Exercise sheet 6

Exercise 1 [Q-linear and r-linear convergence]

Construct a sequence (x_k) which converges r-linear but not q-linear.

Exercise 2 [Problematic cases for Newton's Method]

a) Let $f(x) = |x|^p$, $p > 2$ and $x^0 > 0$. We consider Newton's Method for optimization problems, Algorithm 2.6.1 from the lecture, for finding the global minimum $x^* = 0$ of f with initial guess x^0. Show, that the Newton iteration converges q-linear to x^* and specify the convergence rate γ. Further show, that the convergence is not q-superlinear. Why does this fact not contradict Theorem 2.45 from the lecture?

b) Let $f(x) = \exp(-1/|x|)$ for $x \neq 0$ and $f(0) := 0$. Show, that f is a C^2-function with $f'(0) = f''(0) = 0$. Prove, that Algorithm 2.6.1 for minimizing f converges strictly monotonic to $x^* = 0$ for all $0 < x^0 < 1/3$. Moreover argument that the q-convergence rate is worse than linear.

Exercise 3 [Convergence of inexact Newton's method]

Prove Theorem 2.52, 4)-6) from the lecture. Hint: Use similar arguments as in the proof of Theorem 2.45, 4)-6).

Exercise 4 [Gauß-Newton method]

Let $F\colon \mathbb{R}^n \to \mathbb{R}^m$, $m \geq n$, be a C^2-function. Then we consider the problem

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|F(x)\|_2^2 = f(x). \tag{1}$$

We intend to solve (1) by applying the Gauß-Newton method defined by

for $k = 0, 1, \dots$ **do**
  Solve $F'(x^k)^\top F'(x^k)s^k = -F'(x^k)^\top F(x^k)$ for s^k;
  Set $x^{k+1} = x^k + s^k$;
end

Algorithm 1: Gauß-Newton method

a) Apply Algorithm 2.6.1 to (1) and compare it to Algorithm 1. What are the differences?

b) Construct a convex quadratic function $q_k(s)$ such that $F'(x^k)^\top F'(x^k)s^k = -F'(x^k)^\top F(x^k)$ is equivalent to $\nabla q_k(s^k) = 0$ and interpret s^k as a solution of a suitable quadratic optimization problem.

c) Let $\bar{x} \in \mathbb{R}^n$ be a solution of (1) and $F'(\bar{x})$ has full rank. Argue, that Algorithm 1 is well defined in a neighbourhood of $\bar{x}$.

d) Furthermore let $F(\bar{x}) = 0$ and let (x^k) with $x^k \to \bar{x}$ be generated by Algorithm 1. Show, that (x^k) converges q-superlinear to $\bar{x}$.