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Imaging with Kantorovich-Rubinstein discrepancy

Dirk Lorenz joint work with Jan Lellmann, Carola Schönlieb and Tuomo Valkonen,
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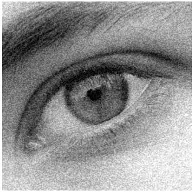
Institut für Analysis und Algebra

- **Image denoising in measure space**
- **Kantorovich-Rubinstein norms**
- KR**-**TV **denoising**
- **Examples**

- **Image denoising in measure space**
- **Kantorovich-Rubinstein norms**
- **KR-TV denoising**
- **Examples**

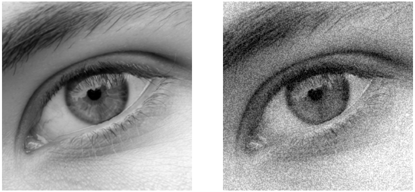
$$\mathrm{TV}(u) = \sup\{\textstyle\int u \operatorname{div}(\phi)\,\mathrm{d}x \;:\; \||\phi|\|_\infty \leq 1\} = \int |\nabla u|\,\mathrm{d}x$$

- Rudin-Osher-Fatemi, 1992 (ROF)

$$\min_u \int_\Omega (u - u^0)^2\,\mathrm{d}x + \lambda \mathrm{TV}(u)$$

- Chan-Esedoglu, 2005 (L^1-TV)

$$\min_u \int_\Omega |u - u^0|\,\mathrm{d}x + \lambda \mathrm{TV}(u)$$



ROF

L^1-TV

ROF L^1-TV

Contrast loss, staircasing, small things disappear...
Who’s to blame for this?

Model images as densities–i.e as measures

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  $(g \in L^1(\Omega) \implies g \in \mathfrak{M}(\Omega), \langle g, f \rangle = \int f \, g \, dx)$

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- Distances for measures: “Wasserstein distances”

$$W_p(\mu,\nu) = \Big(\inf_{\mu,\nu \text{ marginales of } \gamma} \int_{\Omega\times\Omega} |x-y|^p \,\mathrm{d}\gamma(x,y)\Big)^{1/p}$$

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[Evans/Gangbo: Monge–Kantorovich–Rubinstein–Wasserstein–etc. metrics]

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- Optimal transport [Monge 1781]: if γ supported on graph of function $T : \Omega \to \Omega$, T transports μ to ν

$$\mu(A) = \nu(T^{-1}(A))$$

Applications of optimal transport in imaging

- Image registration and warping with W_2 distance [Haker, Tannenbaum 2004]
- Segmentation with W_1 distance for histograms [Ni, Bresson, Chan, Esedoglu 2009]
- Density estimation with W_2 discepancy [Burger, Franek, Schönlieb 2012]
- Active contours with Wasserstein distances [Peyre, Fadili, Rabin 2012]
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- Here: “Kantorovich-Rubinstein-TV” denoising

- Image denoising in measure space
- **Kantorovich-Rubinstein norms**
- KR-TV denoising
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From L^1 to Kantorovich-Rubinstein

- Original Monge problem in Kantorovich form (1942)

$$W_1(\mu,\nu) = \inf_{\mu,\nu \text{ Marginales of } \gamma} \int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\gamma(x,y)$$

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- Kantorovich duality:

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- With $p=1$ Kantorovich-Rubinstein duality: For probability measures

$$\|\mu-\nu\|_{\mathrm{Lip}^*} = W_1(\mu,\nu)$$

with the dual Lipschitz norm

$$\|\mu\|_{\mathrm{Lip}^*} = \sup\{\int f \,\mathrm{d}\mu \;:\; \mathrm{Lip}(f) \leq 1\}$$

Kantorovich-Rubinstein norms

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- Obviously:

$$\|\mu\|_{\mathrm{KR},(\infty,1)} = \|\mu\|_{\mathrm{Lip}^*}, \quad \|\mu\|_{\mathrm{KR},(1,\infty)} = \|\mu\|_{\mathfrak{M}}$$

Dualities

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   $\rightsquigarrow$ **Transport formulation**

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   $\rightsquigarrow$ **Cascading formulation**

- Image denoising in measure space
- Kantorovich-Rubinstein norms
- **KR-TV denoising**
- Examples

L^1-TV and KR-TV

L^1-TV

$$\min_u \lambda \|u - u^0\|_1 + \mathrm{TV}(u)$$

L^1-TV **and** KR-TV

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$$\min_u \max_{|f|\le\lambda} \int f(u - u^0) + \mathrm{TV}(u)$$

L^1-TV and KR-TV

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KR-TV, primal “Lipschitz” formulation

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Relation to TGV denoising

- $\mathrm{TV}(u) = \||\nabla u|\|_{\mathfrak{M}}$

$$\rightsquigarrow \quad \mathrm{TGV}^2_\alpha(u) = \inf_w \alpha_2 \||\nabla u - w|\|_{\mathfrak{M}} + \alpha_1 \||Ew|\|_{\mathfrak{M}}$$

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L^1-TGV

Cascading TV with a vector field, penalize its derivative and obtain a *higher order regularizer* with similar properties.

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L^1-TGV

Cascading TV with a vector field, penalize its derivative and obtain a *higher order regularizer* with similar properties.

KR-TV

Cascading L^1 with the divergence of a vector field, penalize its magnitude and obtain a *lower order discrepancy* with similar properties.

Relation to G-TV cartoon-texture decomposition

- Meyer's G-norm:

$$\|u\|_G = \inf\{\||v|\|_\infty \ : \ \operatorname{div} v = u\}$$

Gets small for oscillating patterns

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$$\min_u \lambda\|u - u^0\|_G + \mathrm{TV}(u) = \min_{u,v} I_{\{0\}}(u - u^0 - \operatorname{div} v) + \lambda\||v|\|_\infty + \mathrm{TV}(u)$$

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KR-TV cascading formulation

$$\min_u \|u - u^0\|_{\mathrm{KR},\lambda} + \mathrm{TV}(u) = \min_{u,v} \lambda_1\|u - u^0 - \operatorname{div} v\|_{\mathfrak{M}} + \lambda_2\||v|\|_{\mathfrak{M}} + \mathrm{TV}(u)$$

Analytical results

Reproduction: For λ_1, λ_2 large enough, minimizer equals u^0.
$\rightsquigarrow$ KR is exact penalty (similar to L^1)

Reduction to the mean value: For λ_1 small enough, minimizer equals mean value of u^0.

Weak maximum principle: If u^0 is bounded, then there is a minimizer $\bar{u}$ such that $\|\bar{u}\|_\infty \leq \|u^0\|_\infty$.
$\rightsquigarrow$ Positivity is preserved (similar to L^1-TV and many others)

Weak mass preservation: For $\frac{\lambda_2}{\lambda_1} \leq \frac{2}{\operatorname{diam}\Omega}$ there is a minimizer that has the same mean as u^0.
$\rightsquigarrow$ Method reconstructs overall density accurately, no overall "intensity loss" (different from L^1-TV)

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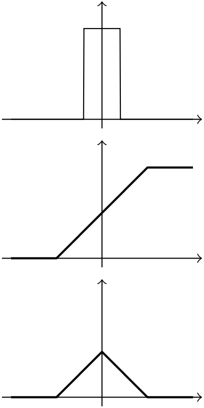
1D examples

Original | μ_2 large, $\mu_1 \to 0$ | μ_1 large, $\mu_2 \to 0$



L^1-TV | Lip^*-TV

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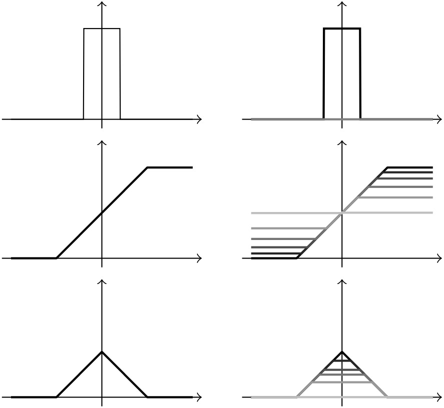

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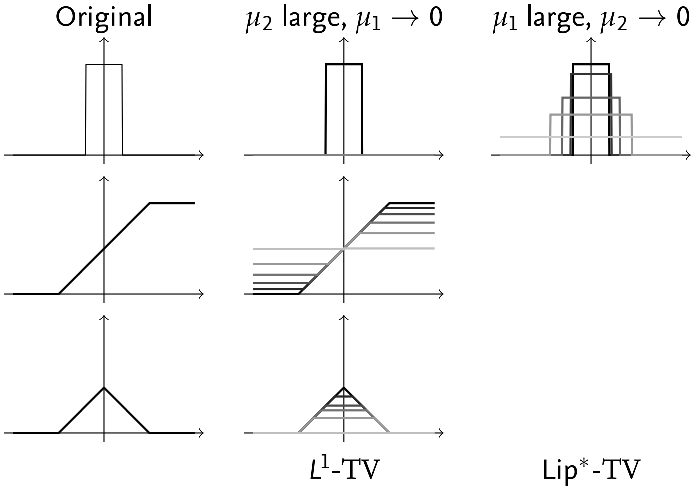

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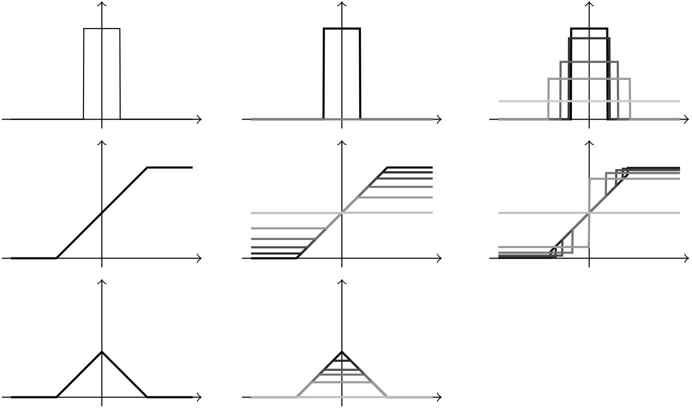

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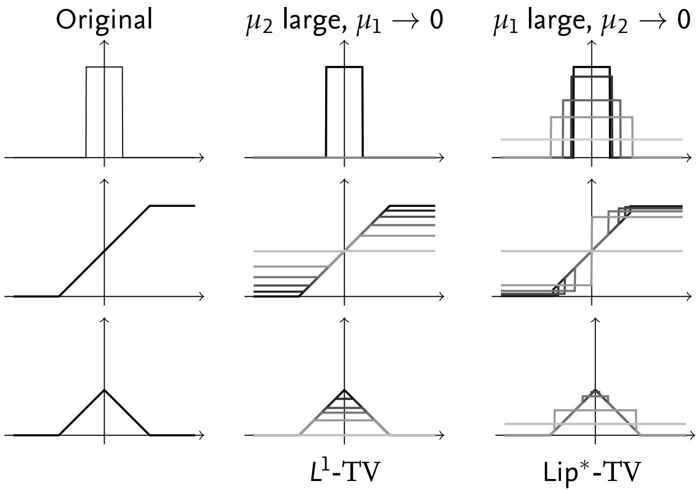

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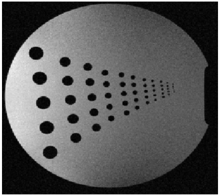
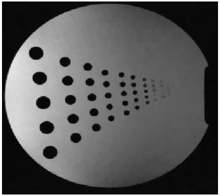
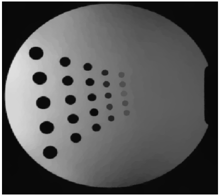
From L^1-TV to KR-TV: Helps with "suddenly disappearing objects"

L^1-TV | KR-TV





$\lambda_1 = 0.6$



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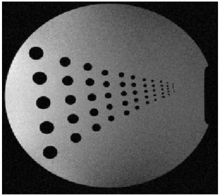
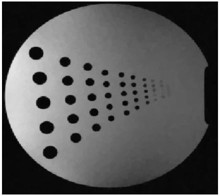
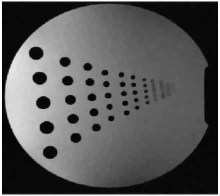
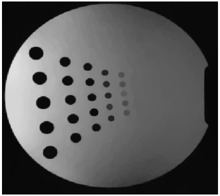
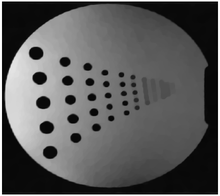






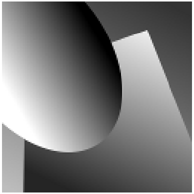
$\lambda_1 = 0.6$ | $\lambda_2 = 0.004$



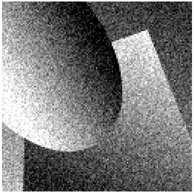


$\lambda_1 = 0.3$ | $\lambda_2 = 0.002$

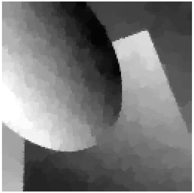
Helps with staircasing, gives small errors



$u^\dagger$

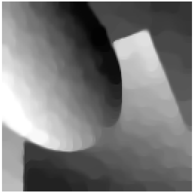


noisy, u^0



L^1-TV

$\|u - u^\dagger\|_{L^1} = 295.7$



KR-TV

$\|u - u^\dagger\|_{L^1} = 253.7$

(Parameters optimized for smallest L^1-error)

Cartoon-texture decomposition

Compare L^1-TV, G-TV and KR-TV

Parameter choice:

- Start with L^1-TV. Choose λ such that most texture is in the texture component, but also some structure.
- In G-TV choose λ such that the TV-seminorm is equal to the result from above.
- In KR-TV set $\lambda_1 = \infty$ and λ_2 such that the TV-seminorm is equal to the result from above.

Cartoon-texture decomposition

Conclusion

- The Kantorovich-Rubinstein norm generalizes the Radon norm and the L^1-norm.
- It can be used as a discrepancy term and the minimization can be formulated as a convex-concave saddle-point problem.
- Cascading reformulation well suited numerically, primal "Lipschitz" formulation suited for analysis
- KR-TV relates texture models and optimal transport.
- KR-TV denoising also preserves edges, may lead to less staircasing, may introduce new edges, performs good in cartoon-texture decomposition.
- Favorable properties: New edges but maximum principle, exact penalty, mean preservation
- Straightforward extension to KR-TV reconstruction

Technische Universität Braunschweig

Imaging with Kantorovich-Rubinstein discrepancy

Dirk Lorenz joint work with Jan Lellmann, Carola Schönlieb and Tuomo Valkonen,
July 4th, 2014

Institut für Analysis und Algebra

- **Image denoising in measure space**
- **Kantorovich-Rubinstein norms**
- KR-TV **denoising**
- **Examples**