?, h-ellipticity?

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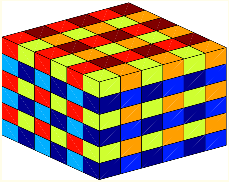
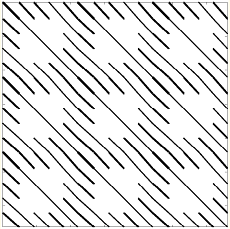
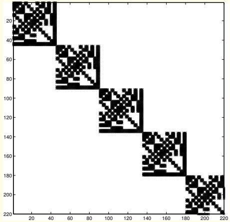
- Main idea: stabilization, change approximation of Hessian to

$$\varphi_s''(v) := \min\left\{\psi''(v),\ s\alpha_1/\alpha_3\right\}$$

Multigrid, Block-Vanka Smoother







- cell-wise reordering of unknowns results block matrix with dense blocks
- five/fifteen tetrahedral nodes per pixel/voxel for two/three components result in 10×10 / 45×45 blocks for 2D/3D
- Gauss-Seidel relaxation sweep with damping $\omega = 2/3$

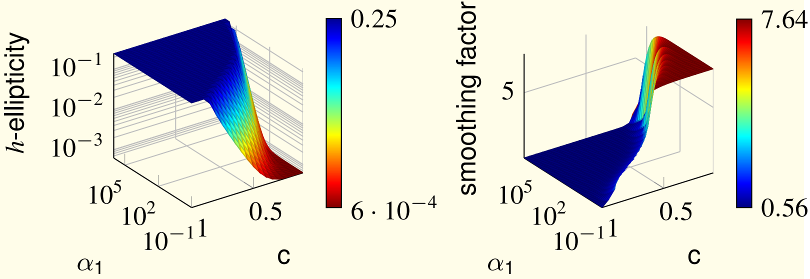
Multigrid, h-Ellipticity

- h-ellipticity, measure for sensitivity of Hessian to high frequencies; ideal: bounded away from zero
- toy example: $y(x) = cx$, c contraction

$$\mathcal{S}[y] = \alpha_1 \mathcal{S}^{\text{length}}[y] + \alpha_2 \mathcal{S}^{\text{area}}[y] + \alpha_3 \mathcal{S}^{\text{volume}}[y]$$

Jan Modersitzki

Local Fourier and Smoothing Analysis, h-Ellipticity



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- larger weight on length term, i.e. α_1 large, implies h-ellipticity larger

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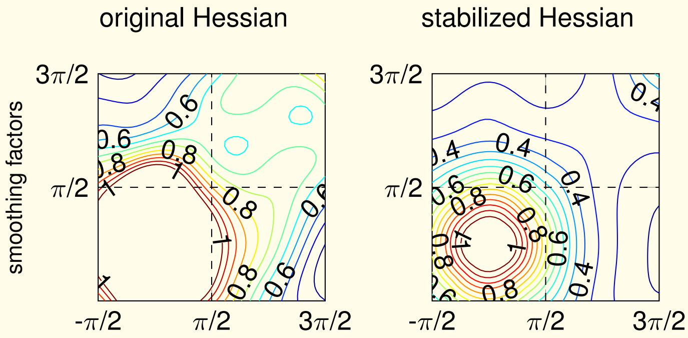
- main idea: stabilization, change approximation of Hessian

$$\varphi_s''(v) := \min\left\{\psi''(v), s\alpha_1/\alpha_3\right\}$$

degrades quality of approximation only for volume term; more outer iterations might be expected

Impact of Stabilization



Convergence history

	GNiter	Jacobi-CG J	Jacobi-CG #iter	Jacobi-CG relres	MG-CG J	MG-CG #iter	MG-CG relres	stabilized MG-CG J	stabilized MG-CG #iter	stabilized MG-CG relres
level-4	-1	8.7e7			8.7e7			8.7e7		
	0	1.3e7			1.3e7			1.3e7		
	1	6.1e6	41	9.94e-3	6.1e6	2	8.93e-3	6.1e6	2	8.93e-3
	2	5.4e6	86	7.77e-3	5.4e6	3	7.80e-3	5.4e6	2	7.79e-3
	3	5.3e6	71	9.97e-3	5.3e6	3	7.41e-3	5.3e6	2	8.95e-3
	4	5.2e6	62	9.58e-3	5.2e6	3	1.31e-3	5.2e6	2	8.65e-3
level-5	-1	1.2e8			1.2e8			1.2e8		
	0	1.0e7			1.0e7			1.0e7		
	1	8.0e6	37	9.49e-3	8.0e6	2	5.38e-3	8.0e6	2	5.37e-03
	2	6.8e6	56	8.49e-3	6.8e6	3	5.61e-3	7.2e6	2	6.35e-03
	3	6.6e6	70	9.52e-3	6.6e6	4	8.45e-3	6.9e6	2	7.63e-03
	4	6.5e6	102	9.65e-3	6.5e6	7	9.47e-3	6.6e6	2	9.00e-03
	5							6.5e6	3	1.65e-03
level-6	-1	1.5e8			1.5e8			1.5e8		
	0	9.5e6			9.5e6			9.5e6		
	1	8.3e6	51	9.49e-3	8.3e6	2	7.79e-3	8.3e6	2	6.41e-03
	2	8.0e6	96	9.98e-3	8.0e6	3	7.55e-3	7.9e6	2	8.07e-03
	4	8.0e6	96	9.88e-3	8.0e6	3	6.05e-3	7.8e6	3	2.43e-03

Summary

Conclusions

- Introduction to image registration
- Case study: motion compensation in PET cardiac imaging
  - Mass preservation
  - Hyperelasticity
- Numerical analysis:
  - Multigrid scheme
  - Ill-conditioned for $\det \nabla y \longrightarrow 0$
  - Stabilizing the Hessian