Exercise sheet 1

Exercise 1 [Optimal design of a building]

We are looking for an optimal design of a box-shaped building. Let l be the length of the building, b its width, h its height (over ground) and t its depth (under ground). In the following we neglect the thickness of the walls, floors and ceilings. The owner has the following demands:

- The building should be at least as long as broad, but should have at most double the length as its width.
- The length l must not be longer than 40 m.
- The height h must not be longer than its length.
- All floors should have the same height of at least 3.50 m.
- At least 10%, but at most 25% of the building should be under ground.
- The ground floor should be flat.
- The combined area of all floors should be at least 10000 m^2.
- The average yearly cost for heating are estimated to be 100 Euro per m^2 of the outer walls of the building over ground. The yearly cost for heating must not be greater than 500000.

The building should be designed in such a way that the amount of soil excavated is minimal.

a) Formulate this problem as constrained optimization problem.

b) Does this optimization problem have admissible points?

c) Does the optimization problem have a solution?

Exercise 2 [First- and second-order necessary and sufficient optimality conditions]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function. Let $\bar{x}$ be a local minimizer of f, that is to say, there exists $\varepsilon > 0$ such that for all $x \in \mathbb{R}^n$,

$$\|x - \bar{x}\| \leq \varepsilon \Longrightarrow f(x) \geq f(\bar{x}).$$

a) Prove that $Df(\bar{x}) = 0$ and that $D^2 f(\bar{x})$ is positive semi-definite.

b) If $Df(\bar{x}) = 0$ and $D^2 f(\bar{x})$ is positive definite, then $\bar{x}$ is a local minimizer of f.

c) We consider now the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \cos(x + y) + \sin(xy) + 2y^2.$$

Is the point $(0, 0)$ a local minimizer?

Exercise 3 [Proving the existence of a minimizer]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. We say that the function f is coercive if the following property holds: for all sequences $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$,

$$\|x_k\| \underset{k\to\infty}{\longrightarrow} +\infty \Longrightarrow f(x_k) \underset{k\to\infty}{\longrightarrow} +\infty.$$

a) Assume that f is coercive and prove the existence of a global minimizer. To this purpose, you can first prove that for a fixed $x_0 \in \mathbb{R}^n$, the following set A is non-empty and compact:

$$A = \big\{x \in \mathbb{R}^n \,|\, f(x) \leq f(x_0)\big\}.$$

b) Prove that the coercivity of f is independent of the choice of the norm.

c) Compute a minimizer of $f(x, y) = x^4 + x^2 + y^2 - xy$. Justify carefully your answer.

Exercise 4 [Strongly convex functions]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. We assume that f is strongly convex, that is to say, that there exists $m > 0$ such that for all x and $y \in \mathbb{R}^n$,

$$\langle \nabla f(y) - \nabla f(x), y - x\rangle \geq m\|y - x\|^2.$$

In this exercise, the norm $\|\cdot\|$ denotes the Euclidean norm.

a) Let $\tilde{x} \in \mathbb{R}^n$. Prove the following inequality: for all $x \in \mathbb{R}^n$,

$$f(x) \geq g(x) := f(\tilde{x}) + \langle \nabla f(\tilde{x}), x - \tilde{x}\rangle + \frac{1}{2}m\|x - \tilde{x}\|^2.$$

b) Prove that g possesses a unique minimizer. Calculate it. Calculate also $\min_{x\in\mathbb{R}^n} g(x)$.

c) Deduce from the above inequality that f possesses a unique minimizer $\bar{x}$.

d) Prove that

$$\|\tilde{x} - \bar{x}\| \leq \frac{\|\nabla f(\tilde{x})\|}{m} \quad \text{and} \quad f(\tilde{x}) - f(\bar{x}) \leq \frac{1}{2m}\|\nabla f(\tilde{x})\|^2.$$

Exercise 5 [Normal equations]

Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ and $b \in \mathbb{R}^m$. We consider the optimization problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2^2 = f(x),$$

where $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^m$.

a) Under which assumption on A is f strongly convex?

b) Conclude that $\min_{x\in\mathbb{R}^n} f(x)$ has a unique solution $\bar{x}$ and characterize it.

c) Prove: $\bar{x}$ solves $\min_{x\in\mathbb{R}^n} f(x)$ if and only if the residuum $Ax - b$ is orthogonal to $\mathrm{im}(A)$. Interpret and illustrate this statement using a sketch.