Exercise sheet 12

Exercise 1 [LICQ]

Prove Corollary 3.28 and Remark 3.29.

Exercise 2 [Optimal control]

We consider in this exercise a system which evolves over time under the control of a 'manager'. We denote by $x_0 \in \mathbb{R}^n$ the state of the system at time 0, by $x_1 \in \mathbb{R}^n$ the state at time 1, and so on until time N. The variable x_i is called *state variable*. In our model, we consider that x_{i+1} depends on the previous value of the state x_i and an additional variable, $u_{i+1} \in \mathbb{R}^m$, called *control variable*:

$$x_{i+1} = f_i(x_i, u_{i+1}), \quad \text{for all } i = 0, ..., N-1.$$

The function $f\colon \{0, ..., N-1\} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is known and assumed to be continuously differentiable with respect to (x, u). For example, the manager can be a gas company: the variable x_i models the level of stocks of gas at a given time and the variable u_i the quantity of gas sold or bought on a market.
Let us consider now the following optimization problem:

$$\inf_{\substack{x_1, x_2, ..., x_N \in \mathbb{R}^n \\ u_1, u_2, ..., u_N \in \mathbb{R}^m}} \sum_{i=0}^{N-1} \ell_i(x_i, u_{i+1}) + \phi(x_N), \quad \text{subject to: } x_{i+1} = f_i(x_i, u_{i+1}), \quad \forall i = 0, ..., N-1.$$

The function $\ell\colon \{0, ..., N-1\} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is given and has the same regularity as f. The function ϕ is given and is continuously differentiable. The initial value of the state x_0 is also known and fixed.
The problem under investigation can be recasted as a *reduced* unconstrained optimization problem: it suffices to eliminate the variables x_1,...,x_N. Indeed, given the parameter x_0 and the controls u_1,...,u_N, the variable x_1 is uniquely defined by $x_1 = f_0(x_0, u_1)$, the variable x_2 by $x_2 = f_1(f_0(x_0, u_1), u_2)$, and so on.
The goal of the exercise is to calculate the gradient of the cost function of the reduced problem. To this purpose, we introduce the following function, for $k \in \{0, ..., N\}$:

$$J_k(x_k, u_{k+1}, ..., u_N) = \Big(\sum_{i=k}^{N-1} \ell_i(x_i, u_{i+1}) \Big) + \phi(x_N),$$

where x_{k+1},...,x_N are defined by: $x_{i+1} = f_i(x_i, u_{i+1})$, for $i = k, ..., N-1$. Note that $J_N(x_N) = \phi(x_N)$. The goal is now to compute the gradient of $J_0(x_0, u_1, ..., u_N)$ with respect to the control variables.

a) Find a relation between J_{k-1} and J_k.

b) Let us fix u_1,...,u_N. Let x_1,...,x_N be such that $x_{i+1} = f_i(x_i, u_{i+1})$, for all $i = 0, ..., N-1$. We set:

$$p_k = \nabla_{x_k} J_k(x_k, u_{k+1}, ..., u_N),$$

for all $k = 0, ..., N$. Note that $p_N = \nabla\phi(x_N)$. Find a relation between p_{k-1} and p_k.

c) Prove the following formula:

$$\nabla_{u_k} J_0(x_0, u_1, ..., u_N) = \nabla_u \ell_{k-1}(x_{k-1}, u_k) + \nabla_u f_{k-1}(x_{k-1}, u_k) p_k.$$

We regard this problem as a problem with constraints and do not follow anymore the approach consisting in eliminating the variables $(x_1, ..., x_N)$. Consider a solution $(x_1, ..., x_N, u_1, ..., u_N)$ to the problem.

d) Prove that the LICQ is satisfied and write the KKT conditions (denote by p_1 the Lagrange multiplier associated with the constraint $f_0(x_0, u_1) - x_1 = 0$, by p_2 the Lagrange multiplier associated with the constraint $f_1(x_1, u_2) - x_2 = 0$, and so on).

Exercise 3 [Second order optimality conditions]

We consider for $n \in \mathbb{N}$ the problem

$$\min_{x \in \mathbb{R}^n} f(x) = \sum_{j=1}^{n} x_j^j \quad \text{s.t.} \quad g(x) = 1 - \|x\|_2^2 \leq 0. \tag{P1}$$

Let $X := \{x \in \mathbb{R}^n : g(x) \leq 0\}$.

a) Is (P1) convex?

b) Show, that in all admissible points of (P1) a CQ is satisfied.

c) Show, that $\bar{x} = (1, 0, 0, \ldots, 0)^T \in \mathbb{R}^n$ with the multiplicator $\bar{\lambda} = \frac{1}{2}$ a KKT pair of (P1) is.

d) Calculate $T(X, \bar{x})$ and $T_+(g, \bar{x}, \bar{\lambda})$ and simplify as far as possible.

e) Show using the second order sufficient and necessary optimality conditions, that $\bar{x}$ in the case $n \leq 2$ a local minimum of (P1) is, but in the case $n \geq 3$ not.

Exercise 4 [KKT conditions in the convex case]

The goal of this exercise is to prove that in the case of a convex problem, the KKT conditions are not only necessary conditions, but also sufficient conditions for optimality.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and convex, let $g : \mathbb{R}^n \to \mathbb{R}^m$ be such that for all $i = 1,...,m$, g_i is a differentiable convex function, let $h : \mathbb{R}^n \to \mathbb{R}^p$ be an affine function. We consider the problem $P2$:

$$\min_{x\in\mathbb{R}^n} f(x), \text{ subject to: } g(x) \leq 0, \; h(x) = 0. \tag{$P2$}$$

Let x^* be a feasible point satisfying the KKT conditions. We denote by $L(x, \lambda, \mu) = f(x) + \langle \lambda, g(x)\rangle + \langle \mu, h(x)\rangle$ the Lagrangian, let (λ^*, μ^*) be the Lagrange multipliers associated with x^*. We also define:

$$N(x^*) = \left\{(\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^n \text{ such that } \lambda^\top g(x^*) = 0\right\}. \tag{1}$$

a) Prove that x^* is a global minimizer of $x \mapsto L(x, \lambda^*, \mu^*)$.

b) Prove the following inequality : for all feasible point x of problem $P2$, for all $(\lambda, \mu) \in N(x^*)$,
$$f(x) - f(x^*) \geq L(x, \lambda, \mu) - L(x^*, \lambda, \mu). \tag{2}$$

c) Deduce that x^* is a global minimizer of problem $P2$.