Exercise sheet 11

Exercise 1 [Properties of the tangential cone]

Let $M \subset \mathbb{R}^n$ be a nonempty set and $x \in M$. Show, that the tangential cone $T(M, x)$ is a nonempty and closed cone.

Exercise 2 [Tangential cone for linear constraints]

a) First we consider the following constraint optimization problem with linear constraints

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad Ax = b,\ Cx \leq d$$

with $f : \mathbb{R}^n \to \mathbb{R}$, $A \in \mathbb{R}^{p \times n}$, $C \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^p$ und $d \in \mathbb{R}^m$. We set

$$\hat{g}(x) := Cx - d \quad \text{and} \quad \hat{h}(x) := Ax - b,$$

so the admissible set has the form $Z := \{x \in \mathbb{R}^n : \ \hat{g}(x) \leq 0,\ \hat{h}(x) = 0\}$.

Let $\bar{x} \in Z$. Show, that $T_l(\hat{g}, \hat{h}, \bar{x}) \subset T(Z, \bar{x})$ and conclude, that for constraint optimization problems with linear constraints $T_l(\hat{g}, \hat{h}, \bar{x}) = T(Z, \bar{x})$ always holds true.

b) Now we consider a general constraint optimization problem of the form

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g(x) \leq 0,\ h(x) = 0 \tag{P1}$$

with continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ und $h : \mathbb{R}^n \to \mathbb{R}^p$.

For a fixed and admissible $\bar{x}$ we consider the following problem

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g^l(x) \leq 0,\ h^l(x) = 0 \tag{P2}$$

with $g^l(x) := g(\bar{x}) + \nabla g(\bar{x})^T (x - \bar{x})$, $h^l(x) := h(\bar{x}) + \nabla h(\bar{x})^T (x - \bar{x})$ and the admissible set $X_l(\bar{x}) := \{x \in \mathbb{R}^n : \ g^l(x) \leq 0,\ h^l(x) = 0\}$.

Show, that $T_l(g, h, \bar{x}) = T_l(g^l, h^l, \bar{x})$ holds..

c) Conclude, that for an admissible $\bar{x}$ the linearized tangetial cone $T_l(g, h, \bar{x})$ of (P1) coincides with the tangential cone $T(X_l(\bar{x}), \bar{x})$ of the linearized problem (P2).

Exercise 3 [Abadie and Guignard constraint qualification]

Calculate for the admissible set $X := \{x \in \mathbb{R}^2 : g(x) \leq 0\}$ the tangetial cone $T(X, \bar{x})$ and the linearized tangential cone $T_l(g, \bar{x})$ in the point $\bar{x}$ and decide, if $T(X, \bar{x}) = T_l(g, \bar{x})$ holds. Moreover check, if $T(X, \bar{x})^\circ = T_l(g, \bar{x})^\circ$ holds.

a) $g(x) = (x_2 - x_1^5, -x_2)^T$, $\bar{x} = (0, 0)^T$.

b) $g(x) = (x_2^2 - x_1 + 1, 1 - x_1 - x_2)^T$, $\bar{x} = (1, 0)^T$.

Exercise 4 [Geometric and arithmetic mean]

a) Show, that the following optimization problem

$$\min \ -\prod_{j=1}^{n} x_j \quad \text{s.t.} \quad \sum_{j=1}^{n} x_j = 1, \quad x \geq 0$$

has a global solution $\bar{x} \in \mathbb{R}^n$.

b) Moreover show, that $\bar{x} > 0$ has to hold and that in $\bar{x}$ the KKT-conditions are satisfied. Use the KKT-conditions in order to calculate $\bar{x}$.

c) Use b), in order to show the following inequality:

$$\left(\prod_{j=1}^{n} x_j\right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{j=1}^{n} x_j \quad \text{für alle} \ \ x \in \mathbb{R}^n, \ x \geq 0.$$