Exercise sheet 2

Exercise 1 [Convex minimization]

Let $X \subseteq \mathbb{R}^n$ be open and $C \subseteq X$ be convex. Moreover the function $f\colon X \to \mathbb{R}$ is assumed to be continuously differentiable as well as convex on C, i.e.

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall \lambda \in [0,1],\ x, y \in C.$$

The function f is strict convex, if the inequality sign is strict for $x \neq y$ and $\lambda \in (0,1)$.

a) Prove: Every local minimum of f on C is a global minimum of f.

b) Prove that f is convex resp. strict convex on C iff there holds:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \quad \forall x, y \in C$$

resp. with $>$ and $x \neq y$.

c) Prove: A strongly convex function on C is strictly convex on C, but not necessarily the other way around.

d) Prove: $\bar{x} \in C$ solves $\min_{x \in C} f(x)$ iff

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \forall x \in C$$

and $\bar{x}$ is unique, if f is strictly convex.

Exercise 2 [Quadratic minimization problems]

Let $A \in \mathbb{R}^{n \times n}$ symmetric, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then we consider the minimization problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\langle Ax, x \rangle + \langle b, x \rangle + c = f(x). \tag{1}$$

a) Prove that the following statements are equivalent:

(1) The point $\hat{x}$ satisfies $\nabla f(\hat{x}) = 0$ and A is positive semi-definite.

(2) The point $\hat{x}$ is a global minimum of (1)

(3) The point $\hat{x}$ is a local minimum of (1).

b) Prove that the following statements are equivalent:

(1) The vector b is an element of column space of A and A is positive semi-definite.

(2) The problem (1) has at least one global solution.

(3) The function f is bounded from below.

c) Let $C \subseteq \mathbb{R}^n$ be closed and convex and A positive definite. Prove that $\min_{x\in C} f(x)$ has a unique solution. Formulate its first order optimality conditions.

Exercise 3 [Regularized Least Squares Problem]

Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{k\times n}$ with $m, k \geq n$, $\alpha > 0$ and $b \in \mathbb{R}^m$. We consider the optimization problem

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \frac{\alpha}{2}\|Bx\|_2^2. \tag{2}$$

a) Prove that $A^\top A + \alpha B^\top B$ is positive definite if $\ker A^\top A \cap \ker B^\top B = \{0\}$. Conclude that (2) then has a unique solution and characterize it.

Next we consider the optimization problem

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \alpha\|x\|_1 \tag{3}$$

with $\|x\|_1 = \sum_{i=1}^n |x_i|$.

b) Prove that problem (3) has a solution. Is it unique? Reformulate (3) as a constrained quadratic optimization problem and formulate its first order optimality conditions.

Exercise 4 [Polynomial Interpolation]

Let $t_1 < ... < t_m$ and $y_1, ..., y_m$ be $2m$ real numbers. In this exercise, the coordinates of a vector $x \in \mathbb{R}^{n+1}$ are denoted x_0,x_1,...,x_n. For all $x \in \mathbb{R}^{n+1}$, we denote by P_x the polynomial function defined by:

$$P_x(t) = \sum_{i=0}^n x_i t^i.$$

We consider the following optimization problem:

$$\min_{x\in\mathbb{R}^n} f(x) := \frac{1}{2}\sum_{j=1}^m \big(P_x(t_j) - y_j\big)^2. \tag{4}$$

a) Motivate the problem.

b) Find a matrix $C \in \mathbb{R}^{m\times(n+1)}$ such that for all $x \in \mathbb{R}^{n+1}$,

$$Cx = (P_x(t_1), ..., P_x(t_m)).$$

c) Find a matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$, a vector $b \in \mathbb{R}^{n+1}$, and a real number c such that (4) has the form (1).

d) Prove that (4) has a unique solution in the case that $m \geq n+1$.

Exercise 5 [Programming exercise: Sparse deconvolution]

Let $u \in \mathbb{R}^n$ be a sparse signal, i.e. the components of u are mostly zero. Moreover we are given a kernel $k \in \mathbb{R}^m$, for example a Gaussian kernel

$$k(i) = \exp\left(\frac{-x(i)^2}{2\sigma^2}\right)$$

for $\sigma > 0$ and $x(i) \in [-L, L]$ for $i = 1, \dots, m$. The convolution of u and k is defined by

$$y(j) = (k * u)(j) = \sum_i k(i)u(j-i) \quad j = 1, \dots, n+m-1.$$

with $u(l) = 0$ for $l \leq 0$. The vector y is a blurred version of the signal u. The convolution between k and u is linear in u and thus can be represented by a matrix $C \in \mathbb{R}^{n+m-1\times n}$. Now we assume that we are given a noisy observation y_n of $y = Cu$ of the form $y_n = y + d$ where d is a random vector in $\mathbb{R}^n$. Now we intend to reconstruct the exact signal u from y_n by solving

$$\min_{u \in \mathbb{R}^n} \|Cu - y_n\|_2^2 \tag{5}$$

as well as the problem (2) with $A = C$, $b = y_n$ and $B = \mathrm{Id}_n$ resp. (3) with $A = C$ and $b = y_n$.

a) Solve the sparse deconvolution problem by completing the following m-file.

b) Compare the reconstruction computed by (5), (2) resp. (3). Which is the best? Why? In which sense is the deconvolution problem ill-posed? How do (3) and (2) compensate for the ill-posedness. Write your answer in the email for the assignment.

c) Change the noise level in your code. Does the quality of the solution change and how can you compensate for that?

Hand in by email (philip.trautmann@uni-graz.at) until 14.10.2019, 23:59 o'clock.

```
function [u]=l1deconv()

n = 256; %Length signal
m = 256; %Length kernel

ue = zeros(n,1); %Exact signal
ue(20) = 100;
ue(100) = 20;
ue(200) = -100;
ue(250) = -20;

alpha = 1; %Regularization parameter

x_k = linspace(-10,10,m); %Build kernel
sigma = 1.0;
k = exp(-x_k.^ 2 / (2 * sigma ^ 2)); %Gaussian kernel
C = toeplitz([k'; zeros(n-1,1)],[k(1),zeros(1,n-1)]); %Convolution matrix

rng(1234)
noise = 0.03; %Relative noise level
dis = rand(n+m-1,1); %Additive noise
y = C*ue+noise*dis*norm(C*ue,inf)*norm(dis,inf)^-1; %Disturbed convolution
figure(1)
plot(y) %Plot disturbed convolution

ul1 = ??? %Calculate L1-reconstruction with quadprog in Matlab or Octave

ul2 = ??? %Calculate L2-reconstruction by solving a linear system

uls = ??? %Calculate the least-squares reconstruction
          %by solving the normal equations.

figure(2)
plot(ul1) %Plot L1-reconstruction
hold on
plot(ul2) %Plot L2-reconstruction
plot(ue) %Plot exact signal
hold off
figure(3)
plot(uls) %Plot least-squares reconstruction
```