Exercise Sheet 11

Exercise 1 [Taylor Polynomial of Multiplication]

(a) Let $n \in \mathbb{N} \cup \{0\}$ and let P be a polynomial of degree at most n. Assume that there exists x_0 such that in a neighbourhood of x_0

$$P(x) = R(x)\,(x - x_0)^n\,,$$

where $R(x)$ is a continuous function at x_0 such that $R(x_0) = 0$. Show that $P \equiv 0$.

Hint: Remember that if $P(x_0) = 0$ then there exists a polynomial $Q(x)$ with $\deg(Q) = \deg(P) - 1$ such that $P(x) = (x - x_0)Q(x)$. Use induction.

(b) Let I be an interval and $x_0 \in I$. For an arbitrary polynomial of degree $k \geq 0$ around x_0,

$$P(x; x_0) = \sum_{j=0}^{k} a_j\,(x - x_0)^j\,, \qquad \{a_j\}_{j=0}^{k} \subset \mathbb{R},$$

we define for $m \in \mathbb{N} \cup \{0\}$, $m \leq k$, the *$m-th$ truncation of P* by

$$\mathrm{Trunc}_m\,(P)\,(x; x_0) = \sum_{j=0}^{m} a_j\,(x - x_0)^j\,.$$

Let $n \in \mathbb{N} \cup \{0\}$ and $f, g \in C^{n+1}(I)$. Show that

$$\mathrm{Trunc}_n\left(T_f^n \cdot T_g^n\right)(x; x_0) = T_{fg}^n(x; x_0).$$

Hint: Use $h(x) = T_h^n(x; x_0) + R_h^{n+1}(x; x_0)$ with R_h^{n+1} in Lagrange form. Apply (a).

Remark: There are cases where one can simplify the computation of the $n-$th truncation of $T_f^n \cdot T_g^n$. Indeed, if

$$T_f^n(x; x_0) = (x - x_0)^l\,Q(x)$$

for $l \leq n$, then

$$\mathrm{Trunc}_n\left(T_f^n \cdot T_g^n\right)(x; x_0) = (x - x_0)^l\,\mathrm{Trunc}_{n-l}\left(Q(x) \cdot T_g^{n-l}\right)(x; x_0).$$

You may use this without proof in (c).

(c) Use (b) to compute $f^{(8)}(0)$ for

$$f(x) := x^6 \sin(x) e^x$$

without differentiating f.

Remark: A similar but more involved argument holds for *composition of functions*.

Exercise 2 [L'Hospital à la Taylor]

(a) Let $f, g \in C^{n+1}(I)$, where I is an interval that contains the point x_0. Assume that

$$f^{(k)}(x_0) = g^{(k)}(x_0) = 0, \quad k = 0, \dots, n-1,$$

and that $g^{(n)}(x_0) \neq 0$. Use the Taylor polynomials

$$T_f^n(x; x_0) = \sum_{j=0}^{n} a_j (x - x_0)^j, \quad T_g^n(x; x_0) = \sum_{j=0}^{n} b_j (x - x_0)^j$$

to show that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{a_n}{b_n} = \frac{f^{(n)}(x_0)}{g^{(n)}(x_0)}.$$

(b) Compute

$$\lim_{x \to 0} \frac{e^x - 1 - \ln(1+x)}{\sin^2(x)}.$$

Exercise 3 [Harmonic Oscillator Solution by Taylor Expansion]

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function that satisfies the differential equation

$$f''(x) + f(x) = 0 \quad \text{for all } x \in \mathbb{R}.^1$$

The goal of this exercise is to establish the representation

$$f(x) = f(0)\cos(x) + f'(0)\sin(x), \qquad \text{for all } x \in \mathbb{R}.$$

(a) Explain why f is continuously differentiable infinitely many times, and show that for all $n \in \mathbb{N}$ we have

$$f^{(2n)}(x) = (-1)^n f(x), \quad \text{and} \quad f^{(2n+1)}(x) = (-1)^n f'(x).$$

(b) Determine $T_f^n(x; 0)$ for $n \geq 0$ and show that $\lim_{n\to\infty} R_f^{n+1}(x; 0) = 0$.

(c) Deduce that $T_f(x; 0) = f(x)$ and conclude the above representation for f.

Exercise 4 [A series representation for π]

We recall from Analysis 1 the *geometric series formula*:

$$\forall n \in \mathbb{N} \cup \{0\} : \qquad \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z} \quad \forall z \in \mathbb{C} \setminus \{1\}.$$

Show that:

[1] This is the governing equation of a one-dimensional harmonic oscillator.

(a) For any $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we have that $\frac{1}{1+t^2} = \frac{(-t^2)^{n+1}}{1+t^2} + \sum_{k=0}^{n}(-t^2)^k$.

(b) For any $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we have that $\arctan(x) = \int_0^x \frac{(-t^2)^{n+1}}{1+t^2}\,\mathrm{d}t + \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{2k+1}$.

(c) For $x \in [-1,1]$ we have that $\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$.

(d) The following formula is valid:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \pm \dots$$

(e) Without any computations (use results from the lecture instead):

- The radius of convergence of the power series in (c) is at least 1.
- The series in (c) is the Taylor series of arctan around $x_0 = 0$.